Designing Monetary Policy:  
Backward-Looking Interest-Rate Rules

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Abstract

When real balances have even a very small productive role, contemporaneous and forward-looking Taylor-Wicksell rules can induce ubiquitous multiplicities of equilibria and lead to consequences that are unintended by policymakers. This raises the issue of whether it is possible to anchor and stabilize the economy through backward-looking rules. We show that the standard uniqueness results that are found in the literature for linearized models with backward-looking, active Taylor rules can be misleading. Analyzing the dynamics of such models without recourse to linearizations or to quadratic approximations, we show that robust multiplicities abound for realistic calibrations. In particular, such rules can give rise to endogenous equilibrium cycles along which the economy oscillates perpetually around the intended steady state equilibrium. The amplitude of these cycles is found to be economically significant. We conclude that further research must be undertaken to design stabilizing monetary policies.

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1 Introduction

The procyclical use of nominal interest rate as an instrument of stabilization policy has been debated and studied in economics since the early work of Wicksell (1907). John Taylor (1993, 1998) has recently suggested that a simple “active” monetary policy rule “in which the reaction of the interest rate to inflation is above a critical threshold shows surprising efficiency and robustness over alternative monetary stabilization policies.” This prescription, maybe in response to the experience of problems with implementing a simple money growth rule, stands in sharp contrast to the earlier prominent views of Friedman (1960) who warned against the long and variable lags of countercyclical monetary policy: “There are persuasive theoretical grounds for desiring to vary the rate of growth of money to offset other factors. The difficulty is that in practice we do not know when to do so and by how much. In practice therefore, deviations from the simple rule have been destabilizing rather than the reverse.” (Friedman, 1960, p. 98). A particular source of the “difficulty in practice” are the multiple equilibria and indeterminacy that can be induced by such policies, giving rise to unintended consequences, and to what has been colorfully termed as “instrument instability.” Indeed a number of recent theoretical studies of “Wicksellian” interest rate policies and “Taylor rules”, set in a variety of standard (flexible price, sticky price, discrete time, continuous time, Ricardian, non-Ricardian, etc.) models, find a plethora of results on the multiplicity of equilibria, together with their implied instabilities and unintended consequences. (See, for example, Benhabib, Schmitt-Groh´e and Uribe, 2001a,b, and 2002; Carlstrom and Fuerst, 2000 and 2001; and Schmitt-Groh´e and Uribe, 2000) Such difficulties have raised the issue of the proper design of Taylor rules, and in particular, the issue of whether they should be forward or backward looking.

The intuition for why contemporaneous interest-rate feedback rules give rise to unique equilibria goes back to Leeper (1991). Consider a very simple endowment economy with an infinitely lived representative agent, where real balances enter the utility function in a separable way, and where the monetary authority sets the nominal rate, $R_t$ as a function of the inflation rate, $\pi_t \equiv P_t / P_{t-1}$. That is, the rule is of the form $R_t = R(\pi_t)$. Because consumption is constant and equal to the endowment, the Euler equation for this model takes the form of a simple Fisher equation, $\pi_{t+1} = \beta R(\pi_t)$, where $0 < \beta < 1$ is the subjective discount rate. The steady-state inflation rate
is implicitly defined by $\pi^* = \beta R(\pi^*)$. Because the price level is free to adjust at time $t$, so is the inflation rate, and uniqueness of the equilibrium requires the steady state to be mathematically unstable: $\beta R'(\pi^*) > 1$. Since $\beta$ is close to unity, the requirement for uniqueness then is an active Taylor rule: $R'(\pi^*) > 1$.

The intuition that follows from this simple model becomes highly misleading as soon as we depart from the simple endowment economy. In particular, dropping the assumption of an endowment economy and allowing even the smallest productive role for money will reverse the results. Benhabib, Schmitt-Grohé, and Uribe (2001a) describe in detail how small and realistic departures from this simplest of models result in local indeterminacies, especially in the context of forward-looking Taylor rules. The hope remains, as argued in Benhabib, Schmitt-Grohé and Uribe (2001a), that a sufficiently backward-looking design of the Taylor rule will avoid such problems, and indeed they show that this is the case when the analysis is limited to equilibria converging to the steady state. Unfortunately, this frequently used local analysis, based on linear approximations to the equilibrium conditions around the steady state, can be deeply misleading. The fact that local multiplicities obtain in part of the parameter space, the region where the rule is not sufficiently backward-looking, implies that local approximations will be invalid in the parameter region where local uniqueness holds: a continuum of equilibrium paths will converge to cycles in the vicinity of the steady state. The exacting assumptions of long-run perfect foresight or rational expectations, operating through transversality conditions, and routinely used to reduce the uniqueness analysis to root-counting and to ruling out divergent trajectories as non-equilibria, are no longer operational. Multiple oscillatory equilibria in the neighborhood of the steady state that satisfy transversality will abound.

The monetary transmission mechanism is often modeled as operating through the demand for money. In this paper we will also model the supply-side, or cost-side effects of money, by allowing real balances to have a mild impact on output, and therefore on the aggregate supply curve that is consistent with and calibrated in accordance to the findings of the literature on money demand. A simple and direct way to model these effects is to introduce real balances into the production function. There exist alternative routes, suggested by Farmer (1984), Christiano and Eichenbaum (1992), Fuerst (1992), and Bernanke and Gertler (1989, 1995), among others, where real interest rates affect the cost of production and the level of output because, given some nominal rigidities,
factors have to be paid in advance of revenues and because borrowing costs go up for collateral-poor firms that are credit constrained\(^1\). As noted in Benhabib, Schmitt-Grohé and Uribe (2001a), even very mild output effects of inflation, acting through interest rates via the Taylor rule, can easily amplify the effects of future expected inflation on the current inflation rates, leading to multiple equilibria and “instrument instability” under active monetary policies. The main question that we address in this paper is whether appropriately designed “backward-looking” Taylor rules can overcome this policy-induced indeterminacy. Our main conclusion is negative: Even though local linear analysis, based on root-counting, suggests that sufficiently backward looking rules yield local uniqueness, by slightly zooming out we see that oscillatory multiplicities, instrument instability and sunspot equilibria necessarily emerge.

2 A sticky-price model

In this section we describe a simple economy which we will use to evaluate the stabilizing properties of backward-looking interest rate feedback rules. To remain close to the related literature that advocates active interest rate feedback rules, we assume that price adjustment is sluggish and that fiscal variables play no role in the determination of prices and inflation. Specifically, in our model price stickiness arises because suppliers of goods face quadratic price adjustment costs as in Rotemberg (1982). An important difference of our modeling approach relative to the related literature is that we assume, following Fischer (1974), Taylor (1977), and Calvo (1979), that marginal costs of production are increasing in the nominal interest rate because money facilitates firms’ production. Introducing a demand for money by firms is motivated by the fact that in industrialized countries firms hold a substantial fraction of the money supply. For example, in the United States, nonfinancial firms held at least 50 percent more demand deposits than households over the period 1970-1990 (see Mulligan, 1997, and the references cited therein). Recent theoretical evaluations of monetary policy rules have restricted attention to the case in which variations in the nominal interest rate affect real variables solely through their effect on aggregate demand. In light of the available evidence on money-holding patterns in the U.S. economy, following in the tradition of

\(^1\)For empirical evidence on the cost-push effects of interest rates and monetary policy at a disaggregated industry level, see Barth and Ramey (2001). We should also note that a large fraction of cash balances are held by firms rather than individuals, at some interest cost, suggesting that the overall liquidity yield of real balances to firms are not nil.
Calvo, Fischer and Taylor appears empirically more compelling. The model developed below is similar to the one laid out in Benhabib, Schmitt-Grohé, and Uribe (2001).

2.1 The household/firm unit

Assume an economy populated by a continuum of household–firm units indexed by \( j \), each of which produces a differentiated good \( Y^j \) and faces a demand function \( Y^d q^j \left( \frac{P_j}{P} \right) \), where \( Y^d \) denotes the level of aggregate demand, \( P^j \) the price firm \( j \) charges for its output, and \( P \) the aggregate price level. Such a demand function can be derived by assuming that households have preferences over a composite good that is produced from differentiated intermediate goods via a Dixit-Stiglitz production function. The function \( d(\cdot) \) is assumed to decreasing and to satisfy \( d(1) = 1 \) and \( d'(1) < -1 \). As will become clear shortly, the restriction imposed on \( d'(1) \) is necessary for the firm’s problem to be well defined in a symmetric equilibrium. The production of good \( j \) is assumed to take real money balances, \( m^j \), as the only input

\[
Y^j = y(m^j),
\]

where the function \( y(\cdot) \) is assumed to be positive, strictly increasing, and strictly concave.

The household’s lifetime utility function is assumed to be of the form

\[
U^j = \int_0^\infty e^{-rt} \left[ u(c^j) - \frac{\gamma}{2} \left( \frac{P^j}{P} - \pi^* \right)^2 \right] \, dt, \tag{1}
\]

where \( c^j \) denotes consumption of the composite good by household \( j \) and \( \pi^* > -r \) denotes the steady-state inflation rate. The utility function \( u(\cdot) \) is assumed to be increasing, twice continuously differentiable, and strictly concave, and the parameter \( \gamma > 0 \) measures the degree to which household-firm units dislike to deviate in their price-setting behavior from the long-run level of aggregate price inflation. In addition to money, the household can hold nominal bonds, \( B \), which pay the nominal interest rate \( R > 0 \). Letting \( a \equiv (M + B)/P \) denote the household’s real financial wealth, \( \tau \) real lump-sum taxes, and \( \pi \equiv \dot{P}/P \) the inflation rate, the household’s instant budget
constraint can be written as
\[ \dot{a}^j = (R - \pi)a^j - Rm^j + \frac{P^j}{P} y(m^j) - c^j - \tau. \] (2)

Household are also assumed to be subject to a no-Ponzi-game constraint of the form
\[ \lim_{t \to \infty} e^{-f_0^t[R(s) - \pi(s)]ds} a^j(t) \geq 0. \] (3)

In addition, firms are subject to the constraint that given the price they charge, their sales are demand-determined
\[ y(m^j) = Y^d d\left( \frac{P^j}{P} \right). \] (4)

The household chooses sequences for \( c^j, m^j, P^j \geq 0, \) and \( a^j \) so as to maximize (1) subject to (2)–(4) taking as given \( a^j(0), P^j(0), \) and the time paths of \( \tau, R, Y^d, \) and \( P. \) The Hamiltonian of the household’s optimization problem takes the form
\[ e^{-rt} \left\{ u(c^j) - \frac{\gamma}{2} \left( \frac{\dot{P}^j}{P^j} - \pi^* \right)^2 + \lambda^j \left[ (R - \pi)a^j - Rm^j + \frac{P^j}{P} y(m^j) - c^j - \tau - \dot{a}^j \right] + \mu^j \left[ Y^d d\left( \frac{P^j}{P} \right) - y(m^j) \right] \right\}. \]

The first-order conditions associated with \( c^j, m^j, a^j, \) and \( P^j \) and the transversality condition are, respectively,
\[ u_c(c^j) = \lambda^j, \] (5)
\[ \lambda^j \left[ \frac{P^j}{P} y'(m^j) - R \right] = \mu^j y'(m^j), \] (6)
\[ \dot{\lambda}^j = \lambda^j (r + \pi - R), \] (7)
\[ \lambda^j \frac{P^j}{P} y(m^j) + \mu^j \frac{P^j}{P} Y^d d\left( \frac{P^j}{P} \right) = \gamma r (\pi^j - \pi^*) - \gamma \dot{\pi}^j, \] (8)

and
\[ \lim_{t \to \infty} e^{-f_0^t[R(s) - \pi(s)]ds} a^j(t) = 0, \] (9)

where \( \pi^j \equiv \dot{P}^j/P^j. \)
2.2 The government

At the center of our analysis is the role played by backward-looking behavior in the conduct of monetary policy. We assume that the central bank follows an interest-rate feedback rule whereby the nominal interest rate is set as an increasing function of an average of past inflation rates. Specifically, we consider the following backward-looking feedback rule

\[ R = \rho(\pi_p); \quad \rho' > 0, \]  

where \( \pi_p \) is a weighted average of past rates of inflation and is defined as

\[ \pi_p = b \int_{-\infty}^{t} \pi(s)e^{b(s-t)}ds; \quad b > 0. \]  

(11)

The function \( \rho(\cdot) \) is assumed to be continuous and non-decreasing. Furthermore, we assume that there exists a \( \pi^* > -r \) such that \( \rho(\pi^*) = r + \pi^* \). Following Leeper (1991), we refer to monetary policy as active if \( \rho'(\pi^*) > 1 \) and as passive if \( \rho'(\pi^*) < 1 \). Differentiating (11) with respect to time yields

\[ \dot{\pi}^p = b(\pi - \pi^p). \]  

(12)

Government purchases are assumed to be zero at all times. Then, the sequential budget constraint of the government is given by \( \dot{B} = RB - \dot{M} - P\tau \), which can be written as

\[ \dot{a} = (R - \pi)a - Rm - \tau. \]  

(13)

Because both the nominal value of initial government liabilities, \( A(0) \), and the initial price level, \( P(0) \), are predetermined, initial real liabilities of the government, \( a(0) = \frac{A(0)}{P(0)} \), are also given.

Government policy is assumed to be of the Ricardian type. Ricardian policies are fiscal-monetary regimes that ensure that the present discounted value of total government liabilities converges to zero—that is, equation (9) is satisfied—under all possible, equilibrium or off-equilibrium, paths of endogenous variables such as the price level, the money supply, inflation, or the nominal interest rate. Furthermore, we restrict attention to one particular Ricardian fiscal policy that takes the
form
\[ \tau + Rm = \alpha a, \]
where the sequence \( \alpha \) is chosen arbitrarily by the government subject to the constraint that it is positive. This policy states that consolidated government revenues, that is, tax revenues plus interest savings from the issuance of money, are always equal to a certain positive fraction of total government liabilities.\(^2\)

### 2.3 Equilibrium

In a symmetric equilibrium all household–firm units choose identical sequences for consumption, asset holdings, and prices. As a result, we can drop the superscript \( j \). In equilibrium, the goods market must clear. That is,

\[ c = y(m). \]

(15)

Using (15) to eliminate \( c \) in (5) yields

\[ u_c(y(m)) = \lambda. \]

(16)

One can then use (16) to express \( m \) as a decreasing function of \( \lambda \).\(^3\)

\[ m = m(\lambda); \quad m_\lambda < 0. \]

(17)

Let \( \eta \equiv d'(1) < -1 \) denote the equilibrium price elasticity of the demand function faced by the individual firm. Using (6), (15), and (17) to eliminate \( m, \mu, \) and \( c \) from equations (7), (8), (9),

\(^2\)A special case of this type of policy is a balanced-budget rule whereby tax revenues are equal to interest payments on the debt, which results when \( \alpha = R \) (provided \( R \) is bounded away from zero). To see that the fiscal policy given by (14) is Ricardian, let \( d \equiv \exp[-\int_0^t(R - \pi)ds] \) and \( x \equiv da \). The definition of a Ricardian fiscal policy requires that \( x \rightarrow 0 \) as \( t \rightarrow \infty \). Note that \( \dot{x} = d[\dot{a} - (R - \pi)a] \). Using equations (13) and (14), this expression can be written as \( \dot{x} = -\alpha x \), which implies that \( x \) converges monotonically to zero.

\(^3\)Note that by (16) one can find the value of \( \lambda \) for any positive value of \( m \). However, the converse may not be true, that is, there may exist values of \( \lambda \) such that no positive value of \( m \) satisfies equation (16). This problem arises when \( u_c \) or \( y(.) \) are either bounded above or bounded below away from zero. Observe also that differentiating equation (16) implies that \( m_\lambda = 1/[y'u_{cc}] < 0. \)
\[\dot{\lambda} = \lambda [r + \pi - R] \quad (18)\]
\[\gamma \dot{\pi} = \gamma r (\pi - \pi^*) - y(m(\lambda)) \lambda \left[ 1 + \eta \left( 1 - \frac{R}{y'(m(\lambda))} \right) \right] \quad (19)\]
\[0 = \lim_{t \to \infty} e^{-\int_{0}^{t} [R - \pi] \, da(t)} \quad (20)\]
\[\dot{a} = (R - \pi) a - Rm(\lambda) - \tau \quad (21)\]
\[\tau = -Rm(\lambda) + \alpha a \quad (22)\]

To characterize the equilibrium dynamics it is convenient to reduce the system of equilibrium conditions further as follows. First note that given any set of functions \(\{\pi, \pi^p, R, \lambda\}\), equations (21) and (22) can be used to construct time paths for \(a\) and \(\tau\). Because the fiscal policy is Ricardian, the so constructed sequences \(\{\pi, a\}\) satisfy the transversality condition (20). Second, use (10) to replace \(R\) in (18) and (19). We then have that any set of functions \(\{\pi, \pi^p, \lambda\}\), satisfying

\[\dot{\lambda} = \lambda [r + \pi - \rho(\pi^p)] \quad (23)\]
\[\dot{\pi} = r (\pi - \pi^*) - \frac{y(m(\lambda)) \lambda}{\gamma} \left[ 1 + \eta \left( 1 - \frac{\rho(\pi^p)}{y'(m(\lambda))} \right) \right] \quad (24)\]
\[\dot{\pi}^p = b(\pi - \pi^p) \quad (25)\]
given \(\pi^p(0)\) constitutes a perfect-foresight equilibrium. We summarize this result in the following definition.

**Definition 1 (Perfect-foresight equilibrium)** A perfect-foresight equilibrium is a set of functions of time \(\{\lambda, \pi, \pi^p\}\) satisfying (23)-(25), given \(\pi^p(0)\).

### 3 Equilibria Converging to the Steady State

Consider first perfect-foresight equilibria in which \(\{\lambda, \pi, \pi^p\}\) converge to a steady-state \(\{\lambda^*, \pi^*, \pi^{p*}\}\). The steady-state values \(\lambda^*\) and \(\pi^*\) are defined as constant values of \(\lambda, \pi\) and \(\pi^p\) that solve (23), (24), and (25). Thus, \(\pi^*\) is a solution to \(r + \pi^* = \rho(\pi^*)\), which by assumption exists though need not be unique. Given a \(\pi^*\), we also know \(\pi^p^*\), since \(\pi^{p*} = \pi^*\). The steady-state value of the marginal utility of consumption, \(\lambda^*\), is given by the solution to \((1 + \eta) / \eta = \rho(\pi^*) / y'(m(\lambda^*))\). Consider the
following change of variables. Let \( p = \ln \lambda - \ln \lambda^* \), \( w = \pi - \pi^* \), and \( z = \pi^p - \pi^* \). One steady-state values of \( \{p, w, z\} \) is then given by \( (p^*, w^*, z^*) = (0, 0, 0) \), and the equilibrium conditions can be expressed as

\[
\begin{align*}
\dot{p} &= r + \pi^* + w - \rho(z + \pi^*) \\
\dot{w} &= r w - \frac{y(m(\lambda^* e^p))\lambda^* e^p}{\gamma} \left[ \frac{1 + \eta}{\eta} - \frac{\rho(z + \pi^*)}{y'(m(\lambda^* e^p))} \right] \\
\dot{z} &= b(w - z)
\end{align*}
\]

In a neighborhood around \( (p^*, w^*, z^*) \), the equilibrium paths of \( p, w, \) and \( z \) converging asymptotically to \( (p^*, w^*, z^*) \) can be approximated by the solutions to the following linearization of (26), (27), and (28) around \( \{p^*, w^*, z^*\} \).

\[
\begin{pmatrix}
\dot{p} \\
\dot{w} \\
\dot{z}
\end{pmatrix} = A \begin{pmatrix}
p - p^* \\
w - w^* \\
z - z^*
\end{pmatrix}
\]

where

\[
A = \begin{bmatrix}
0 & 1 & -\rho' \\
A_{21} & r & A_{23} \\
0 & b & -b
\end{bmatrix}
\]

\[
A_{21} = -\frac{\lambda^* y^* \eta R^* y'' m_\lambda}{\gamma y'} > 0
\]

\[
A_{23} = \frac{\lambda^* y^* \eta \rho'}{\gamma y'} < 0.
\]

Because \( \pi^p \) is a non-jump variable and both \( \lambda \) and \( \pi \) are jump variables, it follows that if \( A \) has exactly one root with a negative real part and two roots with positive real parts, then for any \( \pi^p(0) \) in a small enough neighborhood around \( \pi^* \), there exists a unique perfect-foresight equilibrium converging to \( \{\lambda^*, \pi^*, \pi^*\} \).

Assume that monetary policy is active \( (\rho' > 1) \). Then, depending on the value of the parameter \( b \), which measures the average lag-length in the inflation measure to which the monetary authority
responds, the real allocation is either locally determinate or indeterminate. As long as the feedback rule is sufficiently backward looking \((b \to 0)\), the equilibrium is always unique. To see this, note that when \(\rho' > 1\), the determinant of \(A\), which is given by

\[
\text{Det}(A) = bA_{21} \left(1 - \rho'\right),
\]

is negative. Thus, the number of roots of \(A\) with a positive real part is either zero or two. Therefore, equilibrium is either locally unique or indeterminate. If at the same time the trace of \(A\) is positive, then the number of roots of \(A\) with a positive real part is exactly equal to two. The trace of \(A\) is given by

\[
\text{Trace}(A) = r - b.
\]

Clearly, as \(b\) approaches zero, the trace of \(A\) becomes positive. The following proposition summarizes this results and gives further conditions under which equilibrium is locally unique.

**Proposition 3.1** Suppose that at the steady state monetary policy is active, that is, \(\rho'(\pi^*) > 1\). If \(r + A_{23} > 0\) or \(b < r\), then there exists a unique competitive equilibrium in which \(\lambda, \pi, \text{ and } \pi^p\) converge to the steady state \((\lambda^*, \pi^*, \pi^{p*})\). If \(r + A_{23} < 0\) and \(b > r\), then depending on the value of \(b\), there exist either a continuum or a unique perfect foresight equilibrium, in which \(\lambda, \pi, \text{ and } \pi^p\) converge to the steady state \((\lambda^*, \pi^*, \pi^{p*})\).

**Proof:** Because the equilibrium system features one non-jump variable \((\pi^p)\) and two jump variables \((\lambda \text{ and } \pi)\), local uniqueness requires that exactly two roots of the matrix \(A\) have positive real parts. We have already established that \(A\) has either zero or two roots with positive real parts. We apply Routh’s theorem (see Gantmacher, 1960) according to which the number of roots of \(A\) with positive real parts is equal to the number of variations of sign in the scheme:

\[
-1 \quad \text{Trace}(A) \quad -B + \frac{\text{Det}(A)}{\text{Trace}(A)} \quad \text{Det}(A),
\]

where

\[
B = \text{Sum of the principal minors of } A = -A_{21} - b(r + A_{23}).
\]

This condition implies that in order for no root of \(A\) to have a positive real part it is necessary that
$B$ be positive and that the trace of $A$ be negative. A necessary condition for $B > 0$ is $r + A_{23} < 0$, and a necessary condition for trace$(A) < 0$ is $b > r$. Thus, whenever $r + A_{23} > 0$ or $b < r$, equilibrium is locally unique under active policy.

Assume now that monetary policy is passive ($\rho' < 1$). Again, as shown in Benhabib et al. (2000) passive backward-looking monetary policy cannot bring about local determinacy. To see this, note that if $\rho' < 1$, the determinant of $A$ is positive, so the number of roots of $A$ with a negative real part can never be exactly equal to one. If all roots have positive real parts, a perfect-foresight equilibrium in which the real allocation converges to its steady state does not exist. This could be the case for rules that place a lot of weight on the distant past, that is, rules with small $b$ values. On the other hand, if the feedback rule is highly contemporaneous, that is, as $b$ becomes large, the equilibrium is always locally indeterminate. With $b$ large, the trace of $A$ is negative while the determinant remains positive, therefore $A$ must have two roots with negative real parts, implying indeterminacy of equilibrium. These results are summarized in the following proposition.

**Proposition 3.2** If monetary policy is passive ($\rho'(\pi^*) < 1$), then there does not exist a unique equilibrium converging to the steady state $(\lambda^*, \pi^*, \pi^p*)$. Either there exists a continuum of perfect-foresight equilibria in which $\lambda$, $\pi$, and $\pi^p$ converge asymptotically to the steady state $(\lambda^*, \pi^*, \pi^p*)$ or no local equilibrium exists.

Given the above discussion, one may be led to conclude that as long as one follows the Taylor criterion, that is, the nominal interest rate responds by more than one for one to movements in the inflation rate and the measure of inflation to which the central bank responds is sufficiently backward looking, then an active interest rate feedback rule is stabilizing in the sense that it guarantees local uniqueness of the perfect foresight equilibrium. The central contribution of this paper is to show that even in the range of values of $b$ for which the Taylor criterion ensures a unique equilibrium converging to the steady state, other bounded equilibria do exist. In particular, periodic equilibria (endogenous cycles) become possible.
4 Equilibria Converging to a Cycle

Thus far we have restricted attention to perfect-foresight equilibria in which \( \{\lambda, \pi, \pi^p\} \) converge asymptotically to \( \{\lambda^*, \pi^*, \pi^{p*}\} \). We now investigate the existence of perfect-foresight equilibria in which \( \lambda, \pi, \) and \( \pi^p \) converge asymptotically to a deterministic cycle. In this case the equilibrium dynamics are still bounded and contained in a neighborhood around the steady state but they do not converge to the steady state. The technical reason why cyclical equilibrium dynamics may arise under active monetary policy is that the system of linear differential equations given in (46) can in this case display a Hopf bifurcation for some critical value of the parameter \( b \) describing the average lag-length in the inflation measure used in the interest rate feedback rule. We denote this critical value by \( b^h \). In turn, the existence of a Hopf bifurcation implies that generically (i.e., if the system is non-linear), there will exist a family of cycles for values of \( b \) in a neighborhood located either to the left or to the right of \( b^h \). Furthermore, if the cycle is to the left of \( b^h \) where the steady state is unstable the cycle will be attracting, and the bifurcation is said to be supercritical. The implication is that if the bifurcation is supercritical, then there exist values of \( b \) less than \( b^h \) for which any trajectory \( \{\lambda, \pi, \pi^p\} \) that starts out in a neighborhood of \( \{\lambda^*, \pi^*, \pi^{p*}\} \) will converge to a cycle. Therefore, the perfect foresight equilibrium is indeterminate despite the fact that it is locally unique in the sense described in section 3. The reasons why the indeterminacy of equilibrium identified in this section has been overlooked in the related literature are twofold. First, existing studies have focused on the limiting case in which the nominal interest rate does not affect the cost of production. Second, the majority of previous studies has focused on the dynamics arising from small fluctuations around the steady state that are expected to converge asymptotically to that steady state. Thus, by their very nature, studies of this type are unable to detect equilibria involving bounded fluctuations converging asymptotically to a limit cycle. Readers not interested in the technical details involved in establishing the existence and supercriticality of the Hopf bifurcation may wish to jump directly to the calibration of the model presented at the end of section 4.1 and then jump again to section 4.3, which considers the quantitative aspects of the equilibrium cycles.

\[ b^h \] The Hopf Bifurcation Theorem postulates the existence of a family of cycles, which in the pure linear system pile up at the bifurcation value \( b^h \) and create a center: any nonlinearity will spread them out to either a left or a right neighborhood of \( b^h \). Generically, the amplitude of the cycle varies continuously with \( b - b^h \) and is zero at \( b = b^h \).
4.1 Existence of a Hopf Bifurcation

Formally, a Hopf bifurcation occurs when the real part of two complex roots vanish while the imaginary part does not. A necessary condition for the existence of a Hopf bifurcation in our model is that preferences and technology are such that \( r + A_{23} < 0 \). Otherwise, as we show above, the matrix \( A \) has always two roots with positive real parts. Formally, we have the following result.

**Proposition 4.1 (Hopf Bifurcation)** If monetary policy is active \((\rho'(\pi^*) > 1)\), and \( r + A_{23} < 0 \), then there exists a unique critical value \( b^h > 0 \), such that the dynamical system given in equation (46) displays a Hopf bifurcation. Furthermore, the Hopf bifurcation occurs for \( b^h > r \).

**Proof:** Let \( C \equiv -B + \frac{\text{Det}(A)}{\text{Trace}(A)} \). Consider the scheme given in equation (30). Note that as \( b \to \infty \) the trace of \( A \) becomes negative, \( \text{det}(A)/\text{trace}(A) \) converges to a positive constant and \( B \) converges to \( \infty \). Thus, \( C \) converges to \( -\infty \) implying a pattern of \(- - - -\) so that by Routh’s Theorem, the matrix \( A \) has no root with a positive real part. The function \( C \) is monotonically decreasing in \( b \), approaches \( A_{21} > 0 \) as \( b \) converges to zero and tends to \(-\infty\) as \( b \) becomes arbitrarily large. At \( b = r \), the scalar \( C \) is not well defined. When \( b \) approaches \( r \) from the left, \( C \) tends to \(-\infty\) and when \( b \) approaches \( r \) from the right \( C \to \infty \). It follows that there exists a \( b > r \) such that the sign pattern in equation (30) shifts from \(- - - -\) to \(- - + -\) implying that the real parts of two roots of \( A \) change sign from negative to positive as \( b \) falls below that critical value. We refer to this critical value of \( b \) as \( b^h \). Note that at \( b = b^h \), like for any positive values of \( b \), the determinant of \( A \) is strictly negative. Thus, the two roots that change sign at \( b^h \) and whose real part vanishes at \( b^h \) must be complex, else the determinant would also vanish at \( b^h \). We then have the standard case of a Hopf bifurcation. To see that the system has only one bifurcation, note that for \( b \) close to \( r \) but below \( r \), the sign pattern changes again. This time it changes from \(- - + -\) to \(- + - -\). However, this change is not associated with a change in the number of roots with positive real parts. As \( b \) approaches zero, the pattern changes a last time from \(- + - -\) to \(- + + -\). Again, this change of sign is not associated with a change in the number of roots with positive real parts.

We resort to numerical methods to investigate whether for reasonable parameter values it is indeed the case that \( r + A_{23} < 0 \), so that the dynamical system described by equations (26), (27),

\(^5\)In the limit, when the inflation measure that enters in the feedback rule approaches current inflation, that is, as \( b \to \infty \), there can exist a Hopf bifurcation exactly at the point where \( r + A_{23} = 0 \), and cycles emerge for \( r \) such that \( r + A_{23} > 0 \). For a more detailed discussion see Proposition 7 in Benhabib, Schmitt-Grohé, and Uribe (2001b).
and (28) displays a Hopf bifurcation when monetary policy is active. To this end we assume the following functional forms for preference, technology, and the interest rate feedback rule

\begin{align}
 u(c) & = \frac{c^{1-\sigma} - 1}{1-\sigma}; \quad \sigma > 0 \quad (31) \\
y(m) & = [\alpha m^\rho + (1 - \alpha)\bar{y}^\rho]^{\frac{1}{\rho}}; \quad \rho < 1 \\
\rho(\pi^p) & = R^* + D(\pi^p - \pi^*). \quad (33)
\end{align}

We calibrate the economy as follows. Let the time unit be a quarter. Let the intended nominal interest rate be 6 percent per year \((R^* = \ln(1.06)/4)\), which corresponds to the average yield on 3-month U.S. Treasury bills over the period 1960:Q1 to 1998:Q3. We set the target rate of inflation at 4.2 percent per year \((\pi^* = \ln(1.042)/4)\). This number matches the average growth rate of the U.S. GDP deflator during the period 1960:Q1-1998:Q3. The assumed values for \(R^*\) and \(\pi^*\) imply a subjective discount rate of 1.8 percent per year. Following Taylor (1993), we set the elasticity of the interest-rate feedback rule evaluated at \(\pi^*\) equal to 1.5 (i.e., \(\rho'(\pi^{p*}) = D = 1.5\)). There is a great deal of uncertainty about the value of the intertemporal elasticity of substitution \(1/\sigma\). In the real-business-cycle literature, authors have used values as low as \(1/3\) (e.g., Rotemberg and Woodford, 1992) and as high as 1 (e.g., King, Plosser, and Rebelo, 1988). In the baseline calibration, we assign a value of 2 to \(\sigma\). We will also report the sensitivity of the results to variations in the value assumed for this parameter.

In the steady state, we have by equation (24) that \(\frac{1+\eta}{\eta} = \frac{R^*}{y^*(m^*)}\) which for the particular functional forms assumed above gives rise to the following steady-state ‘money demand’

\begin{align}
m^* = R^* \frac{1}{\rho-1} y^* \left( \frac{\eta}{(1+\eta)\alpha} \right)^{\frac{1}{\rho-1}}
\end{align}

For calibration purposes, we interpret this expression as the long-run money demand function, with a long-run income elasticity of unity and a long-run interest elasticity of \(1/(\rho - 1)\). Using U.S. quarterly data from 1960:Q1 to 1999:Q3, we estimate the following money demand function
Table 1: Calibration

<table>
<thead>
<tr>
<th>r</th>
<th>σ</th>
<th>ρ</th>
<th>α</th>
<th>̄y</th>
<th>π∗</th>
<th>R∗</th>
<th>ρ' (π∗)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0043</td>
<td>2</td>
<td>-3</td>
<td>0.0035</td>
<td>1</td>
<td>0.0103</td>
<td>0.0147</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Note: The time unit is one quarter.

by OLS:⁶

\[ \ln m_t = 0.0446 + 0.0275 \ln y_t - 0.0127 \ln \left( \frac{R_t}{1 + R_t} \right) + 1.5423 \ln m_{t-1} - 0.5918 \ln m_{t-2} \]

\[ t\text{-stat} = (1.8, 4.5, -4.7, 24.9, -10.0) \]

\[ R^2 = 0.998; \quad DW = 2.18. \]

We obtain virtually the same results using instrumental variables.⁷ The short-run log-log elasticity of real balances with respect to its opportunity cost \( R_t/(1 + R_t) \) is -0.0127, while the long-run elasticity is -0.2566.⁸ Because the steady-state relation we use to identify the parameter \( \rho \) has the interpretation of a long-run money demand equation, we set \( \rho \) at -3, so as to be consistent with the estimated long-run money demand elasticity. Given a value for \( \rho \), we can calibrate the parameter \( \alpha \) of the production function by solving the steady-state 'money demand' equation for \( \alpha \) to obtain

\[ \alpha = R^* \frac{\eta}{1 + \eta} \left( \frac{y^*}{m^*} \right)^{\rho - 1}. \]

We set \( y^*/m^* = 5.8/4 \) to match the average quarterly U.S. GDP velocity of M1 between 1960:Q1 and 1999:Q3. Given the baseline value of \( \rho \), the implied value of \( \alpha \) is 0.0035. Finally, we set the fixed factor \( ̄y \) at 1. Table 1 summarizes the calibration of the model.

---

⁶We measure \( m_t \) as the ratio of M1 to the implicit GDP deflator. The variable \( y_t \) is real GDP in chained 1996 dollars. The nominal interest rate \( R_t \) is taken to be the quarterly yield on 3-month Treasury bills. Note that in discrete time, the appropriate measure of the opportunity cost of holding money is given by \( R_t/(1 + R_t) \) rather than simply \( R_t \).

⁷As instruments we choose the first three lags of \( \ln y_t \) and \( \ln R_t/(1 + R_t) \), and the third and fourth lags of \( \ln m_t \).

⁸Ball (2002) estimates a long-run money demand equation of the form \( \ln m_t = \alpha + \theta_y \ln y_t + \theta_R 400 R_t + \epsilon_t \) using Stock and Watson’s (1993) Dynamic OLS Estimator technique with four lags and leads. With a sample of quarterly data from 1959:2 through 1993:4 and measuring the nominal interest rate as the Treasury bill rate, Ball estimates \( \theta_R \) to be -0.040. Ball’s specification implies a long-run log-log interest elasticity of \( 400 \theta_R R = 400 \times -0.040 \times (6.10/400) = -0.24 \), where we used the fact that the average Treasury Bill rate over his sample period was 6.1 percent per year. Ball estimates the long-run money demand equation also with a time series on the rate of return on near-monies that he constructs. These regressions imply a value a log-log interest elasticity of -0.35.
For this parameterization \( r + A_{23} < 0 \), thus we know that if monetary policy is active, a Hopf bifurcation exists for some \( b > r \). In fact, we can compute the exact value of \( b \) at which the system bifurcates. For the baseline calibration, the Hopf bifurcation occurs at a value of \( b \) equal to 2.736. This value of \( b \) implies that the expected lag in the inflation measure is about one month. It follows that for interest rate feedback rules that are less backward looking, equilibrium is locally indeterminate and that for interest rate feedback rules that are more backward looking there exists a unique equilibrium converging to the steady state. However, in the latter case other bounded equilibria may exist. In particular, equilibria in which the economy converges to an attracting cycle. Such cycles are sure to exist when the Hopf bifurcation is supercritical. The next section establishes that this is indeed the case for our baseline calibration.

4.2 Supercriticality of the Hopf Bifurcation

To determine whether the Hopf bifurcation is supercritical, we follow closely Yuri A. Kuznetsov’s (1998) treatment of bifurcations of equilibria and periodic orbits in n-dimensional dynamical systems. Consider the three dimensional system given by (26), (27), and (28). Letting \( x = [p \ w \ z]' \), we can write that system as

\[
\dot{x} = f(x; b); \quad \text{where} \quad x \in \mathbb{R}^3, \ b \in \mathbb{R}, \ \text{and} \ f(0) = 0.
\]

The Jacobian matrix \( A \) of this dynamical system is given in equation (46). We have established above that at \( b = b_h \) and \( x = 0 \), the matrix \( A \) has a simple pair of complex eigenvalues on the imaginary axis, \( \lambda_{1,2} = \pm i\omega \), with \( \omega > 0 \). Let \( q \in C^3 \) be a complex eigenvector corresponding to \( \lambda_1 \):

\[
Aq = i\omega q, \quad A\bar{q} = -i\omega \bar{q},
\]

where a bar over the eigenvector denotes its complex conjugate. Similarly, let \( p \in C^3 \) be the adjoint eigenvector such that

\[
A^T p = -i\omega p, \quad A^T \bar{p} = -i\omega \bar{p}.
\]

Normalize the eigenvector \( q \) so that

\[
< p, q > = 1
\]

16
where \( < p, q > = \sum_{i=1}^{n} \bar{p}_i q_i \).

In order to determine whether the Hopf bifurcation is supercritical we have to compute the first Lyapunov coefficient of the dynamic system (26), (27), and (28) on the center manifold at the critical parameter value \( b = b^h \) and \( x = 0 \). The first Lyapunov coefficient is given by\(^9\)

\[
l_1(0) = \frac{1}{2\omega} \text{Re} \left[ \langle p, C(q, q, \bar{q}) \rangle - 2 \langle p, B(q, A^{-1} B(q, \bar{q})) \rangle + \langle p, B(\bar{q}, (2\omega I_3 - A)^{-1} B(q, q)) \rangle \right],
\]

where \( B(\cdot, \cdot) \) and \( C(\cdot, \cdot, \cdot) \) are multilinear functions. In coordinates, the multilinear functions can be written as

\[
B_i(x, y) = \sum_{j,k=1}^{n} \frac{\partial^2 f_i(\xi)}{\partial \xi_j \partial \xi_k} \mid_{\xi=0} x_j y_k
\]

and

\[
C_i(x, y, z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 f_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \mid_{\xi=0} x_j y_k z_l,
\]

where \( i = 1, 2, \ldots, n \). If \( l_1(0) < 0 \), then the Hopf bifurcation is supercritical and a unique stable limit cycle bifurcates from the origin for \( b < b^h \).\(^10\)

We do not attempt to evaluate the sign of the first Lyapunov coefficient analytically. Instead we determine its sign numerically for our baseline calibration. To compute the first Lyapunov coefficient we proceed as follows. Note that for the functional forms given in equations (31)-(33), the function \( m(\lambda) \) can be written as

\[
m(\lambda) = \left( \lambda^{-\rho/\sigma} - (1 - \alpha)\bar{\gamma}^\rho \right)^{1/\rho}/\alpha.
\]

We then have an exact analytical expression for the function \( f(x) \). We use the Symbolic Math Toolbox of Matlab to find analytical expression for the second- and third-order derivatives of \( f \) needed to compute the multilinear functions \( B(\cdot, \cdot) \) and \( C(\cdot, \cdot, \cdot) \). We then evaluate these expressions at \( x = 0 \) and \( b = b^h \). In this way, we can obtain a number for \( l_1(0) \). For our baseline calibration \( l_1(0) = -11,682 \). Thus the Hopf bifurcation is indeed supercritical and attracting cycles exist for \( b < b^h \).

---

4.3 Implied Dynamics

Figure 1 depicts the phase diagram associated with the equilibrium conditions in the space \((\pi_t^p,\pi_t,\ln(y_t/y^*)).\) To construct the figure, the policy parameter \(b\) was set at 2.5. This means that the average lag length of inflation to which the central bank responds is about five weeks.\(^{11}\) All other parameter values are as shown on table 1. Because the assumed value of \(b\) is below the Hopf bifurcation point of 2.74, the equilibrium system possesses one eigenvalue with a negative real part and two eigenvalues with positive real parts. This means that for each initial value of the state (no-jump) variable \(\pi^p\) there is a unique value of \((y,\pi)\) that guarantees that the equilibrium trajectory converges to the steady state. The resulting map from \(\pi^p\) to \((y,\pi)\), known as the saddle

\[^{11}\text{The average lag length of inflation is given by } b \int_{-\infty}^{0} s e^{bs} ds = -1/b \text{ quarters.}\]
path, is depicted with a broken line in figure 1. The saddle path crosses the steady state, which in
the figure is marked with a bullet. But what if the economy were to start slightly off the saddle
path? The solid lines illustrates that such trajectories diverge from the saddle path and converge
to a limit cycle around the steady state. Along this cycle all variables perpetually fluctuate in an
endogenous, deterministic fashion. The limit cycle is attracting. Any initial value of \((\lambda, \pi, \pi_p)\) in a
three-dimensional neighborhood around the cycle gives rise to an equilibrium trajectory converging
to the cycle. Thus, the equilibrium displays a severe case of indeterminacy.

The amplitude of the endogenous fluctuations shown in figure 1 is significant. To illustrate this,
figure 2 depicts with a solid line the first 20 quarters of the equilibrium dynamics shown in figure 1.

Figure 2: Endogenous Cycles Under Backward-Looking Taylor Rules: Time Paths

Note: All variables are in percent. Inflation, \(\pi_t\), and average lagged inflation, \(\pi_t^p\), are expressed
in annual rates. The smoothing parameter \(b\) is set at 2.5. All other parameter values are as
shown in table 1. The solid line corresponds to equilibria converging to the limit cycle and the
dotted line corresponds the equilibrium converging to the steady state.
The inflation rate fluctuates between -1.7 and 7.7 percent in annual terms. At the same time, the interest rate displays values as low as 0.4 percent and as high as 12 percent per year. Real output also follows a noticeably fluctuating path—although not so pronounced as those of inflation or the nominal interest rate—with peaks of 0.2 percent above trend and troughs of -0.4 percent below trend. It takes a little over 2 quarters to complete one cycle.

The dotted line in each panel of figure 2 shows the dynamics that would arise in an equilibrium in which the economy starts with the same value for the non-jump variable \( \pi_p(0) \) as the economy that converges to the cycle, but where the jump variables \( \lambda \) and \( \pi \) are set such that the economy is initially placed exactly on the saddle path. By construction, the resulting equilibrium trajectory converges to the steady state. Although the initial value of \( \pi_p \) is more than four percentage points above its long-run level (\( \pi_p(0) = 0.085 \)) and the initial interest rate is more than six percentage points above target, the economy converges to the steady state with remarkable speed. As can be seen from the figure, after about one quarter the position of the economy is indistinguishable from the steady state. In the standard analysis of the behavior of sticky-price models with interest rate feedback rules, these dynamics would be the only ones investigated and reported.

5 Sensitivity Analysis

5.1 Parameter Sensitivity

Table 2 displays the value of the parameter \( b \) at which the equilibrium presents a Hopf bifurcation. This critical value is denoted by \( b^h \). The table also indicates whether the Hopf bifurcation is supercritical or not. Recall that supercriticality of the Hopf bifurcation implies that attracting equilibrium cycles exist for values of \( b \) lower than \( b^h \). That is, for Taylor rules that are more backward-looking than the one associated with \( b^h \). Recall also that for values of \( b \) larger than \( b^h \) the equilibrium displays local indeterminacy (of second order), because for such values of \( b \) all eigenvalues of the jacobian matrix \( A \) have negative real parts. Thus, the smaller is \( b \), the larger is the range of backwardness for which the equilibrium is indeterminate. It follows from table 2 that \( b^h \) tends to decreases as households become more risk averse (large \( \sigma \)), as the elasticity of substitution between real money balances and the fixed factor of production increases (\( \rho \) close to 1), as the markup increases (\( \eta \) small in absolute value), and as the share of real balances in the
Table 2: Sensitivity Analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter value</th>
<th>( b^h )</th>
<th>Hopf bifurcation is supercritical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>1</td>
<td>5.47</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>2*</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.82</td>
<td>yes</td>
</tr>
<tr>
<td>( \rho )</td>
<td>-1</td>
<td>1.99</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>-3*</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>-5</td>
<td>3.62</td>
<td>yes</td>
</tr>
<tr>
<td>( \eta )</td>
<td>-3</td>
<td>2.1</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>-5</td>
<td>2.4</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>-21*</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1</td>
<td>2.73</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>350*</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>900</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td>( D )</td>
<td>1.1</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>1.5*</td>
<td>2.74</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.74</td>
<td>no</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.0018</td>
<td>3.23</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>0.0035*</td>
<td>2.73</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>0.007</td>
<td>2.29</td>
<td>yes</td>
</tr>
</tbody>
</table>

Note: * indicates the baseline parameter value. \( b^h \) denotes the value of the parameter \( b \) for which the equilibrium displays a Hopf bifurcation. If the Hopf bifurcation is supercritical, then endogenous cycles exist for values of \( b \) lower than \( b^h \) (i.e., for more backward-looking policies than the one associated with \( b^h \)).
CES production function increases \((\alpha \text{ large})\). The value of \(b\) at which the Hopf bifurcation occurs is quite insensitive to large variations in the degree of price stickiness. Finally, the bifurcation ceases to be supercritical for highly active Taylor rules.

### A Two Factor Model

In this subsection we consider a variation of the theoretical model that allows for endogenous labor supply and assumes that labor is a factor of production. This specification is of interest because it implies a well-defined equilibrium even in the case that money balances become unproductive. Therefore, this model encompasses the standard model as a special case. Suppose that output is produced via the following production function that takes labor and cash holdings as factor inputs

\[
y(m, h) = \left[ \alpha m^\rho + (1 - \alpha)h^\zeta \right]^{\frac{1}{\rho}}; \quad \rho < 1 \quad \text{and} \quad \zeta \leq 1
\]  

(34)

When the parameter \(\zeta\) is zero this production function is the same as that presented in equation (32). If, on the other hand, \(\alpha = 0\), then the production function is similar to that assumed in most of the related literature.\(^{12}\) In the baseline model equilibrium is not well defined when \(\alpha\) approaches zero. By contrast, in the modified model presented here we can consider the case that \(\alpha = 0\) so that by making \(\alpha\) small we are indeed approaching the standard model in which money does not enter in the production function.

To introduce endogenous labor supply we assume that the period utility function takes the form

\[
u(c, h) = \frac{c^{1-\sigma}}{1-\sigma}(1-h)\xi; \quad \sigma > 1 \quad \text{and} \quad \xi < 0,
\]  

(35)

where \(h\) denotes hours of work. We assume that the maximum number of hours the household can supply to the market is one. This type of preferences is commonly used in the real business cycle literature.

The equilibrium conditions are derived in the appendix. Here we simply present the calibration of the model and illustrate the implied dynamics with some numerical examples.

Following the calibration of the baseline model we assign values to \(\sigma, \pi^*, \eta, \gamma, \rho'(\pi^*), R^*,\) and

\(^{12}\)One can show that in steady state \(y \to h^\zeta\) as \(\alpha \to 0\).
GDP velocity. The long-run log-log interest elasticity of money is still given by \(1/(\rho - 1)\). Here we use the slightly higher estimate of Ball (2002) of -0.32, and set \(\rho = -2.1\).\(^{13}\) Further, we assume that \(\zeta = 1\) so that in the limit where money balances cease to be productive, the production function is linear in hours. Finally, we set the preference parameter so that in the steady state the Frisch elasticity of labor supply with respect to the wage rate is 1.3. The implied value is \(\xi = -4\).

For this calibration, the dynamical system experiences a Hopf bifurcation at \(b^h = 1.15\). As in the model without endogenous labor supply, we have that when \(\rho'(\pi^*) > 1\), there exists a unique perfect-foresight equilibrium converging to the steady state for all \(b < b^h\) and a continuum of such equilibria for any \(b > b^h\). Again the Hopf bifurcation is supercritical; the first Lyapunov coefficient of the dynamic system at the critical parameter value \(b = b^h\) is -11,963. It follows that for \(b < b^h\) attracting cycles exist in a neighborhood of \((h^*, \pi^*, \pi_p^*)\).

Figure 3: Two-Factor Model: Endogenous Cycles Under Backward-Looking Taylor Rules

![Diagram](image)

Note: All variables are in percent. Inflation, \(\pi_t\), and average lagged inflation, \(\pi^p_t\), are expressed in annual rates. The smoothing parameter \(b\) is set at 1. \(\rho = -2.1, \xi = -4\) and the remaining parameter values are as shown in table 1.

\(^{13}\)Using the baseline value of -3 does not alter the result in any important way.
to which the central bank responds is one quarter. Because the assumed value of $b$ is below the Hopf bifurcation point of 1.15, the equilibrium system possesses one eigenvalue with a negative real part and two eigenvalues with positive real parts. This means that for each initial value of the state (no-jump) variable $\pi^p$ there is a unique value of $(y, \pi)$ that guarantees that the equilibrium trajectory converges to the steady state. The resulting map from $\pi^p$ to $(y, \pi)$, known as the saddle path, is depicted with a broken line in figure 3. The saddle path crosses the steady state, which in the figure is marked with a bullet. But what if the economy were to start slightly off the saddle path? The solid lines illustrates that such trajectories diverge from the saddle path and converge to a limit cycle around the steady state. Along this cycle all variables perpetually fluctuate in an endogenous, deterministic fashion. The limit cycle is attracting. Any initial value of $(y, \pi, \pi^p)$ in a three-dimensional neighborhood around the cycle gives rise to an equilibrium trajectory converging to the cycle. Thus, the equilibrium displays a severe case of indeterminacy.

Figure 4 shows the time path of a trajectory converging to the limit cycle. The inflation rate fluctuates between -16 and 24 percent in annual terms. At the same time, the interest rate

Figure 4: Two Factor Model: Endogenous Cycles Under Backward-Looking Taylor Rules

Note: All variables are in percent. Inflation, $\pi_t$, and average lagged inflation, $\pi^p_t$, are expressed in annual rates. The smoothing parameter $b$ is set at 1.
displays values as low as 0.16 percent and as high as 12 percent per year. Real output also follows a noticeably fluctuating path—although not so pronounced as those of inflation or the nominal interest rate—with peaks of 0.8 percent above trend and troughs of -0.96 percent below trend. It takes about 1 quarter to complete a cycle.
Appendix

The economy with labor in the utility and production functions

The household chooses sequences for $c^j, h^j, m^j, P^j \geq 0$, and $a^j$ so as to maximize

$$U^j = \int_0^\infty e^{-rt} \left[ u(c^j, h^j) - \frac{\gamma}{2} \left( \frac{\dot{P}^j}{P^j} - \pi^* \right)^2 \right] dt,$$

subject to (3),

$$\dot{a}^j = (R - \pi)a^j - Rm^j + \frac{P^j}{P} y(m^j, h^j) - c^j - \tau$$

and

$$y(m^j, h^j) = Y^d d\left( \frac{P^j}{P} \right).$$

taking as given $a^j(0)$, $P^j(0)$, and the time paths of $\tau$, $R$, $Y^d$, and $P$. The Hamiltonian of the household’s optimization problem takes the form

$$e^{-rt} \left\{ u(c^j, h^j) - \frac{\gamma}{2} \left( \frac{\dot{P}^j}{P^j} - \pi^* \right)^2 + \lambda^j \left[ (R - \pi)a^j - Rm^j + \frac{P^j}{P} y(m^j, h^j) - c^j - \tau - \dot{a}^j \right] + \mu^j \left[ Y^d d\left( \frac{P^j}{P} \right) - y(m^j, h^j) \right] \right\}.$$

The first-order conditions associated with $c^j, h^j, m^j, a^j$, and $P^j$ and the transversality condition are, respectively,

$$u_c(c^j, h^j) = \lambda^j,$$  \hspace{1cm} (39)

$$-u_h(c^j, h^j) = y_h(m^j, h^j)$$  \hspace{1cm} (40)

$$\lambda^j \left[ \frac{P^j}{P} y_m(m^j, h^j) - R \right] = \mu^j y_m(m^j, h^j),$$  \hspace{1cm} (41)

$$\dot{\lambda}^j = \lambda^j (r + \pi - R),$$  \hspace{1cm} (42)

$$\lambda^j \frac{P^j}{P} y(m^j, h^j) + \mu^j \frac{P^j}{P} Y^d d' \left( \frac{P^j}{P} \right) = \gamma r (\pi^j - \pi^*) - \gamma \pi^j,$$  \hspace{1cm} (43)

and

$$\lim_{t \to \infty} e^{-\int_0^t[R(s) - \pi(s)]ds} a^j (t) = 0.$$  \hspace{1cm} (44)
where $\pi^j \equiv \dot{P}^j / P^j$.

**Equilibrium**

As before, in a symmetric equilibrium all household–firm units choose identical sequences for consumption, asset holdings, and prices. As a result, we drop the superscript $j$. In equilibrium, the goods market must clear. That is,

$$c = y(m, h).$$

(45)

Combining (39), (40) and (45) we can express $c$, $m$ and $\lambda$ as functions of $h$.

$$c = c(h)$$

$$m = m(h)$$

$$\lambda = \lambda(h).$$

One can show analytically that $\lambda_h(h^*) < 0$ if $\rho < 0$. Under the assumption that fiscal policy is Ricardian and that monetary policy is given by the backward-looking feedback rule (10) an equilibrium is a set of functions of time $\{h, \pi, \pi_p\}$ satisfying:

$$\dot{h} = \frac{\lambda(h)}{\lambda_h(h)} [r + \pi - \rho(\pi_p)]$$

$$\dot{\pi} = r(\pi - \pi^*) - \frac{y(m(h), h)\lambda(h)\eta}{\gamma} \left[ 1 + \frac{\eta}{\eta} - \frac{\rho(\pi_p)}{y_m(m(h), h)} \right]$$

$$\dot{\pi}_p = b(\pi - \pi^*)$$

Let $p = \ln(h/h^*)$, where $h^*$ is the steady state value of $h(t)$. Then the system of equilibrium conditions can be written as:

$$\dot{p} = \frac{\lambda(h^* e^p)}{\lambda_h(h^* e^p) h^* e^p} [r + \pi^* + w - \rho(z + \pi^*)]$$

$$\dot{w} = r w - \frac{y(m(h^* e^p), h^* e^p)\lambda(h^* e^p)\eta}{\gamma} \left[ 1 + \frac{\eta}{\eta} - \frac{\rho(z + \pi^*)}{y_m(m(h^* e^p), h^* e^p)} \right]$$

$$\dot{z} = b(w - z)$$

In a neighborhood around $(p^*, w^*, z^*)$, the equilibrium paths of $p$, $w$, and $z$ converging asymp-
totically to \((p^*, w^*, z^*)\) can be approximated by the solutions to the following linearization of the above system around \(\{p^*, w^*, z^*\}\).

\[
\begin{pmatrix}
\dot{p} \\
\dot{w} \\
\dot{z}
\end{pmatrix} = B \begin{pmatrix} p - p^* \\
w - w^* \\
z - z^*
\end{pmatrix}
\]

(46)

where

\[
A = \begin{bmatrix}
0 & \frac{\lambda^*}{\lambda^* h^*} & -\rho \frac{\lambda^*}{\lambda^* h^*} \\
B_{21} & r & B_{23} \\
0 & b & -b
\end{bmatrix}
\]

\[
B_{21} = -\frac{\lambda^* y^* h^* \eta R^*}{\gamma y_m^2} [y_{mm} m_h + y_{mh}]
\]

\[
B_{23} = \frac{\lambda^* y^* \eta}{\gamma} \frac{\rho}{y_m} < 0.
\]

Because \(\pi^p\) is a non-jump variable and both \(h\) and \(\pi\) are jump variables, it follows that if \(B\) has exactly one root with a negative real part and two roots with positive real parts, then for any \(\pi^p(0)\) in a small enough neighborhood around \(\pi^*\), there exists a unique perfect-foresight equilibrium converging to \(\{h^*, \pi^*, \pi^*\}\).
References


Fuerst, Timothy S, “Liquidity, Loanable Funds, and Real Activity,” Journal of Monetary Eco-


[available at http://www.econ.upenn.edu/~uribe.]


