Vintage Model

Utility:

$$U(c_t) = ((1-\varepsilon))^{-1}\left(c_t^{((1-\varepsilon))} - 1\right)$$

Production:

$$y = z(a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon})^{\frac{1}{(1-\varepsilon)}}$$

Value function

$$\begin{aligned} &V(k_1, k_2, k_3, z) \\ &= \underset{c,h}{Max}\ ((1-\varepsilon)\eta)^{-1}\left(c_t^{((1-\varepsilon))} - 1\right) \\ &\quad + \delta EV\left(\begin{array}{c} z\left(\begin{array}{c} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} \\ + a_3 k_3^{1-\varepsilon} \end{array} \right)^{\frac{1}{(1-\varepsilon)}} \\ -c, \mu_1 k_1, \mu_2 k_2, z' \end{array} \right) \end{aligned}$$

FOC:

wrt c :

$$c^{-\varepsilon} = \delta E V_1' \left(\begin{array}{c} z \left(\begin{array}{c} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} \\ + a_3 k_3^{1-\varepsilon} \end{array} \right)^{\frac{1}{1-\varepsilon}} \\ -c, \mu_1 k_1, \mu_2 k_2, z' \end{array} \right)$$

Now differentiating $V(k_1, k_2, k_3, z)$, where $'$, $''$, $'''$ indicates variables one, two, three periods ahead:

$$V_3 = \delta E V_1' \cdot MPK_3 = c^{-\varepsilon} MPK_3$$

$$\begin{aligned} V_2 &= c^{-\varepsilon} MPK_2 + \mu_2 \delta E V_3' \\ &= c^{-\varepsilon} MPK_2 + \mu_2 \delta E (c')^{-\varepsilon} MPK_3' \end{aligned}$$

$$\begin{aligned} V_1 &= c^{-\varepsilon} MPK_1 + \mu_1 \delta E V_2' \\ &= c^{-\varepsilon} MPK_1 + \mu_1 \delta E \left(\begin{array}{c} (c')^{-\varepsilon} MPK_2' \\ +\mu_2 \delta E (c'')^{-\varepsilon} MPK_3'' \end{array} \right) \end{aligned}$$

Updating V_1 :

$$\begin{aligned} V_1' &= (c')^{-\varepsilon} MPK_1' + \mu_1 \delta E V_2'' \\ &= (c')^{-\varepsilon} MPK_1' + \mu_1 \delta E \left(\begin{array}{c} (c'')^{-\varepsilon} MPK_2'' \\ +\mu_2 \delta E (c''')^{-\varepsilon} MPK_3''' \end{array} \right) \end{aligned}$$

So FOC becomes:

$$c^{-\varepsilon} = \delta E \left[\begin{array}{c} (c')^{-\varepsilon} MPK_1' + \mu_1 \delta (c'')^{-\varepsilon} MPK_2'' \\ +\mu_1 \mu_2 \delta^2 (c''')^{-\varepsilon} MPK_3''' \end{array} \right]$$

Now we note that:

$$y = z(a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon})^{\frac{1}{(1-\varepsilon)}}$$

$$\begin{aligned} MPK_i &= z(a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon})^{\frac{\varepsilon}{(1-\varepsilon)}} a_i k_i^{-\varepsilon} \\ &= a_i z^{(1-\varepsilon)\omega} \frac{y^\varepsilon}{k_i^\varepsilon} \end{aligned}$$

So FOC is

$$c^{-\varepsilon} = \delta E \left[\begin{array}{c} (c')^{-\varepsilon} a_1 (z')^{(1-\varepsilon)} \frac{(y')^\varepsilon}{(k_1')^\varepsilon} \\ +\mu_1 \delta (c'')^{-\varepsilon} a_2 (z'')^{(1-\varepsilon)} \frac{(y'')^\varepsilon}{(k_2'')^\varepsilon} \\ +\mu_1 \mu_2 \delta^2 (c''')^{-\varepsilon} a_3 (z''')^{(1-\varepsilon)} \frac{(y''')^\varepsilon}{(k_3''')^\varepsilon} \end{array} \right]$$

Now let $c = \lambda y$ so $(c')^{-\varepsilon}(y')^\varepsilon = (\lambda')^{-\varepsilon}$:

$$(\lambda y)^{-\varepsilon} = \delta E \left[\begin{array}{c} (\lambda')^{-\varepsilon} a_1 (z')^{(1-\varepsilon)} (k_1')^{-\varepsilon} \\ +\mu_1 \delta (\lambda'')^{-\varepsilon} a_2 (z'')^{(1-\varepsilon)} (k_1'')^{-\varepsilon} \\ +\mu_1 \mu_2 \delta^2 (\lambda''')^{-\varepsilon} a_3 (z''')^{(1-\varepsilon)} (k_1''')^{-\varepsilon} \end{array} \right]$$

If z is iid or constant, then set
$\lambda = \lambda' = \lambda'' = \lambda'''$:

$$y^{-\varepsilon} = \delta E\begin{bmatrix} a_1(z')^{(1-\varepsilon)}((1-\lambda)y)^{-\varepsilon} \\ +\mu_1\delta a_2(z'')^{(1-\varepsilon)}(\mu_1(1-\lambda)y)^{-\varepsilon} \\ +\mu_1\mu_2\delta^2 a_3(z''')^{(1-\varepsilon)}(\mu_1\mu_2(1-\lambda)y)^{-\varepsilon} \end{bmatrix}$$

$$1 = \delta E\begin{bmatrix} a_1(z')^{(1-\varepsilon)}((1-\lambda))^{-\varepsilon} \\ +(\mu_1)^{1-\varepsilon}\delta a_2(z'')^{(1-\varepsilon)}((1-\lambda))^{-\varepsilon} \\ +(\mu_1\mu_2)^{1-\varepsilon}\delta^2 a_3(z''')^{(1-\varepsilon)}((1-\lambda)y)^{-\varepsilon} \end{bmatrix}$$

$$(1-\lambda)^{\varepsilon} = \delta E\begin{bmatrix} a_1(z')^{(1-\varepsilon)} + (\mu_1)^{1-\varepsilon}\delta a_2(z'')^{(1-\varepsilon)} \\ +(\mu_1\mu_2)^{1-\varepsilon}\delta^2 a_3(z''')^{(1-\varepsilon)} \end{bmatrix}$$

Solve for λ.

Let $x_t = (k_t)^{1-\varepsilon}$. Then,since

$$y = z(a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + a_3 k_3^{1-\varepsilon})^{\frac{1}{(1-\varepsilon)}}$$

we have a linear difference equation system:

$$x_{t+1} = ((1-\lambda)z_t)^{1-\varepsilon}(a_1 x_t + a_2 x_{t-1} + a_3 x_{t-2})$$

NOTE:We can also have work with a simple standard labor supply and solve for L_t:

$$U(c_t, L_t) = ((1-\varepsilon))^{-1}\left(c_t^{((1-\varepsilon))} - 1\right) + V(1 - L_t)$$

$$y = z\left(a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} + (1 - a_1 - a_2)(L)^{1-\varepsilon}\right)^{\frac{1}{(1-\varepsilon)}}$$

$$V(k_1, k_2, z)$$

$$= \underset{c,L}{Max}\ ((1-\varepsilon))^{-1}\left(c_t^{((1-\varepsilon))} - 1\right) + V(1 - L_t)$$

$$+ \delta EV\left(\begin{array}{c} z\left(\begin{array}{c} a_1 k_1^{1-\varepsilon} + a_2 k_2^{1-\varepsilon} \\ +(1 - a_1 - a_2)^{1-\varepsilon} L \end{array} \right)^{\frac{1}{(1-\varepsilon)}} \\ -c, \mu_1 k_1, z' \end{array} \right)$$

See Benhabib and Rustichini, JEDC(1994).