1 Chaotic Dynamics in Economics[1]

Abstract: A new literature in the 1980s studied the possibility that endogenous cycles and irregular chaotic dynamics resembling stochastic fluctuations could be generated by deterministic, equilibrium models of the economy, in particular in overlapping generations models and in models with infinitely lived representative agents. Other empirical studies attempted to identify whether various economic time series were generated by deterministic chaotic dynamics or stochastic fluctuations. While dynamic equilibrium models calibrated to standard parameter values can generate chaotic dynamics and endogenous cycles even under intertemporal arbitrage and without market frictions, definitive empirical evidence for chaos in economics has not yet been produced.

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When a new literature in the 1980s showed that endogenous cycles and chaos can arise in equilibrium models in economics, it came as a surprise. The possibility of deterministic fluctuations, as opposed to fluctuations driven by exogenous stochastic shocks, had been noted in an earlier literature on business cycles, for example in the well-known multiplier-accelerator models, but not in equilibrium models of the economy with complete markets and no frictions. (See for example Frisch (1933) or Samuelson (1939). Yet deterministic fluctuations in equilibrium models with predictable relative price changes should be ruled out by intertemporal arbitrage. Such considerations led to the rejection of regular endogenous cycles in favor of models whose fluctuations are driven by stochastic shocks.

The new literature on chaotic dynamics showed that deterministic cycles and chaos were indeed possible under complete intertemporal arbitrage and without any market frictions, both in standard models of overlapping generations as well as in calibrated models of infinitely lived representative agents. (See for example Benhabib and Day (1980), (1982), Benhabib and Nishimura (1979), Grandmont (1985), and Boldrin and Montrucchio (1986)). Of course relative price fluctuations in such models had to be within the bounds allowed by the discount factor in order to be compatible with intertemporal arbitrage. (For an exploration of the relation between equilibrium cycles, chaos and discount rates in model with infinitely lived agents see Benhabib and Rustichini (1990), Sorger (1992), Mitra (1996) and Nishimura and Yano (1996)). Furthermore, chaotic dynamics could exhibit not only deterministic endogenous cycles, but generate

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trajectories that are irregular, and that are statistically indistinguishable from stable linear stochastic AR1 processes (see Sakai and Tokumaro (1980)).

We can usually describe a dynamical system in discrete time as chaotic if it can generate cycles of every periodicity, where a sequence $\{x_j\}$is of period n if $x_j = x_{j+n}$ but $x_j \neq x_i$,for $j < i < n-1$. In addition, this simple definition of chaos requires the existence of an uncountable number of initial x which give rise to bounded but aperiodic (not even asymptotically) sequences. For example the well-known hump-shaped function, $4x(1-x)$, when iterated, generates such chaotic dynamics. The kind of chaotic dynamics described above is usually referred to as topological chaos. If in addition we require that the set of initial conditions giving rise to aperiodic sequences are not simply uncountable but also have a positive (Lebesgue) measure, then we also have ergodic chaos. A useful sufficient condition to obtain topological chaos with a simple difference equation $x_{t+1} = f(x_t)$, with f continuous and mapping a closed interval into itself, is the existence of some x such that $f(f(f(x))) \leq x < f(x) < f(f(x))$. (See Li and Yorke (1975); for simple sufficient conditions for chaos in higher dimensions, see Diamond (1976), or Maratto(1978).) Note that this condition will be satisfied if the difference equation has a solution of period three. A particularly interesting feature of some dynamic systems that are chaotic is their sensitive dependence on initial conditions: initial conditions that are arbitrarily close can generate sequences that tend to diverge over time. Thus, small measurement errors in initial conditions may cause large forecasting errors, which may explain some of the difficulties associated with business-cycle forecasting.

The aperiodic but bounded trajectories that characterize chaos and exhibit sensitive dependence on initial conditions cannot continue to diverge forever. They converge not to a point or a periodic cycle, but to a bounded chaotic or "strange" attractor. The dynamical system which induces the local separation and instability of the trajectories must eventually bend them back. The combination of local stretching and global folding generates the complex nature of the dynamics. Such dynamic behavior is in fact a familiar theme in economics that highlights the self-correcting nature of the economic system. Shortages create incentives for increased supply; dire necessities give rise to inventions as the invisible hand guides the allocation of resources. An equally familiar theme is that of instability: the multiplier interacts with the accelerator, leading to explosive or implosive investment expenditures; self-fulfilling expectations give rise to bubbles and crashes. In combination, these two themes suggest a nonlinear system, somewhat unstable at the core, but effectively contained further out. The contribution of the new literature on chaotic dynamics starting in the early 80s has been to demonstrate the compatibility of endogenous irregular fluctuations with equilibrium dynamics in economics.

For a very simple example of chaotic dynamics consider a simple overlapping generations model where each generation lives two periods. The utility function of a generation born at t is $U(c_0(t), c_1(t+1))$, where $c_0(t)$ is consumption when young and $c_1(t+1)$ is consumption when old. This generation faces a budget constraint $c_1(t+1) = w_1 + r(t)(w_0 - c_0(t))$, where w_0 is the endowment when young, w_1 is the endowment when old, and $r(t)$ is the rate of return on savings.

The first order condition to the problem of maximizing utility subject to the budget constraint, assuming interiority, yields $r(t) = \frac{U_1(c_0(t), c_1(t+1))}{U_2(c_0(t), c_1(t+1))}$. Here U_1 and U_2 denote the derivatives of the utility function U with respect to the first and second arguments. During each period t, market clearing requires that sum of the endowments of the young and the old add up to the sum of their consumptions: $w_1 + w_0 = c_1(t) + c_0(0)$. Now consider the quadratic utility function $U(c,(t),c,(t+1)) = ac_0(t) - 0.5b(c_0(t))^2 + c_1(t)$, $0 \leq c_0 \leq a/b$, and $a, b > 0$. Substituting the first order condition into the budget constraint, and using the market clearing condition, the difference equation describing the dynamics is given by $c_1(t+1) = ac_0(t)(1 - (b/a)c_0(t))$. Note that $c_0(t) \in (0, a/b)$ for all $c_0(0) \in (0, a/b)$, provided $a \leq 4$. This difference equation will exhibit chaotic dynamics in c_0 for $a \in [3.53, 4]$, $b = a$. For example if $a = 3.83$, the difference equation has a 3-period cycle for $c_0(t) = 0.1561$,where $c_0(t+1) = 0.5096$ and $c_0(t+2) = 0.9579$. In this simple example utility saturates at $c_0 = a/b$, but the chaotic trajectories and those with period greater than one never attain b/a, since if $c_0(t) = b/a$, $c(t+i) = 0$ for all $i = 1, 2, ...$ Another simple example of an exponential utility function that will generate chaotic dynamics in this simple overlapping generations model, for $a > 2.692$ and $w_1 > e^{a-1}$, is $U(c,(t),c,(t+1)) = A - e^{a+w_0-c_0(t)} + c_1(t)$. (See Benhabib and Day (1982), section 3.4.)

Techniques to empirically distinguish between data generated by non-chaotic stochastic systems and deterministic chaotic systems have been developed by physicists and mathematicians (see for example Ruelle and Eckmann (1985)). These techniques have been further refined into statistical tests for applications to economic data by Brock (1986), Brock, Dechert, LeBaron and Scheinkman (1996), among others. Very roughly, these methods exploit the idea that deterministic systems will generate trajectories that are of lower dimension than those generated by stochastic systems which have more scattered trajectories. For example if we consider a one-dimensional difference equation that generates chaotic dynamics, say $x_{t+1} = 4x_t(1 - x_t)$ for initial $x_0 \in (0,1)$, plotting x_{t+1} against x_t will yield a curve. By contrast, if the dynamics were generated by a linear or nonlinear stochastic system with noise, the same plot would produce a scatter of points, which could not be captured by a "relatively smooth", one-dimensional line. By formalizing this idea, we may attempt to distinguish data generated by deterministic chaotic systems and non-chaotic stochastic systems, even without explicit knowledge of the underlying economic system generating the data. In general however such a method is hard to apply because unlike data generated by scientific experiments, economic time series are often not long enough. If the order of underlying dynamical system generating the data is high-dimensional, say of the order of five or higher, or alternatively if we can only observe the realizations of a subset of the variables of the underlying economic model, distinguishing between stochastically and chaotically generated data becomes very difficult. The difficulty of empirically identifying chaos in high dimensional economic systems may be particularly important if chaotic dynamics is more likely to be manifested in disaggregated sectoral or industry data whose components, because of resource constraints or other scarcities, can move in ways that partially offset one another's cyclic or irregular movements. It would therefore be fair to say that at this point, while we know that standard dynamic equilibrium models with parameters calibrated to values often used in the literature may well generate chaotic dynamics, more definitive empirical evidence for chaos in economics has not yet been produced.

While it may be instructive to set the theories of endogenous economic fluctuations in opposition to the theories of fluctuations driven by stochastic shocks, in practice it is more helpful to consider endogenously oscillatory dynamics as complementary to stochastic fluctuations. In certain environments it may make little difference if endogenous mechanisms by themselves generate regular and irregular persistent fluctuations, or whether they give rise to damped oscillations that are sustained by stochastic shocks. On the other hand, if the underlying equilibrium system is subject to distortions and there is room for stabilization policy, correctly identifying the source of the fluctuations becomes much more important. (See for example Benhabib, Schmitt-Grohe and Uribe (2002)). Furthermore recognizing the role of oscillatory dynamics may diminish our reliance on unrealistically large shocks to explain economic data, for example, in real business cycle theory.

2 References

Benhabib, J. and Nishimura, K. (1979): "The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth", Journal of Economic Theory 21,421-444.

Benhabib, J. and Day, R. H. (1980): "Erratic Accumulation," Economic Letters, 6, 113-117.

Benhabib, J. and Day, R. H. (1982): "A Characterization of Erratic Dynamics in the Overlapping Generations Model." Journal of Economic Dynamics and Control 437-55.

Benhabib, J. and Rustichini, A.(1990): "Equilibrium Cycling with Small Discounting," Journal of Economic Theory, 52,423-432.

Benhabib, J., Schmitt-Grohe, S., and Martin Uribe, (2002) : "Chaotic Interest Rate Rules," American Economic Review, Papers and Proceedings, 92, 72-78.

Boldrin, M., and Montrucchio, L. (1986) : "On the Indeterminacy of Capital AccumulationPaths. " Journal of Economic Theory, 40, 24-39.

Brock, W. A. (1986): "Distinguishing Random and Deterministic Systems," Journal of Economic Theory, 40, 68-195.

Brock, W. A., Dechert, W.D., LeBaron B., and J. Scheinkman (1996) : "A Test for Independence Based Upon the Correlation Dimension," Econometric Reviews, 15 (3), 197-235.

Diamond, Phil, (1976): "Chaotic Behavior of Systems of Dilference Equations", International Journal of Systems Science 7, 953-956.

Eckmann, J. P. and Ruelle, D. (1985): ”Ergodic theory of chaos and strange attractors,” Rev. Modern Phys. 57 , no. 3, 617–656.

Frisch, R. (1933): “Propagation Problems and Impulse Problems in Dynamic Economics”, in Economic Essays in Honor of Gustav Cassel, Allen and Unwin, London 171-205 [reprinted as Chapter 9 in Readings in Business Cycles (eds. R.A. Gordon and L. Klein), Richard D. Irwin, Homewood, Illinois,155-185].

Grandmont, J. M. (1985) : ”On Endogenous Competitive Business Cycles. ”Econometrica 53, 995-1045.

Li, T. and Yorke, J. A. (1975) : ”Period Three Implies Chaos,” American Mathematical Monthly 82,985-992.

Marotto, F.R., (1978): ”Snapback Repellers Imply Chaos in R”, Journal of Mathematical Analysis and Applications 63, 199-223.

Mitra, T. (1996): “An Exact Discount Factor Restriction for Period Three Cycles in Dynamic Optimization Models,” Journal of Economic Theory 69, 281–305.

Nishimura, Kazuo and M. Yano (1996): “On the Least Upper Bound of Discount Factors that are Compatible with Optimal Period-Three Cycles,” Journal of Economic Theory 69, 306–333.

Sakai, H. and Tokumaru, H. (1980): ”Autocorrelations of a Certain Chaos”, IEEE Transactions in Acoustic Speech Signal Process 28, 588-590.

Samuelson, P.A. (1939): “Interactions between the Multiplier Analysis and the Principle of Acceleration”, Review of Economic Statistics, 21,75-78

Sarkovskii, A. N. (1964) : ”Coexistence of Cycles of a Continuous Map of the Line Into Itself,” Ukrain. Matem. Zhur. 16, 61-71.

Sorger, Gerhard (1992): “On the Minimum Rate of Impatience for Complicated Optimal Growth Paths,” Journal of Economic Theory 56, 160–179.

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