Interest Rate Policy in Continuous Time with Discrete Delays*

Jess Benhabib
Department of Economics
New York University
New York, NY 10003
Tel. 212 998-8971; Fax 212 995-4824
e-mail: jess.benhabib@nyu.edu

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Abstract

We study the design of monetary policy in a continuous time framework with delays. More explicitly, we consider a linear, flexible price model where inflation and nominal interest rates chage continously, but where nominal rates are set by the Central bank in response to a lagged inflation measure, and where the measure of inflation can be constructed as a flexible distributed delay. Therefore the central bank has, in addition to the choice of an "active" or "passive" response to inflation, two additional parameters to select: the lag of the inflation measure, and the coefficient for the distributed delay to construct the inflation measure. The pure continuous time and discrete time frameworks emerge as special cases of our differential-delay system. This richer framework also allows us to reconcile results on the local uniqueness and multiplicity of equilibria that obtain in the two pure cases, to uncover special assumptions embedded in the pure cases, and to prescribe effective policy options to avoid the problem of local indeterminacy and its unintended consequences.

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1 Introduction

A fundamental tool of monetary policy is the setting and control of the nominal rate of interest for the purpose of stabilizing inflation and employment. Whether the nominal rate should be an instrument of stabilization, and if so how it should be used has been the subject of debate in economics for a long time, and has been extensively studied by Wicksell (1907), Friedman (1960) and others. Recently John Taylor (1993, 1998) has suggested that a simple "active" monetary policy rule in which the reaction of the interest rate to inflation is above a critical threshold shows surprising efficiency and robustness over alternative monetary stabilization policies. Such a rule requires further elaboration, and the specification of its details that may have important consequences for its successful implementation. For example, the fiscal policy accompanying a simple interest rate rule may determine whether other "liquidity trap" equilibria are possible or not. Forward looking rules where the monetary authority sets the interest rate based on a forecast of the future inflation rates may be much more prone to multiple equilibria, with unintended economic consequences, than are backward looking rules where the interest rate is set as a function of past inflation rates. In particular, whether the monetary authority reacts to a weighted average of past inflation rates or not, and how these weights are constructed may be critical to the success of the stabilization policy.

Recently there have been many studies in the literature to evaluate the effects of "Wicksellian" interest rate policies, set either in a discrete-time or a continuous-time framework, with a plethora of results on the multiplicity or uniqueness of equilibria, which at times are difficult to reconcile. (See for example Benhabib, Schmitt-Grohe and Uribe (2001a) and Carlstrom and Fuerst (2000, 2001).) Neither a discrete-time nor a continuous-time framework for describing and evaluating monetary policy may be entirely appropriate however. Inflation may be changing very frequently and possibly continuously with time, while the nominal rates, which also may be adjusted with high frequency, may be responding to delayed measures of inflation. Thus the description of a model of the economy and its monetary policy may involve a complex and long lag structure. For example if inflation changes every day, but the nominal interest rate is set according to an inflation measure lagged even thirty days, we would have a thirtieth order difference equation describing the equilibrium. If inflation changes twice daily, the order of the difference equation would be sixty, and if inflation changes continuously, we would be in the framework of differential-delay equations involving transcendental polynomials with an infinite number of roots. The situation would be further complicated if the nominal rate is set not only as a function of lagged inflation, but as a distributed lag on past inflation rates that starts at some delayed value. Such a structure can introduce not only unanticipated short-term fluctuations, but more perverse instabilities into the dynamics of the economy, as we discuss later.

While these complications can be analyzed within the context of differential-delay equations, they would only amount to technicalities if a simple discrete-time or continuous-time structure provided a good approximation for effective policy advice. However such approximations may be misleading or incomplete, and hide some of the structural assumptions embedded in the pure models. Furthermore the pure cases may cloud the full options available to a central bank, which can decide not only if its interest rate policy is forward or backward looking, but also has the flexibility to pick the lag or lead of its response to inflation rates, as well as the distributed lag structure of past or forecasted inflation rates to which it responds. Considering the full range of options available to he central bank in a general framework can help the design of monetary policy to avoid problems of multiple equilibria, and its unintended consequences on the economy. Our model will provide a simple unified framework in which a richer menu of monetary policy options can be studied.

For simplicity, we will restrict our analysis to a flexible-price economy with simple linear policy rules that are backward looking, in part because extending the analysis to forward looking rules is straightforward, and also because forward-looking rules are known to be prone to multiple equilibria and instabilities in more standard contexts (see Benhabib, Schmitt-Grohe and Uribe (2001a,b, 2002 a,b), or Wooodford(1999)). We will also try to derive the results for the pure discrete and continuous-time cases by taking limits when possible, although the results for the pure cases may not always correspond to the limiting results, precisely because the pure cases may not be good and robust approximations. In the sections that follow we will set up the model, discuss some special cases and relate them to the pure discrete and pure continuous time models. We will save the main Proposition and its Corollary, which characterize the conditions and policy configurations that generate unique and indeterminate equilibria, for the last section.

2 The model

We follow the flexible price model introduced in Benhabib, Schmitt-Grohe and Uribe (2001a). The household's utility function is given by

$$U = \int_0^\infty e^{-rt} u\left(c, m^{np}\right) dt \tag{1}$$

where $r > 0$ denotes the rate of time preference, c consumption, $m^{np} \equiv m^{np}/P$ real balances held for non-production purposes, m^{np} nominal money balances held for non-production purposes, and P the nominal price level. The instant utility function $u(\cdot,\cdot)$ is strictly increasing and strictly concave, and satisfies $u_{cc} - u_{cm}u_c/u_m < 0$ and $u_{mm} - u_{cm}u_m/u_c < 0$ which implies that c and m^{np} are normal goods. Output $y(m^p)$ may be produced with real balances $m^p \equiv m^p/P$ held by firms We assume that $y(m^p)$ is positive, non-decreasing, and concave. This specification allows us to study some canonical special cases. When money is unproductive, we have a simple endowment economy and we can take $y(m^p)$ to be a constant. When money is productive, we can assume that $y(m^p)$ is non-decreasing, and that it satisfies Inada conditions. Money and consumption can be complements ($u_{cm} > 0$) or substitutes ($u_{cm} < 0$) in the utility function, or utility can be separable in money and consumption so that ($u_{cm} = 0$).Each of these cases will have distinct implications for the characterization of equilibrium.

The household can hold nominal bonds, B, which pay the nominal interest rate $R > 0$. Let $\infty a \equiv (m^{np} + m^p + B)/P$ denote the household's real financial wealth, τ the real lump-sum taxes, and $\pi \equiv \dot{P}/P$ the inflation rate. The household's instant budget constraint is:

$$\dot{a} = (R - \pi)a - R(m^{np} + m^p) + y(m^p) - c - \tau. \tag{2}$$

The household chooses sequences for c, m^{np}, $m^p \geq 0$ and a so as to maximize (1) subject to (2) and the following no-Ponzi-game condition

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a(t) \geq 0, \tag{3}$$

taking as given $a(0)$ and the time paths of τ, R, and π. The optimality conditions associated with the household's problem are

$$\begin{aligned}
u_c(c, m^{np}) &= \lambda & (4)\\
m^p\left[y'(m^p) - R\right] &= 0 & (5)\\
\frac{u_H(c, m^{np})}{u_c(c, m^{np})} &= R & (6)\\
\lambda\left(r + \pi - R\right) &= \dot{\lambda} & (7)\\
\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]ds} a(t) &= 0 & (8)
\end{aligned}$$

where λ is the Lagrange multiplier associated with the household's instant budget constraint. When money is productive $R > 0$ implies that m^p is a strictly decreasing function of R:

$$m^p = m^p(R), \tag{9}$$

with $m^{p\prime} \equiv dm^p/dR < 0$. If money is not productive, $R > 0$ implies that $m^p = m^{p\prime} = 0$. Using equation (6) and the normality assumption of money and consumption, m^{np} can be expressed as a function of consumption and the nominal interest rate that is increasing in c and decreasing in R:

$$m^{np} = m^{np}(c, R). \tag{10}$$

3 Government Policies

Whether the government's fiscal policy is Ricardian (as in the case of a balanced budget) or only locally Ricardian (so the government is solvent only on bounded equilibrium trajectories) need not concern us here as we will restrict ourselves to studying local equilibria around steady states (See Benhabib, Schmitt-Grohe and Uribe (2002b)). Monetary policy is Wicksellian, so that nominal interest rates are set as a function of past inflation rates:

$$R = \rho(\pi^p); \qquad \rho' > 0 \tag{11}$$

where π^p is a weighted average of past rates of inflation and is defined as[1]

$$\pi^p = \beta \int_{-\infty}^{t-w} \pi(s) e^{\beta(s-(t-w))} ds; \qquad \beta > 0 \tag{12}$$

Here the central bank sets the nominal rate as a function of an exponentially decreasing weighted average of past inflation rates, but with a time delay of w The constant β is the weighting coefficient, with the limit $\beta = 0$ representing equal weights on all past inflation rates, and $\beta = \infty$ corresponds to the full weight concentrated at time $t-w$, with zero weight attached to inflation rates at all other times. Thus $\beta = \infty$, $w = 1$ would be the standard discrete time setup, as in Leeper (1991), where the Central Bank sets the nominal rate as a function of last period's inflation rate. It is important to note that in this economy while agents make full use of their information to forecast next period's inflation for their consumption-savings decisions, the Central Bank does not forecast, but sets the inflation rate as a fixed function of last period's inflation. This should be viewed as a deliberate policy chosen by design in order to avoid the multiplicity problems that arise under forward-looking rules, rather than a lack of foresight or of informational rigidity. Thus our analysis may also help in the proper design of backward-looking rules to avoid indeterminacies.

A linear specification of rule (11) is given by:

$$R = \alpha \left[\beta \left(\int_{-\infty}^{t-w} \pi(s) e^{\beta(s-(t+w))} ds \right) - \pi^* \right] + R^*$$

where $\alpha > 0$, π^* is the target inflation rate, and $R^* = r + \pi^*$.

4 Equilibria

In equilibrium the goods market must clear:

$$c = y(m^p). \tag{13}$$

Using equations (9)–(11) and (13) to replace m^p, m^{np}, R, and c in equation (4), λ can be expressed as a function of π,

$$\lambda = \lambda(\rho(\pi^p)) \tag{14}$$

with

$$\lambda'(\pi^p) = \rho' \left[u_{cc} y' m^{p\prime} + u_{cm}(m_c^{np} y' m^{p\prime} + m_R^{np}) \right] \tag{15}$$

[1] This is a "backward-looking" policy. Our analysis can easily be extended to "forward-looking" monetary policy where nominal rates are set as function of forecasted inflation rates,and $w < 0$. See also footnote 4.1. Note also that we could specify he central bank to have a limited backward horizon by seeting the lower bound of the integral to $t - w - \omega$, which could be handled in the linear differential-delay framework with relative ease.

where m_c^{np} and m_R^{np} denote the partial derivatives of m^{np} with respect to c and R, respectively. Using this expression, (9)–(11), and (13), equations (7) can be rewritten as

$$\dot{\pi} = \frac{\lambda(\pi^p)}{\lambda'(\pi^p)}\left[r + \pi - \rho(\pi^p)\right] \tag{16}$$

Differentiating (12) with respect to time yields

$$\begin{aligned} \dot{\pi}^p(t) &= \beta(\pi(t-w) - \pi^p(t)) && (17) \\ \dot{\pi}^p(t+w) &= \beta(\pi(t) - \pi^p(t+w)) && (18) \end{aligned}$$

Substituting this into equation (16) we obtain the equilibrium trajectory for π^p, and therefore for π by solving [2]:

$$\begin{aligned} \dot{\pi}^p(t) &= \left(\frac{\lambda(\rho(\pi^p(t)))}{\rho'(\pi^p(t))\lambda'(\rho(\pi^p(t))}\right)\left[r + \pi(t) - \rho(\pi^p(t))\right] && (19) \\ \dot{\pi}^p(t) &= \left(\frac{\lambda(\rho(\pi^p(t))}{\rho'(\pi^p(t))\lambda'(\rho(\pi^p(t))}\right)\left[r + \frac{\dot{\pi}^p(t+w)}{\beta} + \pi^p(t+w) - \rho(\pi^p(t))\right] && (20) \end{aligned}$$

Let

$$H(t) = \left(\frac{\lambda'(\pi^p(t))}{\lambda(\pi^p(t))}\right)$$

and H denote the steady state value. Note that the sign of H is the same as that of λ'. In the case where money enters production only, or when it enters utility with $u_{cm} < 0$, $H > 0$. If money enters utility only and $u_{cm} > 0$ $(u_{cm} = 0)$, then $H < 0$ $(H = 0)$. Linearizing at the steady state, $\pi^p(t) = \pi(t) = \pi^*$, where we define $\rho(\pi^*) = \alpha$, we have:

$$\begin{aligned} d\dot{\pi}^p(t) &= H^{-1}[\beta^{-1}d\dot{\pi}^p(t+w) + d\pi^p(t+w) - \alpha d\pi^p(t)] && (21) \\ 0 &= H^{-1}\beta^{-1}d\dot{\pi}^p(t+w) - d\dot{\pi}^p(t) + H^{-1}\pi^p(t+w) - \alpha H^{-1}d\pi^p(t) \end{aligned}$$

This is a difference-delay equation of the neutral type, with an infinite number of roots. Its characteristic equation is given as

$$0 = \left(H^{-1}\beta^{-1}\tilde{s} + H^{-1}\right)e^{w\tilde{s}} - \tilde{s} - \alpha H^{-1} \tag{22}$$

or

$$0 = \left(H^{-1}\beta^{-1}e^{s} - 1\right)s + H^{-1}we^{s} - \alpha H^{-1}w \tag{23}$$

where $s = \tilde{s}w$.[3]

[2] To solve this differential-delay equation, we need a set of the initial conditions given by $\pi(\tau)$, $\tau = [t_0 - w, t_0]$. We will further discuss this issue a little later.

[3] The characteristic equation is transcendental with a finite number of real roots which solve:

$$f(s) = \left(H^{-1}\beta^{-1}e^{s} - 1\right)s + H^{-1}we^{s} - \alpha H^{-1}w = 0$$

4.1 The case where $H = 0,\ w > 0$ and $\beta \to \infty$

To address the issue of the local uniqueness of the equilibrium trajectory, we will start by considering a very special case which reduces to the model studied by Leeper (1991) in discrete time. Assume that we have an endowment economy where money is unproductive, and where money and consumption are separable in the utility function. This implies that $H = 0$. Furthermore assume that $\beta \to \infty$ so that the whole weight in determining π^p is given to the inflation rate at $t - w$. Then our differential-delay equation

$$Hd\dot{\pi}^p(t) = \beta^{-1}d\dot{\pi}^p(t+w) + d\pi^p(t+w) - \alpha d\pi^p(t)$$

reduces to:

$$0 = d\pi^p(t+w) - \alpha d\pi^p(t)$$

This is a standard difference equation which yields the standard results obtained by Leeper (1991) in a discrete time framework: We have a unique equilibrium corresponding to $\pi^p = \pi^*$ if monetary policy is active $(\alpha > 1)$, but a continuum of equilibria if $\alpha < 1$. In the latter case all trajectories starting in the neighborhood of $\pi^p = \pi^*$ converge to π^*, and satisfy the market clearing conditions as well as the optimization conditions of agents. This result depends on the not unreasonable assumption that if the Central Bank were to credibly commit to an active policy at time t_0, the initial value of $\pi^p(t_0 - w)$ will not pin down the time t_0 nominal interest rate. The public and the Central Bank are forward looking at time t_0; they realize that any trajectory not starting at π^* would diverge, so the central bank sets the nominal rate at R^* over the interval $[t_0, t_0 + w]$, as if $\pi(t)$ had been at its steady state value π^* over the interval in the past, not unlike the prescription of a "timeless perspective" discussed by Woodford and Gianonni (2001) (see also Woodford (1999)).[4] From then onwards, a backward-looking active monetary policy sets

$$R(t) = \alpha[\pi(t-w) - \pi^*] + R^*,$$

which yields the locally unique equilibrium trajectory $\pi(t) = \pi^*$ for an active policy $(\alpha > 1)$. It follows that to coincide with the standard results on uniqueness versus multiplicity obtained by Leeper (1991) in discrete time, we must

This equation also has an infinite number of complex roots, $s = a \pm bi$, which solve

$$\left(H^{-1}\beta^{-1}e^a e^{\pm bi} - 1\right)(a \pm bi,) + H^{-1}we^a e^{\pm bi} - \alpha H^{-1}w = 0$$

If we use the transformation $e^{\pm bi} = \cos b \pm i \sin b$ we have:

$$\begin{aligned} & \left(H^{-1}\beta^{-1}e^a(\cos b \pm i\sin b) - 1\right)(a \pm bi,) \\ = \quad & -H^{-1}we^a(\cos b \pm i\sin b) + \alpha H^{-1}w \end{aligned}$$

We can get explicit equations that define a and b by equating real and imaginary parts, but these involve trigonometric functions and are therefore also transcendental, with an infinite number of solutions.

[4]If $w = 1$, inflation at time the initial time t_0 is $\frac{p(t_0)}{p(t_{0-1})}$, so we may assume that $p(t_0)$ adjusts, and $\pi^p(t_0)$ is free. However if $w > 1$, this reasonong fails, so we rely on the timeless perspective.

explicitly assume that the initial conditions at the time corresponding to the implementation of a new policy are viewed by the monetary authority from a "timeless perspective," as if the economy had been at its steady state $\pi(t) = \pi^*$ over the period of the time delay, $[t_0 - w, t_0]$.[5], [6].

The special case $\beta \to \infty$, which focuses the weights of the inflation measure at the single date $t - w$, is of course unrealistic: given the possibility of measurement error or occasional outliers, the Central Bank will not want to rely on the realization of inflation at a single date in the past to set the nominal rate, but will try to smooth its inflation measure by some averaging over past rates. We therefore proceed with the analysis of equilibria by analyzing the solutions of (21) for $\beta < \infty$. We will rely on some basic results from Bellman and Cooke (1963) (see chapter 5 and p. 159, 176, and 190-191). A necessary and sufficient condition for all the continuous solutions of (21) to approach zero as $t \to \infty$ is for the least upper bound of the real parts of its characteristic roots to be negative[7]. If we let time run backwards, then a necessary and sufficient condition for all the continuous solutions of (21) to approach zero as $t \to -\infty$ is for the lower bound of the real parts of its characteristic roots to be positive. Then in the forward direction for $t \to \infty$, none of the solutions can converge to zero, implying local uniqueness. Note that this implies that a necessary condition for local uniqueness is the positivity of the lower bound of the real parts of the characteristic roots of (21).

4.2 The case where $w = 0$.

This very special case of "distributed delay" corresponds to the one studied Benhabib, Schmitt-Grohe and Uribe (2001a), (2002). The characteristic equation (22) reduces to:

$$0 = \left(H^{-1}\beta^{-1}\tilde{s} + H^{-1}\right) - \tilde{s} - \alpha H^{-1} \tag{24}$$

$$\tilde{s} = \frac{H^{-1}(1-\alpha)}{1-\beta^{-1}H^{-1}} = \frac{\beta(1-\alpha)}{\beta H - 1}; \tag{25}$$

Thus, the system has only a single root $\tilde{s}$ due to the special exponential distributed delay structure with $w = 0$. As in Benhabib, Schmitt-Grohe and Uribe (2001a), if $H > 0$ (money enters only into production or $u_{cm} < 0$), an active policy ($\alpha > 1$) implies local indeterminacy ($\tilde{s} < 0$) and a passive policy ($\alpha < 1$) implies local uniqueness ($\tilde{s} > 0$), unless the the policy rule is sufficiently backward looking ($\beta H < 1$) ,in which case the results are reversed. If $H < 0$, an active policy implies local uniqueness, and a passive policy implies indeterminacy.

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[5] See however Corollary 1 in Section 5.

[6] Note that if we have a forward-looking rule instead of a backward-looking one, which would result in a differential-delay equation of he advanced type, past initial conditions become irrelevant for setting the nominal rate.

[7] This requires the initial conditions $\pi(t)$ over $t \in [t_0 - w, t_0]$ be of class C^1, which is the case under our postulate of the "timeless perspective."

4.3 The case where $w > 0$, $H = 0$, $0 < \beta < \infty$

This case corresponds to an endowment economy with preferences separable in consumption and money, and with a positive delay w as in Leeper's (1991) discrete time formulation, but where the nominal rate is set in response to an exponentially distributed delay of inflation rates starting at $t - w$. The weights are no longer concentrated on the single lagged date $t - w$, although they may be skewed towards $t - w$ if β is large. The differential-delay equation (21) describing equilibrium trajectories is now of the retarded type:

$$0 = r + \frac{\dot{\pi}^p (t + w)}{\beta} + \pi^p (t + w) - \alpha (\pi^p(t) - \pi^*) - R^* \tag{26}$$

Its characteristic equation can be written as

$$0 = \left(\beta^{-1}\tilde{s}w + w\right) e^{\tilde{s}w} - \alpha w$$

or

$$s = -\beta w + \beta\alpha w e^{-s}.$$

where $s = \tilde{s}w$. Its real roots are solutions to $f(s) = s + \beta w - \beta\alpha w e^{-s} = 0$. Note that $f(0) = w(1-\alpha)$, $f(\infty) = \infty$, $f(-\infty) = -\infty$ and $f'(s) = 1 + \beta\alpha w e^{-s} > 0$. It immediately follows that there is a single real root, which is positive if monetary policy is active ($\alpha > 1$) and negative if monetary policy is passive ($\alpha < 1$). However, we would be wrong to conclude that we have local uniqueness under active policies because now there are also an infinite number of complex roots. For this particular "retarded" case, we can get a handle on the local dynamics by studying "asymptotic roots" as $\mid \tilde{s} \mid \to \infty$. We are looking to characterize complex solutions of

$$0 = \beta^{-1} w\tilde{s} + w - \alpha w e^{-\tilde{s}w} = \tilde{s}\left(1 + \frac{\beta}{\tilde{s}}\right) - \alpha\beta e^{-\tilde{s}w}$$

for $\mid \tilde{s} \mid$ large, which can be approximated by the zeros of $\tilde{s} - \alpha\beta e^{-\tilde{s}w} = 0$ (see Bellman and Cooke, p.99). This yields :

$$\mathrm{Re}(\tilde{s}) = -\log w^{-1} \mid \tilde{s} \mid + \log w^{-1} \mid -\beta\alpha \mid$$

So asymptotically as $\mid \tilde{s} \mid \to \infty$, for finite and positive β and w, we have $\mathrm{Re}(\tilde{s}) \to -\infty$ which implies that (26) has complex roots with negative real parts, unless of course $\beta = \infty$ to start with.[8] This result, where the nominal rate is set as a

[8] Using a theorem by Burger (1956) we can also provide a complete set of conditions for all the roots of (26) to have negative real parts. The conditions for all s to have negative real parts, applied to our case, reduces to $\alpha < 1$. They are slightly more complex for the general case.

function of a distributed delay over inflation rates starting at $t-w$, is in sharp contrast to the earlier result with $\beta=\infty$, because it implies indeterminacy: there are initial conditions $\pi(t)$ defined over $t\in[t_0-w,t_0]$ (other than the constant steady state $\pi(t)=\pi^*$) that converge to the steady state. Only when we collapse the system to the model of Leeper (1991) by focusing weights onto a single date do we obtain local uniqueness. We will see below that a necessary condition for uniqueness in the general case with $w>0$ and $H\neq 0$ will be $\mid \beta H \mid > 1$, which cannot be satisfied in this case if $H=0$ and β is finite.

4.4 The discretized model and the discrete-time case

To explore the relation of the standard pure discrete-time case to the continuous-time case, we may discretize (21) (ignoring constant terms) as

$$d\pi_{t+w+1}-(1-\beta)\,d\pi_{t+w}-H\beta d\pi_{t+1}+(H-\alpha)\,\pi_t, \tag{27}$$

with characteristic equation (28):

$$\lambda^{w+1}-(1-\beta)\,\lambda^{w}-H\beta\lambda+(H-\alpha)\,\beta=0 \tag{28}$$

When $w=1$ and $H=0$, (27) is of second order with real roots, both of which are outside the unit circle only if $\alpha>1$ and $\beta(1+\alpha)>2$: a sufficiently active monetary policy, $\left(\alpha>Max\left(2\beta^{-1}-1,1\right)\right)$, yields a unique equilibrium, while a passive one ($\alpha<1$) does not.[9] However, to properly approximate continuous time, we must allow inflation to change very frequently relative to the delay w. For example if inflation changes with a unit lag, say daily, and w is the lag on inflation used to set the nominal rate, say a month, then $w=30$. If inflation changes twice daily, then $w=60$, and of course for continuous time we have the limit, $w=\infty$. For equation (28), roots within the unit circle, and therefore indeterminacy, can arise for even small values of w for active monetary policies with $\alpha>1$. For example, simple computations show that for $w\geq 7$, $H=1.4$, $\alpha=1.5$, $\beta=.85$, the number of complex root within the unit circle increase with w.

We can also formulate the discrete-time version of our model. For simplicity, let $H=0$. The Euler equation for this discrete-time case is

$$U'(c_t)=U'(c_{t+1})\,\frac{(1+r)^{-1}R\,(S_{t-w})}{\pi_{t+1}} \tag{29}$$

where

$$S_t=(1-b)\left(\pi_t+b\pi_{t-1}+b^2\pi_{t-2}.....\right),\quad b\in(0,1) \tag{30}$$

Note that when $w=b=0$ and we have an endowment economy with $c_t=\bar{c}$, the model collapses to the simple case considered by Leeper (1991). Differencing (30) we have $S_t-bS_{t-1}=(1-b)\,\pi_t$ For the endowment economy with $H=$

[9] If $\beta=1$, $w>1$, and $H=\alpha$, on the the hand, one root is always zero, and the others are the $w'th$ roots of α.

0, and a linear Taylor rule, $R = \alpha(S_{t-w} - \pi^*) + R^*$, equation (29) becomes (ignoring constant terms)[10]:

$$S_{t+w+1} - bS_{t+w} - \alpha(1-b)(1+r)^{-1} S_t = 0, \tag{32}$$

For $\alpha = 1.5$, $(1+r)^{-1} = .96$, $b = .9$ and $w = 10$, the modulus of the roots of (32) range from 1.1934 to 0.977, with only one real root which is the largest one in modulus. If we set $b = .1$, the roots range from 1.0476 to 1.0292, with the only real roots 1.0476 and -1.0276. These observations suggest that local indeterminacy easily sets in if $w > 0$, but setting b closer to zero, implying a backward-looking distributed lag structure, may help avoid local indeterminacy problems. However, formal results that characterize the linear dynamics when w is large seem easier to obtain in a differential-delay structure, so we return to our analysis in a continuous-time framework.

4.5 The General Case with $H \neq 0,\ w > 0,\ 0 < \beta < \infty$

In this case, the characteristic equation (22) corresponding to the equilibrium inflation rates given by equation (21) can be written as:

$$0 = \left(H^{-1}\beta^{-1}\right)\tilde{s}e^{\tilde{s}w}\left(1 - \frac{\beta}{\tilde{s}}\right) - \tilde{s}\left(1 + \frac{\alpha H^{-1}}{\tilde{s}}\right).$$

The asymptotic roots of this equation as $\mid \tilde{s} \mid \rightarrow \infty$ are approximated by the roots of

$$0 = \tilde{s}\left(\left(H^{-1}\beta^{-1}\right)e^{\tilde{s}w} - 1\right)$$

(excluding $\tilde{s} = 0$), whose real parts are given by

$$\mathrm{Re}(\tilde{s}) = w^{-1}\log \mid \beta H \mid$$

(Bellman and Cooke, p.417 and p. 153.)[11] To rule out roots with negative real parts and avoid indeterminacy, it is necessary that $\mid \beta H \mid > 1$, which fails if $H = 0$, that is if money enters only the utility function with $u_{cm} = 0$. Even if $\mid H \mid$ is large however, giving too much weight to distant inflation rates (β small) will generate indeterminacy.

To rule out multiple equilibria, we must also rule out negative real roots. We must therefore characterize the real roots of

$$f(s) = \left(H^{-1}\beta^{-1}e^{s} - 1\right)s + H^{-1}we^{s} - \alpha H^{-1}w.$$

[10] As in the discretized continuous-time case considered above, the roots of (**??**) are now bounded by

$$\mid \lambda \mid \leq \left\{\mid \alpha(1-b)^{-1}(1+r)^{-1} \mid w\right\}^{\frac{1}{w+1}}. \tag{31}$$

[11] Of course the approximation is invalid if $w = 0$.

We have:

$$
\begin{aligned}
f(0) &= H^{-1}w(1-\alpha) \\
f'(s) &= \left(H^{-1}\beta^{-1}s + H^{-1}w\right)e^{s} + H^{-1}\beta^{-1}e^{s} - 1 \\
f''(s) &= \left(H^{-1}\beta^{-1}s + H^{-1}w\right)e^{s} + 2H^{-1}\beta^{-1}e^{s}
\end{aligned}
$$

Consider first the case where $H < 0$. Then $\lim_{s\to\infty} f(s) = -\infty$, $\lim_{s\to-\infty} f(s) = \infty$ and $f'(s) < 0$. It follows that if monetary policy is active, $f(0) = H^{-1}(1-\alpha) > 0$, so that we have one positive real root. If monetary policy is passive on the other hand, $f(0) = H^{-1}(1-\alpha) < 0$, so that we have one negative real root. For the case where $H > 0$, we have $\lim_{s\to\infty} f(s) = \infty$ and $\lim_{s\to-\infty} f(s) = \infty$. If monetary policy is active, $f(0) < 0$ and we have at least one positive and one negative real root, while if monetary policy is passive, $f(0) > 0$,and $f'(0) \lesseqgtr 0$ depending on parameters, so that we may have two zero real roots, two positive real roots or two negative real roots.

5 Main Results

In a simple continuous time framework with delays, the monetary authority has three parameters, α, β and w at its discretion for setting a simple Wicksellian rule specifying how the nominal interest rate should be set as a function of a measure of past inflation rates. The following Proposition summarizes conditions on policy parameters α, β, and w that result in local indeterminacy:

Proposition 1 *: The steady state equilibrium π^* for (21), with $w > 0$ and $H \neq 0$, will be locally indeterminate if either i) $H(1-\alpha) < 0$, or ii) $\mid \beta H \mid < 1$, or both. When $H = 0$ (money only enters utility, and is separable from consumption), the condition $\mid \beta H \mid < 1$ is always satisfied and the equilibrium is always indeterminate unless $\beta \to \infty$, in which case (21) collapses to a difference equation and the steady state π^* is indeterminate if $\alpha < 1$ and locally unique if $\alpha > 1$. If $w = 0$, the equilibrium is determinate (indeterminate) if $(1-\alpha)(\beta H - 1) > 0$ (< 0).*

As noted by El'sgol'ts and Norkin (1973, p.142), neutral differential-delay systems like (21) can discontinously switch stability as the time delay $w \to 0$. For example, if $H(1-\alpha) > 0$ but $\mid \beta H \mid < 1$, the equilibrium steady state π^* is indeterminate for all $w > 0$, no matter how small w is, but if $\alpha > 1$, it is locally unique when $w = 0$.[12] However, setting $w = 0$ may not be feasible due to informational lags, and therefore determinacy results obtained for $w = 0$ can be misleading. In this flexible price environment, setting β large so that distant

[12] Another revealing example, given by El'sgol'ts and Norkin (1973, p.142) is the equation

$$\dot{x}(t) + ax(t) - b\dot{x}(t-w) - abx(t-w) = 0$$

with $a > 0$, $b > 1$. It has characteristic equation $(s+a)\left(1 - be^{-ws}\right) = 0$, and roots with real parts $\mathrm{Re}(s) = w^{-1}\log(b)$, and therefore the stationary point $x = 0$ is unstable. When $w = 0$ however, it has as its solution $x(t) = x(0)e^{-at}$, which converges to zero.

inflation rates do not receive a large weight in the construction of an inflation measure to which the monetary authority responds is a good strategy to avoid indeterminacy, except in the case where the Central Bank can set $w = 0$ and $\alpha > 1$. For any $w > 0$, setting the parameter α to avoid indeterminacy requires a knowledge of the sign of H, which is determined by the degree to which money plays a role in the production process by providing cost-saving liquidity to firms. The fact that a large fraction of $M1$ is held by firms suggests that they are indeed willing to forgo some interest and pay for liquidity in order cut their costs of transaction.

Since the non-asymptotic roots of (21), for $\tilde{s}$ small, cannot in general be signed[13], we can only give necessary conditions for local uniqueness, which we summarize in the Corollary below.

Corollary 1 *: For the equilibrium steady state π^* to be locally unique when $w > 0$ and $H \neq 0$, it is necessary that i) $H(1-\alpha) > 0$, and ii) $\mid \beta H \mid > 1$. When $H = 0$ (money only enters utility, and is separable from consumption), the condition $\mid \beta H \mid > 1$ cannot be satisfied, so for local uniqueness it is necessary that $\alpha > 1$ and either $\beta \rightarrow \infty$ or $w = 0$.*

So far the analysis has assumed that the central bank could initially set the nominal rate from a timeless perspective, under the assumption that it would keep the nominal rate at R^* over, $(t_0, t_0 + w)$, implementng its Wickesellian rule as if the inflation rate had been at its steady state level π^* in the past. The alternative assumption is for the central bank to take the path of past inflation as given in setting the nominal rate. This amounts to taking initial conditions as given, which means that existence of a local equilibrium converging to the steady state requires the local stability of the steady state. In the usual model with $w = 0$, this assumption would require a passive monetary policy, $\alpha < 1$, for the existence of a local equilibrium. When $w > 0$, this is no longer sufficient, since local stability or the negativity of the asymptotic roots also requires $\mid \beta H \mid < 1$: that is unless $H = 0$, the Wicksellian rule must give sufficient weight to past inflation rates. The following corollary states the result.

Corollary 2 *If the central bank sets the nominal interest rate according to a Wicksellian rule, and use rates of past inflation at the time it first implements the rule (rather than applying the timeless perspective, as if the inflation rate had always been at its steady state level), then for a local equilibrium to converge to the steady state for any set of initial conditions, it is necessary that i) $\alpha < 1$ (monetary policy be passive), and, ii) $\mid \beta H \mid < 1$.*

Finally, the above analysis is only local in nature and may be severely misleading. Local analysis is effective in identifying problems of local indeterminacy, but cannot rule out global indeterminacy in cases where local uniqueness

[13] The asymptotic approximation is reasonably accurate for small or moderate values of $\tilde{s}$, so that there may only be a few non-asymptotic roots that are hard to sign. See also Wright (1960, 1961) for a characterization of the roots and stability critera for neutral equations that can be very useful.

holds. Benhabib, Schmitt-Grohe and Uribe (2002c) have shown, in the context of a continuous time model with distributed lags (with $w = 0$) and sticky prices, that under active monetary policy ($\alpha > 1$) with $H > 0$, local uniqueness holds with β sufficiently small, but that a continuum of equilibrium trajectories converge to a limit cycle. Similar cycles are likely to bifurcate in the current nonlinear model with $w > 0$ and $H \neq 0$ since the stability, or the dimension of the stable manifold of the steady state, changes as we vary parameters.

6 Conclusion

Since Taylor's (1993) seminal paper a large literature has focused on the problem of avoiding mutliple equilibria by the proper design of monetary policy. While active vs. passive policies ($\alpha \gtrless 1$), forward or backward looking rules, and the precise modelling of the role of money in the economy ($H \gtrless 0$) have been shown to affect the results on indeterminacy, some of the results seem to depend on whether the analysis is conducted in discrete or continuous time. We have tried to reconcile the differences in results by providing a model that can nest discrete and continuous time analysis. We allow monetary policy to react to a measure of past inflation rates (weighted by β) with a lag w which can differ from the frequency at which inflation changes: in fact except for the illustrative section 4.4, we allow inflation to change continuously. With this additional flexibility that is provided by the parameters β and w, we can span the results obtained in continuous and discrete time, so that particular parametrizations will reproduce the results known to hold in continuous and discrete time. Proposition 1 and its Corollaries show how the weighting parameter β and the lag w will interact with the monetary policy parameter α and the parameter H whose sign identifies the particular role of money, to determine whether the equilibrium is unique or indeterminate. We learn in particular that if interest rates are set so as to respond to a weighted measure of past inflation rates with weights that do not decay quickly (small β), and if monetary policy has a lagged response ($w > 0$), problems of multiple equilibria and indeterminacy are more likely to emerge in the standard contexts.

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