Indeterminacy Under Constant Returns in Multisector Economies*

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Abstract

The purpose of this paper is to give a theoretical characterization of the size of external effects required for indeterminacy in multisector models with constant social returns, and decreasing private returns. Our characterization makes clear that there is a large class of constant returns to scale economies, with standard Cobb-Douglas production technologies and linear utility functions, which when perturbed to incorporate external effects, exhibit indeterminacy or multiple equilibria. The perturbations that introduce external effects are constrained to maintain constant returns to scale at the social level, and therefore imply that there are decreasing returns to scale from the private perspective. For a large class of constant returns Cobb-Douglas technologies, we provide a a method to construct indeterminate economies by such perturbations, and we characterize the magnitude of the external effects that yield multiple equilibria in terms of the parameters of the unperturbed economy. We show that it is very easy to construct large and plausible classes of economies that exhibit indeterminacy with constant returns to scale, and with external effects that are arbitrarily small. We also provide several examples of such economies, both with linear and nonlinear (logarithmic) utility functions.

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Key words: indeterminacy, multiple equilibria.

1 Introduction

Recently there has been a renewed interest in indeterminacy, or alternatively put, in the existence of a continuum of equilibria in dynamic economies that exhibit some market imperfections[1]. One of the primary concerns of this literature has been the empirical plausibility of indeterminacy, which arises in markets with external effects or with monopolistic competition, often coupled with some degree of increasing returns. While the early results on indeterminacy relied on relatively large increasing returns and high markups, more recently Benhabib and Farmer ([5]) showed that indeterminacy can also occur in two-sector models with small sector-specific external effects and very mild increasing returns. Nevertheless, a number of empirical researchers, refining the earlier findings of Hall ([20], [21]) on disaggregated US data, have concluded that returns to scale seem to be roughly constant, if not decreasing[2]. While one can argue whether the degree of increasing returns required for indeterminacy in Benhabib and Farmer ([5]) falls within the standard errors of the recent empirical estimates, one may also inquire whether increasing returns are at all needed for indeterminacy to arise in a plausible manner.

The purpose of this paper is to give a theoretical characterization of the size of external effects required for indeterminacy in multisector models with constant social returns, and decreasing private returns. Our characterization makes clear that there is a large class of constant returns to scale economies, with standard Cobb-Douglas production technologies and linear utility functions, which when perturbed to incorporate external effects, exhibit indeterminacy or multiple equilibria. The perturbations that introduce external effects are constrained to maintain constant returns to scale at the social level, and therefore imply that there are decreasing returns to scale from the private perspective. For a large class of constant returns Cobb-Douglas technologies, we provide a method to construct indeterminate economies by such perturbations, and we characterize the magnitude of the external effects

[1] A long but incomplete list of the recent literature includes Beaudry and Deveruex ([3]), Benhabib and Farmer ([4]), ([5]), Benhabib and Perli ([9]), Benhabib, Perli and Xie ([10]), Boldrin and Rustichini ([11]), Chatterjee and Cooper ([14]), Christiano and Harrison ([15]), Farmer and Guo ([16]), ([17]), Gali ([18]), Perli ([22]), Rotemberg and Woodford ([23]), Schmitt-Grohé ([24]), Weder ([26]) and Xie ([27]).

[2] See for example Basu and Fernald ([1]), ([2]), Burnside, Eichenbaum and Rebello ([12]) or Burnside ([13]), among others.

that yield multiple equilibria in terms of the parameters of the unperturbed economy. We show that it is very easy to construct large and plausible classes of economies that exhibit indeterminacy with constant returns to scale, and with external effects that are arbitrarily small. We also provide several examples of such economies, both with linear and nonlinear (logarithmic) utility functions.

Constant social returns coupled with small external effects imply that some sectors must have a small degree of decreasing returns at the private level. This is in contrast to models of indeterminacy with social increasing, but private constant returns to scale. An implication of decreasing private returns is of course positive profits. In the parameterized examples given in the sections below, these profits will be quite small because the size of external effects, and therefore the degree of decreasing returns needed for indeterminacy, will also be small. Nevertheless positive profits would invite entry, and unless the number of firms are fixed, a fixed cost of entry must be assumed to determine the number of firms along the equilibrium in the neighborhood of the steady state. Such a market structure would then exhibit increasing private marginal costs but constant social marginal costs, which is in line current empirical work on this subject. It seems therefore that models of indeterminacy based on market imperfections which drive a wedge between private and social returns must have some form of increasing returns, no matter how small, either in variable costs as in some of the earlier models of indeterminacy, or through a type of fixed cost that prevents entry in the face of positive profits. (See also [18], and [19].) The point is that while some wedge between private and social returns is necessary for indeterminacy, this in no way requires decreasing marginal costs, or increasing marginal returns in production.

Indeterminacy or multiple equilibria in dynamic models with small market distortions emerge as a type of coordination problem. Roughly speaking, what is needed for indeterminacy is a mechanism that, starting from a arbitrary equilibrium, if all agents were to simultaneously increase their investment in an asset, the rate of return on the asset would tend increase, justifying the higher investment. One such simple mechanism in one-sector models is increasing returns, typically sustained in a market context via external effects or monopolistic competition (see also the footnote above). In a multisector model however, the rates of return and marginal products depend not only on stocks of assets, but also on the composition of output across sectors. Increasing the production and the stock of a capital asset, say due to an increase in its price, may well increase its rate of return. It is therefore possible to have constant aggregate returns in all sectors at the social level, and to still obtain indeterminacy with minor or even negligible external effects in some of the sectors.

The next two sections introduce the model and describe the dynamics. In section 3.1 we give a proposition describing conditions for indeterminacy in terms of the discount rate and the parameters of the Cobb-Douglas production functions. In section 3.2 provide a constructive method to produce multiple equilibria by perturbing a large class of Cobb-Douglas economies with the introduction of external effects. The propositions in this section characterize the size of the external effects required to produce indeterminacy, as a function of the parameters of the unperturbed economy. In section 3.3 we provide a number of examples of indeterminate economies, both with linear and with non-linear (logarithmic) utility functions. Proofs are relegated to the appendices.

2 The Model

A representative agent optimizes an additively separable utility function with discount rate $(r-g)>0$. This problem can be described as:

$$Max \int_0^\infty U\left(y_0\right)e^{-(r-g)t}dt \tag{1}$$

subject to:

$$y_j = e_j \prod_{i=0}^{n} \left(x_{ij}\right)^{\beta_{ij}}, \qquad j=0,1,\cdots,n \tag{2}$$

$$\frac{dx_i}{dt} = y_i - gx_i \qquad i=1,\cdots,n \tag{3}$$

$$\sum_{j=0}^{n} x_{ij} = x_i \qquad i=0,1,\cdots,n \tag{4}$$

Here $U\ (y_j)$ is a twice differentiable, concave and increasing instantaneous utility function of the consumption good y_0; x_i is the stock of the $i'th$ capital good; x_{ij} is the allocation of the $i'th$ capital good to the production of the $j'th$

good for $j = 1, \cdots, n$; $i = 0, 1, \cdots, n$; and $g > 0$ is the depreciation rate. We denote total labor as $x_0 = 1$, so that it is in fixed supply. The allocations of labor to the production of the i goods are given by x_{0i}. Equations (3) represent the accumulations of the n capital goods x_i. The initial values of the stocks at time zero, $x_i(0)$, are given. The optimization is with respect to the inputs $x_{ij}(t)$ for all i, j, t. Production is subject to an external effect e_j, treated as a constant by the agent, and equal to $\prod_{i=0}^{n} (x_{ij})^{b_{ij}}$. Therefore the true production functions are

$$y_j = \prod_{i=0}^{n} (x_{ij})^{\beta_{ij}+b_{ij}} \qquad j = 0, 1, \cdots, n \tag{5}$$

We can write the Hamiltonian associated with the problem given by (1) as:

$$\begin{aligned} H &= U\left(e_0 \prod_{i=0}^{n} (x_{i0})^{\beta_{i0}}\right) \\ &\quad + \sum_{j=1}^{n} p_j \left(e_j \prod_{i=0}^{n} (x_{ij})^{\beta_{ij}} - g x_j\right) \\ &\quad + \sum_{i=0}^{n} w_i \left(x_i - \sum_{j=0}^{n} x_{ij}\right), \end{aligned}$$

Here p_j and w_i are Lagrange multipliers, representing utility prices of the capital goods and their rentals, respectively. The static first order conditions for this problem are given by:

$$\begin{aligned} w_s &= p_j \left(\beta_{sj} \prod_{i=0}^{n} (x_{ij})^{\beta_{ij}+b_{ij}}\right) (x_{sj})^{-1} \\ &= U'(y_0) \left(\beta_{s0} \prod_{i=0}^{n} (x_{i0})^{\beta_{i0}+b_{i0}}\right) (x_{s0})^{-1} \end{aligned} \tag{6}$$

for $j = 1, ...n$ and $s = 0, 1...n$.

It is easily shown that under constant returns and a Cobb-Douglas technology, static efficiency conditions given by (6) imply that factor rentals are uniquely determined by output prices, and that outputs can be expressed as a function of aggregate stocks and prices . Therefore, taking the price of the consumption good as numeraire, we can express factor rentals as $w_i(p)$ and outputs as $y_i(x,p)$, where $x = (x_1, x_2 ... x_n)$ and $p = (p_1, p_2 ... p_n)$.(See Lemma 3 in the Appendix.)

The necessary conditions that describe the solution to the problem (1) are given by the equations of motion:

$$\frac{dx_i}{dt} = y_i(x,p) - gx_i \qquad i = 1, \cdots, n \tag{7}$$

$$\left(\frac{d(U'p_i)}{dt}\right) = rU'(c)p_i - U'(c)w_i(x,p) \qquad i = 1, \cdots, n \tag{8}$$

Except in section3.3.3 at the end of the paper, we will assume that the utility function is linear. We state this explicitly as:

Assumption 1: $U(c) = c$.

We define the $(n+1)\times(n+1)$ non-negative matrices $B_0 = [\beta_{ij}]$, $E = [b_{ij}]$ and $\hat{B}_0 = B_0 + E$ .We assume that the economy exhibits overall constant returns to scale:

Assumption 2: $\sum_{i=0}^{n} (\beta_{ij} + b_{ij}) = 1 \qquad j = 0, 1 ... n.$

Furthermore we assume:

Assumption 3: The matrices B_0 and $\hat{B}_0$ are non-singular.

Assumption 4: $\beta_{00} > 0$.(The production of the consumption good requires labor.)

Let $\mathcal{B}$ denote the set of pairs of matrices (B_0, E) satisfying Assumptions 2-4, and for $g > 0,\ r - g > 0$, let $\Psi(\mathcal{B}; r, g)$ be the class of Cobb-Douglas economies satisfying the Assumptions 1-4, that is Cobb-Douglas economies with linear utility functions and exponent matrices $(B_0, E) \subset \mathcal{B}$. For $g > 0,\ r - g > 0$, we define the union of $\Psi(\mathcal{B}; r, g)$over r, g as $\Omega(\mathcal{B}) = \cup_{r,g} \Psi(\mathcal{B}; r, g)$.

3 Dynamics

It is easy to show that under our assumptions the equations (7) and (8) have a unique steady state (x^*, p^*) (see Benhabib and Nishimura ([8])). Linearizing around the steady state we obtain:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \left[\frac{\partial y(x^*, p^*)}{\partial x}\right] - gI & \left[\frac{\partial y(x^*, p^*)}{\partial p}\right] \\ [0] & -\left[\frac{\partial w(x^*, p^*)}{\partial p}\right] + rI \end{bmatrix} \begin{bmatrix} x - x^* \\ p - p^* \end{bmatrix}$$
$$= J \begin{bmatrix} x - x^* \\ p - p^* \end{bmatrix}$$

The matrix $\left[\frac{\partial w(x^*, p^*)}{\partial x}\right]$ is identically zero because the factor rentals are uniquely determined by prices, independently of factor stocks (See Lemma 3 in the appendix). We note that the matrix J is quasi-triangular, so that its roots are the roots of $\left[\frac{\partial y(x^*, p^*)}{\partial x}\right] - gI$ and $-\left[\frac{\partial w(x^*, p^*)}{\partial p}\right] + rI$.

Let B and $\hat{B}$ be $(n \times n)$ matrices given by

$$B = [\beta_{ij} - (\beta_{0j}\beta_{i0})/\beta_{00}] \tag{9}$$

$$\hat{B} = [(\beta_{ij} + b_{ij}) - (\beta_{0j} + b_{0j})(\beta_{i0} + b_{i0})/(\beta_{00} + b_{00})] \tag{10}$$

where $i, j = 1, ...n$ and β_{00} is a scalar. Let W denote the $n \times n$ diagonal matrix with diagonal elements w_i, $i = 1, ...n$ and zero iff-diagonal elements. Similarly let P denote the $n \times n$ diagonal matrix with diagonal elements p_i, $i = 1, ...n$ and zero iff-diagonal elements. We show in the appendix (Lemma 6 and Lemma 7) that

$$\left[\frac{\partial y(x^*, p^*)}{\partial x}\right] = P^{-1}B^{-1}W$$

and

$$\left[\frac{\partial w(x^*, p^*)}{\partial p}\right] = W\left[\hat{B}'\right]^{-1}P^{-1}$$

Furthermore, we show in Lemma 8 in the appendix, that evaluated at the steady state, the roots of the matrix $\left[\frac{\partial y(x^*, p^*)}{\partial x}\right]$ and the matrix B have the same sign structure while the roots of the matrix $\left[\frac{\partial w(x^*, p^*)}{\partial p}\right]$ and the matrix $\hat{B}$ also have the same sign structure.[3] Therefore, when there are no

[3]Note also that the root strucures of B and $\hat{B}$ are directly related to factor intensity matrices at the steady state , given in equations 34 and 48 in the appendix by $\left[\frac{\partial y}{\partial k}\right] = \left[a_i^j - (a_0^j a_i^0)/a_0^0\right]$ and and 48 in the appendix by $\left[\frac{\partial w}{\partial p}\right] = \left[\widehat{a}_i^j - (\widehat{a}_0^j \widehat{a}_i^0)/\widehat{a}_0^0\right]$.

external effects we have $B = \hat{B}$, so that the roots of J come in pairs of $((\mu_i - g), (-\mu_i + r))$.

3.1 Indeterminacy

In the economy described above, there are n capital goods whose initial values are given. The evolution of capital stocks x and prices p in equilibrium are determined by equations (7) and (8). Any trajectory that converges to the stationary point (x^*, p^*) will automatically satisfy the transversality conditions and constitute an equilibrium. Therefore given $x(0)$, if there are more than one set of initial prices $p(0)$ in the stable manifold of (x^*, p^*), the equilibrium trajectory from $x(0)$ will not be unique. In particular, if the dimension of the local stable manifold of (x^*, p^*) is greater than n,that is if the Jacobian J has more than n negative roots, there will be a continuum of initial prices $p(0)$ for each $x(0)$ in the neighborhood of (x^*, p^*) that will give rise to trajectories which converge to (x^*, p^*) and constitute a continuum of equilibria.

Definition 1 *Let (x^*, p^*) be the steady state of the economy whose dynamics are described by equations (7) and (8). If the dimension of the locally stable manifold of (x^*, p^*) has dimension greater than n, then the economy and its steady state (x^*, p^*) are said to be indeterminate.*

As the following Proposition makes clear, it is now easy to construct Cobb-Douglas technologies that exhibits indeterminacy.

Proposition 1 *Suppose the matrix $\left[B^{-1} - \left(\frac{g}{r}\right) I\right]$ has s roots with negative real parts but the matrix $\left[\left[\hat{B}\right]^{-1} - I\right]$ has less than s roots with negative real parts, where $s \in \{1, ..., n\}$. Then the economy and its steady state are indeterminate.*

Proof. : See section 4.0.6 in the Appendix. **QED**

Inspecting the definitions of the matrices B and $\hat{B}$, we see that $\hat{B}$ and its roots can be quite different from B and its roots, even if the size of the external effects are small. As long as the matrix $\left[B^{-1} - \left(\frac{g}{r}\right) I\right]$ has at least one root with a negative real part, there are many degrees of freedom in the choice of externalities to satisfy the hypotheses of the above Proposition, and indeterminacy is easy to obtain. In the next subsection we provide a systematic method to construct indeterminate economies and give a characterization of the magnitude of the external effects required for indeterminacy.

3.2 Size of Externalities and Indeterminacy

In this subsection we will show that there is a large class of constant returns economies that become indeterminate when perturbed by introduction of external effects, even though the perturbed economies are constrained to exhibit overall constant returns to scale. Our results characterize the magnitude of external effects required for indeterminacy, and make clear that it is possible to construct classes of standard Cobb-Douglas economies that are indeterminate, with external effects that are arbitrarily small.

In order to characterize the magnitude of external effects required to obtain indeterminacy while still retaining constant returns to scale production functions, we have to impose some restrictions on the initial set of economies without external effects that we start out with. We now formally describe these restrictions.

Assumption 5: The matrix B has at least one negative real root.

In a two-sector model B is a scalar, and Assumption 5 simply implies that the capital good is labor intensive.

A sufficient condition for Assumption 5 to hold is for the determinant of B to be negative, since the determinant is the product of the roots. Since $\beta_{00} > 0$, this is equivalent to requiring the exponent matrix B_0 to have a negative determinant because

$$|B_0| = \beta_{00} \cdot \left|\beta_{ij} - (\beta_{00})^{-1} \beta_{i0}\beta_{0j}\right| = \beta_{00} \cdot |B| < 0$$

Therefore as an alternative to Assumption 5,we may also use the stronger assumption:

Assumption $5'$: $|B_0| < 0$.

We now introduce a final assumption which needs some elaboration. We want to show how a determinate steady state may become indeterminate with the introduction of small external effects. For this purpose we want to begin with a determinate economy (without external effects) in $\Omega(\mathcal{B})$ that has a steady state which is stable in the saddle-point sense. Saddle-point instability can occur in multisector economies with unique steady states under constant returns to scale and without any externalities or market distortions. In such cases the equilibrium trajectory is unique, but is associated with the existence of cycles or chaotic dynamics (see [6])[4]. When exploring indeterminacies we would like to avoid such complications , which occur when the Jacobian matrix of the linearized dynamics, J, has more than half of its roots with negative real parts.

For an economy without external effects, that is for an economy for which all the elements of E are zero, the roots of J come in pairs $\left(\sigma_i^{-1} - g,\ -\sigma_i^{-1} + r\right)$, where σ_i^{-1} is a root of the matrix $P^{-1}B^{-1}W$. Let $\lambda_i = \sigma_i r$. Since at the steady state $w_i/p_i = r$, it follows that λ_i will be a root of the matrix B. Saddle-point stability of this economy without external effects is assured by Assumption 6 below.

Assumption 6: (The saddle point stability for the initial economy) Let λ_i be an eigenvalue of the matrix B. Then $\left(r \operatorname{Re}\left(\lambda_i^{-1}\right) - g\right)\left(-r \operatorname{Re}\left(\lambda_i^{-1}\right) + r\right) < 0$ *for all* i.

Assumptions 5 and 6 impose additional restrictions on the economies in $\Omega(\mathcal{B})$. Since we start with a class of constant returns to scale economies with no external effects, the matrix E is initially set to have only zero elements, and of course constant returns implies that the column sums of $B_0 = [\beta_{ij}]$ are equal to 1. Let E_0 be the matrix E with all its elements set to zero, and let $\mathcal{B}_0$ be the set of pairs of matrices (B_0, E_0) that satisfy Assumptions 1-5, as well as Assumption 6. We denote this class of Cobb-Douglas economies satisfying assumptions 1-6 by $\Omega(\mathcal{B}_0) \subset \Omega(\mathcal{B})$.

Given any economy in $\Omega(\mathcal{B}_0)$, we will now construct another economy in $\Omega(\mathcal{B})$ with external effects, as described below. Starting with an economy in $\Omega(\mathcal{B}_0)$,for $m \in (0,1)$, we define a corresponding economy for which the capital goods have decreasing returns to scale of degree $(1-m)$ from the private perspective. This is equivalent to multiplying all columns of B_0 except

[4]More precisely, it is known that an economy without distortions and externalities, even with decreasing returns in some sectors, will collapse to a social planners problem, and will have its eigenvalues defining its local dynamics around the steady state come in pairs $\left(\sigma_i^{-1} - g, -\sigma_i^{-1} + r\right)$, where σ_i^{-1} is a root of B^{-1} and depends on the discount rate $r-g$. In some circumstances when $r > g$, both roots in the pair can become positive $\left(\sigma_i^{-1} - g, -\sigma_i^{-1} + r\right)$. The equilibrium path is still unique from given initial conditions due to the concavity of the planner's problem, but the unique steady state is unstable and the optimal path may be cyclic or chaotic. See [6].

the first by $(1-m)$:

$$X_m = \left[\begin{array}{cc} \beta_{00} & (1-m)\cdot\beta_{0.} \\ \beta_{.0} & (1-m)\cdot\beta \end{array} \right]$$

Every column of X_m except the first one adds up to $(1-m)$. Let $b_{00} = 0,\ b_{0.} = 0,\ b_{.0} = 0$. We now construct a matrix of externalities so that all goods are produced with constant returns from the social perspective. Let

$$E_m = \left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & mI \end{array} \right]$$

Note that m defines a measure of the size of overall external effects for this economy. Furthermore, the column sums of the exponent matrix $X_m + E_m$ below are equal to unity, implying constant overall returns in each industry:

$$X_m + E_m \equiv \left[\begin{array}{cc} \beta_{00} & (1-m)\beta_{0.} \\ \beta_{.0} & (1-m)\beta \end{array} \right] + \left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & mI \end{array} \right]$$

We now have an economy defined by $(r, g, X_{m,} E_m)$, constructed from an economy in $\Omega_0(\mathcal{B}_0)$. Given m, let $\mathcal{B}_m$ be the set of matrices constructed as above from matrices $(B_0, E_0) \subset \mathcal{B}_0$, so that $(X_m, E_m) \subset \mathcal{B}_m$. For any $m \in (0,1)$, the construction above defines a mapping from an economy in $\Omega(\mathcal{B}_0)$ to a unique constant returns economy in $\Omega(\mathcal{B})$ that exhibits external effects of magnitude m in the production of capital goods . We denote this mapping by: $\Upsilon_m : \Omega(\mathcal{B}_0) \to \Omega(\mathcal{B})$. For any $m \in (0,1)$, the class of economies in $\Omega(\mathcal{B})$ constructed as above from $\Omega(\mathcal{B}_0)$ are given by $\Upsilon_m(\Omega(\mathcal{B}_0))$.

Definition 2 *For an economy in $\Omega(\mathcal{B}_0)$, it follows from Assumption 5 $(5')$ that the exponent matrix B has at least one negative real root. Let λ_B denote the negative real root of B that is smallest in absolute value. Furthermore, if the matrix B^{-1} has roots with real parts in the interval $(0, g/r)$, we denote by $\mathrm{Re}\left(\lambda_M^{-1}\right)$ the largest such real part. If B^{-1} does not have a root with real part in the interval $(0, g/r)$, then λ_M^{-1} is defined as identically zero.*

Proposition 2 *Any economy in $\Upsilon_m(\Omega(\mathcal{B}_0))$ which satisfies*

$$(i) \quad \frac{-\lambda_B}{1-\lambda_B} < m$$

$$(ii) \quad 0 < m < 1 - (r/g)Re\left(\lambda_M^{-1}\right), \quad OR, \quad 1 - Re\left(\lambda_M^{-1}\right) < m < 1$$

is indeterminate.

Proof. See section 4.0.6 in the Appendix.

Assumption 5 in the proposition above requires the matrix B to have a negative real root. This assumption can be relaxed to the require that the matrix B have at least one root with negative real part, provided an additional constraint on m is postulated.

Assumption $5''$: The matrix B has at least one root with a negative real part.

We denote the class of economies in $\Omega(\mathcal{B}_0)$ satisfying Assumptions $5''$ and 6 by $\hat{\Omega}(\mathcal{B}_0) \subset \Omega(\mathcal{B}_0)$.

Definition 3 *For an economy in $\hat{\Omega}(\mathcal{B}_0)$ it follows from Assumption $5''$ that the exponent matrix B has at least one root with negative real part: let $\operatorname{Re}(\lambda_B)$ denote the negative real root of B that is smallest in absolute value. Furthermore, if the matrix B^{-1} has roots with real parts in the interval $(0, g/r)$, we denote by $\operatorname{Re}(\lambda_M^{-1})$ the largest such real part . If B^{-1} does not have a root with real part in the interval $(0, g/r)$, then λ_M^{-1} is defined as identically zero.*

Corollary 1 *Any economy in $\Upsilon_m\left(\hat{\Omega}(\mathcal{B}_0)\right)$ which satisfies*

$$
\begin{array}{llll}
(i) & \dfrac{-\operatorname{Re}(\lambda_B)}{1-\operatorname{Re}(\lambda_B)} & < & m \\
(ii) & 0 & < & m < 1-(r/g)Re\left(\lambda_M^{-1}\right), \;\; OR, \;\; 1-Re\left(\lambda_M^{-1}\right) < m < 1
\end{array}
$$

and

$$
(iii) \quad \operatorname{Re}(\lambda_B) + \frac{(Im(\lambda_B))^2}{(1-m)\left[(1-m)\operatorname{Re}(\lambda_B)+m\right]} < 0
$$

is indeterminate.

Proof. : See section 4.0.6 in the Appendix.

3.3 Examples

3.3.1 The two sector model

If we confine our attention to a two-sector model with one capital good, we can obtain a result that can be stated directly in terms of factor intensities. As shown in the appendix (see the proof of Lemma 6), comparing the ratios of Cobb-Douglas exponents of the production functions amounts to comparing factor intensities between capital goods and the consumption good. In the two sector case the matrix B reduces to a scalar reflecting these factor intensities which can be defined by the Cobb-Douglas exponents, both with and without the external effects. We may therefore say that the capital good is labor intensive **from the private perspective** if $(\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0)$, but that it is capital intensive **from the social perspective** if $(\beta_{11}+b_{11})(\beta_{00}+b_{00}) - (\beta_{10}+b_{10})(\beta_{01}+b_{01}) > 0$. The expressions above allow us to state the following simple result:

Corollary 2 *In the two-sector model, if the capital good is labor intensive* **from the private perspective,** *but capital intensive* **from the social perspective**, *then the steady state is indeterminate.*

Proof. : See section 4.0.6 in the Appendix.

Note that the assumption that the capital good is labor intensive from the private perspective, that is $(\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0)$, corresponds to Assumption 5 or $5'$ in the previous section. A simple example based on the above corollary illustrates the possibility of indeterminacy in the two-sector model, for any $r > 0$, $g \geq 0$, and only a small externality of the labor (0.05) in the production of the consumption good. Let:

$$\begin{bmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.34 \\ 0.65 & 0.66 \end{bmatrix}$$

and

$$\begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0 \end{bmatrix}$$

Then we have

$$\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0 \tag{11}$$

$$(\beta_{10} + b_{10})(\beta_{01} + b_{01}) - (\beta_{11} + b_{11})(\beta_{00} + b_{00}) < 0 \tag{12}$$

and therefore both roots of J are negative. Note also that without some external effects both of the above conditions cannot hold simultaneously. It is clear nevertheless that examples satisfying the above conditions for indeterminacy can be constructed with arbitrarily small external effects.

A slight modification of the above example can serve as an illustration of Proposition 2. We eliminate the labor externality on the consumption good so that $b_{00} = 0$, we set $\beta_{00} = 0.33, \beta_{10} = 0.67,\ \beta_{11} = 0.66\,(1-m)\,,\ \beta_{01} = 0.34\,(1-m)\,$, and we introduce a capital externality in the investment good, $b_{11} = m$. Constant returns to scale from the social perspective is therefore maintained in both industries. The matrix B in Proposition 2 in this case is a scalar, equal to $-0.01\,(1-m)\,$. It is easy to compute that for any for $m \in [0.0295, 1)$, the hypotheses of Proposition 2 will be satisfied (and that the inequalities 11 and 12 will hold), so that the economy will be indeterminate .

3.3.2 An example with generalized factor intensities

In the trade literature a number of results that apply to two-sector models have been generalized to multisector models by imposing special structures on the production technology. The special structures are typically imposed on the input coefficient matrices so that factor intensity conditions can be defined in a multisector framework and the Rybczinski and Stolper-Samuelson theorems can be generalized. We use these special structures to derive conditions for indeterminacy.

Definition 4 *Let the* $((n+1)\times(n+1))$ *input coefficient matrix for our Cobb-Douglas economy be defined by* $A_0 = W_0^{-1} B_0 P_0 = [a_{ij}]\,.$

Definition 5 *Let* $\hat{a}_{ij} = a_{ij}(\beta_{ij} + b_{ij})/\beta_{ij}$ *and define* $\widehat{A_0} = [\hat{a}_{ij}]\,.$

Definition 6 *A non-negative matrix* A_0 *is an* $SSS-I$ *matrix if* A_0 *is non-singular and* A_0^{-1} *is a Minkowski matrix with all off-diagonal elements negative and all diagonal elements positive.*

Definition 7 *A non-negative matrix* A_0 *is an* $SSS-II$ *matrix if* A_0 *is non-singular and* A_0^{-1} *is a Metzler matrix with all off-diagonal elements positive and all diagonal elements negative.*

Note that since W_0^{-1} and P_0 are nonnegative diagonal matrices, if B_0 is a Minkowski or Metzler matrix, so is A_0. It follows that for the proposition below, the definitions of SSS-I and SSS-II can be equivalently stated in terms of the exponent matrix B_0. While the matrix B_0 defines a Cobb-Douglas technology, the input coefficient matrix A_0 reflects steady state input coefficients that could in principle come from technologies other than Cobb-Douglas.

Proposition 3 *If A_0 is SSS-II and $\widehat{A_0}$ is SSS-I, then the steady state is indeterminate.*

Proof. See section 4.0.6 in the Appendix.

The last proposition implies that all the roots of the Jacobian of the linearized dynamics have negative real parts, and therefore the indeterminacy is of order n. This proposition may be seen as a generalization of the results of the two-sector model in the previous section, with $n = 1$.The restrictions imposed on the input coefficients however are overly strong because, as is clear from Proposition 2, only one negative root of the Jacobian is sufficient for indeterminacy.

3.3.3 Non-Linear Utility

Suppose the instantaneous utility function is

$$U(c) = Mc^{(1-\sigma)} \qquad \sigma \in (0,1)$$

and the production functions are

$$c = y_0 = \prod_{i=0}^{n} (x_{i0})^{\beta_{i0}+b_{i0}} \tag{13}$$

$$y_j = \prod_{i=0}^{n} (x_{ij})^{\beta_{ij}+b_{ij}}, \qquad j = 1, \cdots, n \tag{14}$$

Let the production functions for the capital goods be homogenous of degree 1, and let the production of the consumption good be homogenous of degree $(1-\sigma)^{-1}$, so that $\sum_i (\beta_{ij} + b_{ij}) = 1,\ for\, j = 1, ..., n$, and $\sum_i (\beta_{i0} + b_{i0}) = (1-\sigma)^{-1}$. This economy will be equivalent to an economy with a linear utility in consumption good $\tilde{c}$, and a redefined constant returns to scale production function for the consumption good, scaling the Cobb-Douglas exponents by $(1-\sigma)$:

$$\begin{aligned} M\tilde{c} &= Mc^{(1-\sigma)} \\ \tilde{c} &= \left(\prod_{i=0}^{n} (x_{i0})^{\beta_{ij}+b_{i0}} \right)^{(1-\sigma)} \end{aligned}$$

The two economies above are equivalent in the sense that maximizing the sums of discounted utility subject to the technology constraints will produce identical solutions. Therefore we can apply the results of the earlier section to the transformed economy to characterize indeterminacy in the case of non-linear utility.

The construction above is non-robust because the curvature in the utility function, given by $(1-\sigma)$, is exactly offset by the increasing returns in the production of the consumption good. Generalizing our results on the characterization of indeterminacy to cases of variable returns to scale in production would allow us to have arbitrary returns to scale in consumption, and then to scale these returns to redefine a linear utility function as above. We will pursue this in future research.

If we rely on numerical computations however it is easy to construct calibrated and robust examples of economies with non-linear utility and constant returns to scale in all sectors, that display indeterminacy. The following example illustrates this point. Consider a three sector economy with constant returns and an instantaneous utility function that is logarithmic: $U(y_0) = \ell n\, y_0$. Labor supply is fixed to unity. Let $r = 0.05$ and $g = 0.01$, so that the discount rate is 0.04. The Cobb-Douglas exponents for the production functions are given as follows. For the consumption good y_0 let

$$\beta_{00} = 0.66,\ b_{00} = 0.00,\ \beta_{10} = 0.24, b_{10} = 0.00, \beta_{20} = 0.10,\ b_{20} = 0.00,$$

for the first investment good y_1 let

$$\beta_{01} = 0.61,\ b_{01} = 0.00,\ \beta_{11} = 0.20, b_{11} = 0.06,\ \beta_{21} = 0.13,\ b_{21} = 0.00,$$

and for the second investment good y_2 let

$$\beta_{02} = 0.61,\ b_{02} = 0.00,\ \beta_{12} = 0.23, b_{12} = 0.00,\ \beta_{22} = 0.10,\ b_{22} = 0.06.$$

It is clear that this economy has pretty standard labor shares in each industry, a standard discount rate, small external effects only in the production of the two investment goods (given by $b_{11} = 0.06$ and $b_{22} = 0.06$), and overall constant returns to scale in each industry. The dynamics of this economy can be expressed with four differential equations, two for the capital stocks and two for their relative prices. The four roots of the associated Jacobian of the linearized dynamics are given by: $\{0.0957,\ -0.0464,\ -1.68,\ -1.62\}$. The three negative roots indicate local indeterminacy.

For calibrated RBC examples of three sector models exhibiting indeterminacy in discrete time that introduce sunspots, and match the standard moments of US time series on consumption, investment, output and employment, see the [7].

4 Appendix

4.0.4 The functions $\mathbf{w}_i(p)$ and $\mathbf{y}_i(x,p)$

Each firm maximizes its profit given output price p_j and input prices $w_0,\cdots,w_n$. Its profit is,

$$\pi_j = p_j y_j - \sum_{i=0}^{n} w_i x_{ij} \tag{15}$$

The first order condition subject to the private production function (1) is the following:

·

$$p_j \beta_{ij} \frac{y_j}{x_{ij}} = w_i \quad i = 0,\cdots,n \tag{16}$$

The cost minimizing solutions x_{ij} as a function of $w_0, ..., w_n$ and y_j are given in the following lemma:

Lemma 1 Given y_j and $w_0,\cdots,w_n$, the input level to minimize the production cost is

$$x_{ij} = y_j \prod_{t=0}^{n} \left(\frac{w_t}{\beta_{tj}} / \frac{w_i}{\beta_{ij}} \right)^{\beta_{tj}+b_{tj}} \tag{17}$$

Proof. From equation (4),

$$x_{ij} = p_j \frac{\beta_{ij} y_j}{w_i} \tag{18}$$

We substitute (6) into (2) and use $1 = \sum_{t=0}^{n} (\beta_{tj} + b_{tj})$,

$$1 = p_j \prod_{t=0}^{n} (\beta_{tj}/w_t)^{\beta_{tj}+b_{tj}} \tag{19}$$

Substitute $p_j = (x_{ij}/y_j) \cdot (w_i/\beta_{ij})$ into (7),

$$y_j = x_{ij} \frac{w_i}{\beta_{ij}} \prod_{t=0}^{n} \left(\frac{\beta_{tj}}{w_t} \right)^{\beta_{tj}+b_{tj}} \tag{20}$$

By solving (8) with respect to x_{ij}, we obtain the following.

$$x_{ij} = y_j \prod_{t=0}^{n} \left(\frac{w_t}{\beta_{tj}} / \frac{w_i}{\beta_{ij}} \right)^{\beta_{tj}+b_{tj}} \tag{21}$$

QED

Let us denote the input coefficient to produce one unit of the output y_j by a_{ij}, and let p_0 be the price of the consumption good and w_0 the wage rate for labor. Then we have

$$a_{ij} = \prod_{t=0}^{n} \left(\frac{w_t}{\beta_{tj}} / \frac{w_i}{\beta_{ij}} \right)^{\beta_{tj}+b_{tj}} \tag{22}$$

Let $A_0 = [a_{ij}]\,, B_0 = [\beta_{si}]$ and $\hat{B}_0 = [\beta_{si} + b_{si}]\,.$ Let W_0 denote the $(n+1) \times (n+1)$ diagonal matrix with diagonal elements $w_i,\ i = 0, 1, ...n$ and zero iff-diagonal elements. Similarly let P denote the $(n+1) \times (n+1)$ diagonal matrix with diagonal elements $p_i,\ i = 0, 1, ...n$ and zero iff-diagonal elements.

Lemma 2: $A_0 = W_0^{-1} B_0 P_0$ and it is nonsingular.

Proof. From 18 we have

$$a_{ij} = \frac{p_j \beta_{ij}}{w_i} \tag{23}$$

for $i = 0, 1, ...n$ and $j = 0, 1, ...n$, which yields the expression in the Lemma. Under Assumption 3 A_0 is nonsingular since it is obtained by premultiplying and postmultiplying B_0 with diagonal matrices with positive elements.

QED

Lemma 3: Factor rentals are functions of output prices, $w_i = w_i\,(p)$ and outputs are functions of factor stocks, labor and output prices, $y_i = y_i\,(x, p)\,.$

Proof. : Setting $j = i$ we obtain $n+1$ equations by substituting (y_i/x_{ii}) from equations 17 into equations (16). Taking logs, and noting that $\sum_{s=0}^{n} (\beta_{si} + b_{si}) = 1$ for all i, we obtain:

$$\ell n\ p_i = \sum_{t=0}^{n} (\beta_{ti} + b_{ti})\,(\ell n\ w_t - \ell n\ \beta_{ti}) \qquad i = 0, 1...n$$

Since $\hat{B}_0$ is nonsingular by Assumption 3, this can be rewritten as:

$$w_i(p) = K_i \prod_{s=0}^{n} (p_s)^{(\beta_{si}+b_{si})} \qquad i = 0, 1...n \tag{24}$$

where K_i is a constant, and p_0 can be taken as the numeraire.

Let $y = (y_1, \cdots, y_n)'$ and $x = (x_{1,} \cdots, x_n)'$, where x_i is the endowment of $i - th$ factor in the economy. Also, x_0 is the labor endowment and y_0 is the output of the consumption good. Full employment conditions are written,

$$A_0 \begin{bmatrix} y_0 \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ x \end{bmatrix} \tag{25}$$

Since A_0 in equation (25)is non-singular from Lemma 2, we have $\begin{bmatrix} y_0 \\ y \end{bmatrix} = A_0^{-1} \begin{bmatrix} x_0 \\ x \end{bmatrix}$; since the elements of A_0 are the input coefficients a_{ij} which by equation (22) are functions of input prices $w_i(p_0; p)$, it follows that $y_i = y_i(x_0; x, p_0; p), i = 0, 1....n$, where p_0 can be taken as numeraire.

QED

In order to obtain the dual relationship of price and output in the context of externalities, we express the price function in terms of input coefficients and Cobb-Douglas exponents. Let $\hat{a}_{ij} = a_{ij}(\beta_{ij} + b_{ij})/\beta_{ij}$ and define $\hat{A}_0 = [\hat{a}_{ij}]$.

Lemma 4 : $\hat{A}_0 = W_0^{-1} \hat{B}_0 P_0$ and it is nonsingular.

Proof. : Since from 18

$$a_{ij} = \frac{p_j \beta_{ij}}{w_i},$$

it follows that

$$\hat{a}_{ij} = \frac{p_j(\beta_{ij} + b_{ij})}{w_i} \tag{26}$$

and Lemma 4 follows immediately from Assumption 3.

QED

Lemma 5

$$p_j = \sum_{i=0}^{n} \hat{a}_{ij} w_i \qquad j = 0, ...n \tag{27}$$

Proof. From (6) and (11),

$$p_j \beta_{ij} = w_i a_{ij} \tag{28}$$

$$p_j(\beta_{ij} + b_{ij}) = w_i \hat{a}_{ij} \tag{29}$$

Taking the sum over $i = 0, \cdots, n$

$$p_j = \sum_{i=0}^{n} w_i \ \hat{a}_{ij} \tag{30}$$

QED

Let $p = (p_1, \cdots, p_n)'$ and $w = (w_1, \cdots, w_n)'$. Also, let p_0 be the price of the consumption good and w_0 the wage rate for labor. Then price and cost relation may be summarized into the following.

$$\begin{bmatrix} p_0 \\ p \end{bmatrix} = \widehat{A}_0' \begin{bmatrix} w_0 \\ w \end{bmatrix} \tag{31}$$

QED

4.0.5 Local Dynamics

Let us define the $n \times n$ matrix:

$$B = (\beta_{ij} - (\beta_{0j}\beta_{i0})/\beta_{00})$$

Define the matrices $W = Iw$ and $P = Ip$, where I is the $n \times n$ identity matrix.

Lemma 6

$$\left[\frac{\partial y(x^*, p^*)}{\partial x}\right] = P^{-1}B^{-1}W$$

Proof.

We first take the total derivative of the full employment equation (12). Let $w = (w_0, w_{1,} \cdots, w_n)'$.

$$A_0 \begin{bmatrix} dy_0 \\ dy \end{bmatrix} + \sum_{j=0}^{n} y_j \frac{\partial a^j}{\partial w} dw = \begin{bmatrix} dx_0 \\ dx \end{bmatrix} \tag{32}$$

where

$$\left[\frac{\partial a^j}{\partial w}\right] = \left[\frac{\partial a_{ij}}{\partial w_s}\right] \tag{33}$$

where $i, j, s = 0, 1, ...n$. It follows that

$$\frac{\partial(y_0; y)}{\partial\,(x_0, ; x)} = A_0^{-1}$$

and from the expression for the inverse of A_0,

$$\frac{\partial y}{\partial x} = \left[a_{ij} - \frac{a_{0j}a_{i0}}{a_{00}}\right]^{-1} \tag{34}$$

From equation 26 the above can be written as:

$$\frac{\partial y}{\partial x} = P^{-1}\left[\beta_{ij} - \frac{\beta_{0j}\beta_{i0}}{\beta_{00}}\right]^{-1} W = P^{-1}B^{-1}W$$

QED

Let

$$\hat{B} = (\beta_{ij} + b_{ij} - (\beta_{0j} + b_{0j})(\beta_{i0} + b_{i0})/(\beta_{00} + b_{00}))$$

Lemma 7:

$$\frac{\partial w(x^*, p^*)}{\partial p} = W\left[\hat{B}'\right]^{-1} P^{-1}$$

Proof. :

Let $\overline{w}_{ij} = w_i(\beta_{ij} + b_{ij})/\beta_{ij}$ and $\overline{w} = (\overline{w}_0, \cdot\cdot\cdot, \overline{w}_n)$. We rewrite a_{ij} as a function of $\overline{w}$ instead of w and denote it by $\overline{a}_{ij}$. From equation 22 we have:

$$\overline{a}_{ij} = \prod_{t=0}^{n}\left[\left(\frac{\overline{w}_{tj}}{\beta_{tj} + b_{tj}}\right) / \left(\frac{\overline{w}_{ij}}{\beta_{ij} + b_{ij}}\right)\right]^{\beta_{tj}+b_{tj}} \tag{35}$$

It follows that:

$$\hat{a}_{ij}w_i = \overline{a}_{ij}\overline{w}_{ij} \tag{36}$$

From Lemma 4 by total differentiation we obtain:

$$dp_j = \sum_t \left(\sum_i \frac{\partial \overline{a}_{ij}}{\partial \overline{w}_{ij}}\overline{w}_{ij}\right) d\overline{w}_{sj} + \sum \overline{a}_{ij} d\overline{w}_{ij} \qquad j = 0, ...n \tag{37}$$

We will show that the first term above is zero:

$$\sum_{i=0}^{n} \frac{\partial \overline{a}_{ij}}{\partial \overline{w}_{tj}} \overline{w}_{ij} = 0 \tag{38}$$

Note that since $\sum_{i=0}^{n} (\beta_{ij} + b_{ij}) = 1$, $\overline{a}_{ij}$ is equal to the following.

$$\overline{a}_{ij} = \left(\frac{\overline{w}_{ij}}{\beta_{ij} + b_{ij}} \right)^{-1} \prod_{t=0}^{n} \left(\frac{\overline{w}_{tj}}{\beta_{tj} + b_{tj}} \right)^{\beta_{tj} + b_{tj}} \tag{39}$$

Also the following relation always holds.

$$\overline{a}_{ij} = \overline{a}_{sj} \left(\frac{\overline{w}_{sj}}{\beta_{sj} + b_{sj}} \right) / \left(\frac{\overline{w}_{ij}}{\beta_{i0} + b_{ij}} \right) \tag{40}$$

We differentiate $\overline{a}^j_0, \cdots, \overline{a}^j_n$ with respect to $\overline{w}_{sj}$ and obtain,

$$\begin{aligned} \frac{\partial \overline{a}^j_0}{\partial \overline{w}_{sj}} &= \left(\frac{\overline{w}_{0j}}{\beta_{0j} + b_{0j}} \frac{\overline{w}_{sj}}{\beta_{sj} + b_{sj}} \right)^{-1} \prod_{t=0}^{n} \left(\frac{\overline{w}_{tj}}{\beta_{tj} + b_{tj}} \right)^{\beta_{tj} + b_{tj}} \\ &= \left(\frac{\beta_{0j} + b_{0j}}{s_j \overline{w}_{0j}} \right) \overline{a}_{sj} \end{aligned} \tag{41}$$

$$\frac{\partial \overline{a}_{sj}}{\partial \overline{w}_{sj}} = (\beta_{sj} + b_{sj} - 1) \frac{\overline{a}_{sj}}{\overline{w}_{sj}} \tag{42}$$

and for $t = 1, ...s - 1, s + 1, ...n$,

$$\frac{\partial \overline{a}_{tj}}{\partial \overline{w}_{sj}} = \left(\frac{\beta_{tj} + b_{tj}}{\overline{w}_{tj}} \right) \overline{a}_{sj} \tag{43}$$

Hence

$$\begin{aligned} \sum_{t=0}^{n} \overline{w}_{tj} \frac{\partial \overline{a}_{tj}}{\partial \overline{w}_{sj}} &= \left(\sum_{t} (\beta_{tj} + b_{tj}) - 1 \right) \overline{a}_{sj} \\ &= 0 \end{aligned} \tag{44}$$

Therefore from 37 and 36 we have

$$dp_j = \sum_{i=0}^{n} \overline{a}_{ij} d\overline{w}_{ij} \qquad j = 0, ...n \tag{45}$$

$$= \sum_{i=0i}^{n} \hat{a}_{ij} dw_i \tag{46}$$

or

$$\begin{bmatrix} dp_0 \\ dp \end{bmatrix} = \widehat{A_0}' \begin{bmatrix} dw_0 \\ dw \end{bmatrix} \tag{47}$$

It follows that

$$\frac{\partial(w_0; w)}{\partial\left(p_0, ; p\right)} = \left[\hat{A}_0'\right]^{-1}$$

and from the expression for the inverse of $\hat{A}_0'$,

$$\frac{\partial w}{\partial p} = \left[\hat{a}_{ji} - \frac{\hat{a}_{i0}\hat{a}_{j0}}{\hat{a}_{00}}\right]^{-1} \tag{48}$$

From equation 23 the above can be written as:

$$\frac{\partial w}{\partial p} = W\left[\beta_{ji} + b_{ji} - \frac{\left(\beta_{0i} + b_{0i}\right)\left(\beta_{j0} + b_{j0}\right)}{\beta_{00} + b_{00}}\right]^{-1} P^{-1} = W\left[\hat{B}'\right]^{-1} P^{-1}$$

QED

Definition: The inertia of a matrix A is a triplet $I = \{\pi(A), v(A), \delta(A)\}$ where $\pi(A)$ is the number of roots of A with positive real parts, $v(A)$ is the number of roots of A with negative real parts, and $\delta(A)$ is the number of roots of A with zero real parts.

Lemma 8: At the steady state where $w_i/p_i = \rho$ for all i,the inertia of B is the same as $P^{-1}B^{-1}W$, and the inertia of $\hat{B}'$ is the same as $W\left[\hat{B}'\right]^{-1} P^{-1}$.

Proof. : We first note that

$$\left|P^{-1}B^{-1}W\right| = \prod_{t=1}^{n} \left(w_t/p_t\right)\left|B^{-1}\right| \tag{49}$$

Since $w_t = rp_t$ at the steady state for $t = 1, ...n$.

$$\left|P^{-1}B^{-1}W\right| = r^n \left|B^{-1}\right| \tag{50}$$

Furthermore, every principle minor of $P^{-1}B^{-1}W$ of order i will be given by the corresponding principle minor of B^{-1} multiplied by r^i. If the characteristic equation of B^{-1} is $f(\lambda) = (-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + ... + b_{11}(-\lambda) + b_0 = 0$, the coefficients b_{n-i} will be the sum of principal minors of order i. Therefore, the characteristic polynomial of $P^{-1}B^{-1}W$ will have coefficients $r^i b_{n-i}$. If the characteristic equation of $P^{-1}B^{-1}W$ is given by $g(\nu) = 0$, then

$$r^{-n}g(\nu) = r^{-n}(-\nu)^n + r^{1-n}b_{n-1}(-\nu)^{n-1} + ... + r^{-1}b_{11}(-\nu) + b_0 = f(\upsilon/r)$$

Therefore if λ is a root of B^{-1}, then λ/r is a root of $P^{-1}B^{-1}W$ and the inertia of B and B^{-1} is the same as that of $P^{-1}B^{-1}W$. The proof that the inertia of $\hat{B}'$ is the same as $W\left[\hat{B}'\right]^{-1}P^{-1}$ is identical.

QED

4.0.6 Proofs of Propositions and Corollaries

Proof of Proposition 1:

Proof. : The roots of J are the roots of its diagonal sub-matrices. Since at the steady state $(w_i/p_i) = r$, using the relationships derived in the appendix we can derive

$$\begin{aligned} \left[\frac{\partial y(x^*, p^*)}{\partial x}\right] - gI &= P^{-1}B^{-1}W - \left(\frac{g}{r}\right)P^{-1}W \\ &= P^{-1}\left[B^{-1} - \left(\frac{g}{r}\right)I\right]W \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} -\left[\frac{\partial w(x^*, p^*)}{\partial p}\right] + rI &= -W\left[\hat{B}'\right]^{-1}P^{-1} + WP^{-1} \\ &= W\left[-\left[\hat{B}'\right]^{-1} + I\right]P^{-1} \end{aligned}$$

By the same argument as in Lemma 8 in the appendix, the roots of $\left[B^{-1} - \left(\frac{g}{r}\right)I\right]$ have the same sign pattern as $P^{-1}\left[B^{-1} - \left(\frac{g}{r}\right)I\right]W$, and the roots of $\left[-\hat{B}^{-1} + I\right]$

have the same sign pattern as $W\left[-\left[\hat{B}'\right]^{-1}+I\right]P^{-1}$. Thus the sign pattern of the roots of J is the same as the roots of $\left[B^{-1}-\left(\frac{g}{r}\right)I\right]$ and $\left[-\left[\hat{B}\right]^{-1}+I\right]$. Therefore the total number of roots of J that have negative real parts is greater than n.This completes the proof.

QED

Proof of Proposition 2:

Let

$$B_m=(1-m)\left(\beta-(\beta_{00})^{-1}\beta_{0.}\beta_{.0}\right)$$

Note that if λ_B is a root of B, $\lambda_{Bm}=(1-m)\lambda_B$ is a root of $B_m=(1-m)\cdot B$, so that the roots of B and B_m have the same sign pattern. We also define

$$\hat{B}_m=B_m+b=(1-m)\left(\beta-(\beta_{00})^{-1}\beta_{.0}\beta_{0.}\right)+mI$$

Lemmas 6 and 7 in the appendix now immediately imply that for any economy in $\Upsilon_m\left(\Omega\left(\mathcal{B}_0\right)\right)$, we have

$$\left[\frac{\partial y(x^*,p^*)}{\partial x}\right]=P^{-1}B_m^{-1}W$$

and

$$\left[\frac{\partial w(x^*,p^*)}{\partial p}\right]=W\left[\hat{B}'_m\right]^{-1}P^{-1},$$

where $\left[\frac{\partial y(x^*,p^*)}{\partial x}\right]$ and $\left[\frac{\partial w(x^*,p^*)}{\partial p}\right]$ are the submatrices of the Jacobian matrix J that define the linearized dynamics around the steady state. We now have to show that $\left[\hat{B}_m^{-1}-I\right]$ has one less root with a negative real part than $\left[B_m^{-1}-\left(\frac{g}{r}\right)I\right]$: then the indeterminacy of the economy in $\Upsilon_m\left(\Omega\left(\mathcal{B}_0\right)\right)$ follows immediately from Proposition 2.

We first show that the matrix $\hat{B}_m$ has one less root with a negative real part than the matrix B_m. Since

$$\hat{B}_m=B_m+b=(1-m)\,B_m+mI$$

It follows that

$$\lambda_{\hat{B}_m}=(1-m)\,\lambda_B+m$$

where $\lambda_{\hat{B}_m}$ is a real root of $\hat{B}_m$. Now if $\lambda_B < 0$ by assumption, $\lambda_{\hat{B}_m} > 0$ if and only if

$$\frac{-\lambda_B}{1-\lambda_B} < m$$

Therefore if B_m or $[B_m]^{-1}$ have s' roots with negative real parts, $\hat{B}_m$ and $\left[\hat{B}_m\right]^{-1}$ have at most $s'-1$ roots with negative real parts.

Next we show that $\left[\hat{B}_m^{-1} - I\right]$ has at least one less root with a negative real part than $\left[B_m^{-1} - I\right]$. This will be true if $\left(\lambda_{\hat{\beta}m}^{-1} - 1\right) > 0$, or if $\lambda_{\hat{\beta}m} < 1$. Since $\lambda_{\hat{B}_m} = (1-m)\,\lambda_B + m$, we will always have $\lambda_{\hat{\beta}m} < 1$ provided $m \in (0,1)$ and $\lambda_B < 0$, as has been postulated.

Finally we must show that the sign pattern of the roots of $\left[B_m^{-1} - I\right]$ and $\left[B_m^{-1} - \left(\frac{g}{r}\right) I\right]$ are the same under the saddlepoint property of the steady state, given by Assumption 6 above. (Note here that the roots of B_m are equal to the roots of B multiplied by $(1-m)$.) It is clear that if the real part of a root of B_m^{-1} is negative, subtracting 1 or g/r will not alter its sign. Therefore we focus the roots B_m^{-1} whose real parts are positive. If these real parts are larger than unity, again subtracting 1 or g/r will not alter their sign, and again $\left[B_m^{-1} - I\right]$ and $\left[B_m^{-1} - \left(\frac{g}{r}\right) I\right]$ will have roots with the same sign pattern. If there exists a root of B_m^{-1} whose real part is less than unity, but larger than g/r, then $\left[B_m^{-1} - I\right]$ and $\left[B_m^{-1} - \left(\frac{g}{r}\right) I\right]$ will have roots with differing sign structures. If $r\lambda_i^{-1}$ is a root of $P^{-1}B^{-1}W$,then the corresponding root of B_m^{-1} is $\lambda_i^{-1}/(1-m)$, since at the steady state $w_i/p_i = r$ *for all* i. Therefore, we need a condition that will assure that either $\text{Re}\left(\lambda_i^{-1}\right)/(1-m) > 1$, or, that $\text{Re}\left(\lambda_i^{-1}\right)/(1-m) < g/r$ will hold whenever $\text{Re}\left(\lambda_i^{-1}\right)/(1-m) < 1$. The latter is alternatively stated as follows: if $\text{Re}\left(\lambda_i^{-1}\right) < (1-m) < 1$,then $\text{Re}\left(\lambda_i^{-1}\right) < (g/r)(1-m)$. We must show therefore that either $m > 1 - \text{Re}\left(\lambda_i^{-1}\right)$, or that whenever $m < 1 - \text{Re}\left(\lambda_i^{-1}\right)$, we also have $m < 1 - (r/g)\,\text{Re}\left(\lambda_i^{-1}\right)$. It is easily seen by inspection that under Assumption 6 whenever $\text{Re}\left(\lambda_i^{-1}\right) < 1$,we must have

$$1 - (r/g)\,\text{Re}\left(\lambda_i^{-1}\right) > 0.$$

Therefore m satisfies either $m > 1 - \text{Re}\left(\lambda_i^{-1}\right)$ or $m < 1 - (r/g)\,\text{Re}\left(\lambda_i^{-1}\right)$ *for all* i, if it holds for the root with the largest positive real part $\text{Re}\left(\lambda_M^{-1}\right)$ of B that lies in the interval $(0, g/r)$, as is postulated in the proposition.

QED

Proof of Corollary 1:

The proof of the Proposition 2 now has to be modified because $\left(\mathrm{Re}\left(\lambda_{\hat{\beta}m}^{-1}\right)-1\right) > 0$, does not necessarily follow from $\mathrm{Re}\left(\lambda_{\hat{\beta}m}\right) < 1$ if $\lambda_{\hat{\beta}m}$ is not real. Since $\lambda_{\hat{B}_m} = (1-m)\,\lambda_B + m$, $\left(\mathrm{Re}\left(\lambda_{\hat{\beta}m}^{-1}\right)-1\right) > 0$ will hold only if:

$$\mathrm{Re}\,(\lambda_B) + \frac{(Im\,(\lambda_B))^2}{(1-m)\left[(1-m)\,\mathrm{Re}\,(\lambda_B)+m\right]} < 0,$$

which reduces to $\mathrm{Re}\,(\lambda_B) < 0$ if $Im\,(\lambda_B) = 0$.

QED

Proof of Corollary 2:

The matrix $\left[\frac{\partial y(x^*,p^*)}{\partial x}\right] - gI$ now reduces to a scalar and $\left(\frac{\partial y(x^*,p^*)}{\partial x}\right)$ has the same sign as $(\beta_{11}\beta_{00}-\beta_{10}\beta_{01})$, which is negative by the assumption that the capital good is labor intensive from the private perspective. Similarly

$$-\left[\frac{\partial w(x^*,\,p^*)}{\partial p}\right] + r = r\left(1 - \frac{(\beta_{00}+b_{00})}{(\beta_{11}+b_{11})\,(\beta_{00}+b_{00}) - (\beta_{10}+b_{10})\,(\beta_{01}+b_{01})}\right)$$

Noting that $\beta_{00}+b_{00}+\beta_{10}+b_{10} = \beta_{01}+b_{01}+\beta_{11}+b_{11} = 1$, the expression above becomes

$$-\left[\frac{\partial w(x^*,\,p^*)}{\partial p}\right] + r = r\left(\frac{(\beta_{01}+b_{01})}{(\beta_{10}+b_{10})\,(\beta_{01}+b_{01}) - (\beta_{11}+b_{11})\,(\beta_{00}+b_{00})}\right)$$

which is also negative by the assumption that the capital good is capital intensive **from the social perspective.** This completes the proof.

QED

Proof of Proposition 3:

Proof. We will use the following result from the trade literature in our proof :

Result 1: Let A_0^{-1} be a Metzler matrix. Then the real parts of all the eigenvalues of A_0^{-1} are negative if and only if there is an $x \geqq 0$ such that $A_0^{-1}x < 0$. (For a proof see [25] p.393, Theorem 4.D.3)

Since the quantity of factors used must add up to their total, we have:

$$A_0\begin{pmatrix} y_0 \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ x \end{pmatrix} \tag{51}$$

By definition if A_0 is SSS-II, A_0^{-1} is Metzler and has negative diagonals and positive off diagonals. Let

$$A_0 = \begin{pmatrix} a_{00} & a_{0.} \\ a_{.0} & a \end{pmatrix} \tag{52}$$

where $a_{00}$is a scalar. Then

$$A_0^{-1} = \begin{pmatrix} a_{00}^{-1}(I + a_{0.}A^{-1}a_{.0}) & , & -a_{00}^{-1}a_{0.}A^{-1} \\ -A^{-1}a_{.0}a_{00}^{-1} & , & A^{-1} \end{pmatrix} \tag{53}$$

$$where\ A = a - a_{.0}a_{00}^{-1}a_{0.}$$

$$y = (-A^{-1}a_{.0}a_{00}^{-1})x_0 + A^{-1}x \tag{54}$$

At the steady state

$$y = gx \tag{55}$$

Hence

$$(A^{-1} - gI)x = A^{-1}a_{.0}a_{00}^{-1}x_0 < 0 \tag{56}$$

Therefore, from Result 1 above, since $A^{-1} - gI$ is a Metzler, it has roots with negative real parts only.

Next we look at the price-cost relationship, that is,

$$\begin{pmatrix} p_0 \\ p \end{pmatrix} = \widehat{A_0'}\begin{pmatrix} w_0 \\ w \end{pmatrix} \tag{57}$$

If $\widehat{A_0}$ is SSS-I, $\widehat{A_0}^{-1}$ is a Minkowski matrix and has positive diagonals and negative off diagonals. Let

$$\widehat{A_0}' = \begin{pmatrix} \widehat{a}_{00} & \widehat{a}_{0.} \\ \widehat{a}_{.0} & \widehat{a} \end{pmatrix} \tag{58}$$

where a_{00} is a scalar. Then

$$\widehat{(A_0')}^{-1} = \begin{pmatrix} \widehat{a}_{00}^{-1}(I + \widehat{a_{0.}}\widehat{A}^{-1}\widehat{a_{.0}}) & , & -\widehat{a}_{00}^{-1}\widehat{a}_{0.}\widehat{A}^{-1} \\ -\widehat{A}^{-1}\widehat{a}_{.0}\widehat{a}_{00}^{-1} & , & \widehat{A}^{-1} \end{pmatrix} \tag{59}$$

$$where\ \widehat{A} = \widehat{a} - \widehat{a}_{0.}\widehat{a}_{00}^{-1}\widehat{a}_{0.}$$

$$w = (-\widehat{A}^{-1}\widehat{a}_{.0}\widehat{a}_{00}^{-1})p_0 + \widehat{A}^{-1}p \tag{60}$$

At a steady state

$$w = rp \tag{61}$$

$$(rI - \widehat{A}^{-1})p = (-\widehat{A}^{-1}\widehat{a}_{.0}\widehat{a}_{00}^{-1})p_0 < 0 \tag{62}$$

Since $-\widehat{A^{-1}}$ has all positive off-diagonals, $(rI - \widehat{A^{-1}})$ has all negative diagonals and all positive off-diagonals. Therefore, from Result 1 above, $rI - \widehat{A^{-1}}$ has roots with negative real parts only.

Q.E.D.

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