Model in Continuous Time

- Goal: Solve dynamics with arbitrary ICs
- Translate the discrete-time version as directly as possible
- Describe as an optimal stopping problem
  - Sequential
  - Recursive
- Some notation:
  - r is a discount rate
  - Δ is a small time interval
  - $f(t+) = \lim_{\tau \downarrow t} f(\tau)$, $f(t-) = \lim_{\tau \uparrow t} f(\tau)$
  - $W'(t) = \frac{\mathrm{d}W(t)}{\mathrm{d}t}$

Firm's Problem

- As before, firm chooses to search vs. produce
  - If they produce, they get flow value z
  - If they search, they pay $c_s z$ and draw with certainty
- As before, searchers only meet producers in $f(t, z)$
- Define the following to cast as an optimal stopping problem
  - $V(t, z)$: Value of production (i.e continuation)
  - $W(t)$: Value of search (i.e. stopping) net of costs
  - $h(t)$: The threshold such that $z \leq h(t)$ searches. Right-continuous.
  - $S(t)$: "Flow" of searchers at time t

Searching

- Agents will draw conditional on meeting producers:

$$\frac{f(t,z)}{1-F(t,h(t))}$$

- Support evolves with $h(t)$

$$\min \operatorname{support}\{f(t+,z)\} = h(t)$$

- Hence, where $h(t)$ is continuous, $F(t,h(t)) = 0$
  - Agents draw from $f(t,z)$
  - Only a flow of agents search
- Assume $h(t)$ is continuous $\forall t$

Sequential Problem

$$
\begin{aligned}
V(t,z) &= \max_{T\geq 0}\left\{\int_0^T e^{-r\tau} z \mathrm{d}\tau + e^{rT}\left[W(t+T) - cz\right]\right\} \\
&= \max_{T\geq 0}\left\{\tfrac{1-e^{-rT}}{r} z - ce^{-rT} z + e^{-rT} W(t+T)\right\}
\end{aligned}
$$

- Given a $W(t)$
- Agent chooses time until searching, $T(t,z)$
- Costs and value of searching discounted at time T

Recursive Continuation Value

$$V(t,z) = z\Delta + (1 - r\Delta)V(t+\Delta, z)$$

Rearrange and divide by Δ

$$rV(t+\Delta, z) = z + \frac{V(t+\Delta,z) - V(t,z)}{\Delta}$$

Take the limit

$$rV(t,z) = z + \frac{\partial V(t,z)}{\partial t}$$

Optimal Stopping Sufficiency Conditions

$V(t, z)$, $h(t)$, and $W(t)$ must satisfy:

$$rV(t,z) = z + \frac{\partial V(t,z)}{\partial t} \qquad \text{(Bellman)}$$

$$V(t, h(t)) = W(t) - ch(t) \qquad \text{(Value Matching)}$$

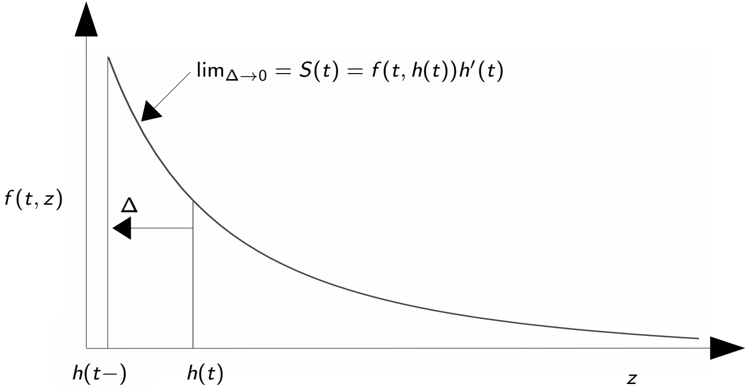
$$\frac{\partial V(t, h(t))}{\partial t} = W'(t) \qquad \text{(Smooth Pasting)}$$

Value of Search

$$W(t) = \int_{h(t)}^{\infty} V(t, \tilde{z}) f(t, \tilde{z}) \mathrm{d}\tilde{z}$$

- Assuming $h(t)$ is continuous,
- Value is the expected continuation value of the new draw

Flow of Searchers



Law of Motion for $f(t, z)$

New draws have pdf $f(t, z)$,

$$\frac{\partial f(t, z)}{\partial t} = S(t)f(t, z)$$

Using the formula for $S(t)$

$$\frac{\partial f(t, z)}{\partial t} = f(t, h(t))h'(t)f(t, z)$$

A solution for any $h(t)$ and $f_0(z)$ initial condition

$$f(t, z) = \frac{f_0(z)}{1 - F_0(h(t))}, \quad \forall z > h(t)$$

Solution Approach

i) Simplify the sequential problem to generate an ODE in $W(t)$

ii) Simplify the $W(t)$ to eliminate $V(t, z)$ and get an integral equation

iii) Solve the system of equations in $W(t)$, $h(t)$

Simplifying the Sequential Problem

$$V(t,z) = \max_{T \geq 0} \left\{ \frac{1-e^{-rT}}{r} z - ce^{-rT} z + e^{-rT} W(t+T) \right\}$$

Taking the FOC for T

$$0 = e^{-rT} \left(z + rcz - rW(t+T) + W'(t+T) \right)$$

Evaluate at indifference point, $h(t)$, where $T = 0$

$$rW(t) = (1+cr)h(t) + W'(t)$$

By definition, at the optimum: $T(t,z) = h^{-1}(z) - t$

$$V(t,z) = \frac{1-e^{-r(h^{-1}(z)-t)}}{r} z - ce^{-r(h^{-1}(z)-t)} z + e^{-r(h^{-1}(z)-t)} W(h^{-1}(z))$$

Simplifying the Value of Search

Substitute $V(t,z)$ and LOM into $W(t)$

$$
\begin{aligned}
W(t) &= \int_{h(t)}^{\infty} V(t,\tilde{z}) f(t,\tilde{z}) \mathrm{d}\tilde{z} \\
&= \int_{h(t)}^{\infty} \left(\tfrac{1}{r} z - \tfrac{1+cr}{r} e^{-r(h^{-1}(z)-t)} z + e^{-r(h^{-1}(z)-t)} W(h^{-1}(z)) \right) \tfrac{f_0(z)}{1-F_0(h(t))} \mathrm{d}z
\end{aligned}
$$

Summary of Equations

$V(t,z)$, $h(t)$, and $W(t)$ must satisfy:

$$rV(t,z) = z + \frac{\partial V(t,z)}{\partial t}$$
$$V(t,h(t)) = W(t) - ch(t)$$
$$\frac{\partial V(t,h(t))}{\partial t} = W'(t)$$

and/or

$$rW(t) = (1+cr)h(t) + W'(t)$$
$$W(t) = \int_{h(t)}^{\infty} \left(\tfrac{1}{r} z - \tfrac{1+cr}{r} e^{-r(h^{-1}(z)-t)} z + e^{-r(h^{-1}(z)-t)} W(h^{-1}(z)) \right) \tfrac{f_0(z)}{1-F_0(h(t))} \mathrm{d}z$$

Last 2 equations are an integral-differential system in $h(t), W(t)$

Balanced Growth Path Guess

Guess and Verify:

- $F_0(z) = 1 - \left(\frac{x_0}{z}\right)^k$, a Pareto distribution
- $W(t) = W_0 e^{gt}$
- $h(t) = x_0 e^{gt}$
  - Note that $h(0)$ is chosen as the minimum of support
- Hence $h^{-1}(z) = \frac{1}{g} \log\left(\frac{z}{x_0}\right)$

Plug into our system of 2 equations and use undetermined coefficients

BGP Solution

Solving the system for g and W_0,

$$g = \frac{1 - cr(k-1)}{ck(k-1)}$$

$$W_0 = x_0 \frac{ck(k-1)(1+cr)}{cr(k^2-1)-1}$$

Substituting into sequential $V(t,z)$

$$V(t,z) = \frac{z}{r} + \frac{g(1+cr)}{r-g}\frac{z}{r}\left(\frac{z}{x_0}\right)^{-r/g} e^{rt}$$

Solution with Arbitrary Initial Conditions

With an arbitrary $F_0(z)$,

- Let $x = h(t)$, a transformation of variables
- Let $f_x(z) = \frac{f_0(z)}{1-F_0(x)}$, truncated of the IC
- Proposition: The growth rate of x in equilibrium is

$$g(x) = \frac{\frac{1}{x}\int z f_x(z)\mathrm{d}z - (1+cr)}{c x f_x(x)}$$

- Proposition: If $\exists\, x$ such that $g(x) = 0$, then growth stops