General Dynamic Programming Problem:

$$V(x) = \underset{c}{Max} \ \frac{M}{\alpha}(c+\eta)^{\alpha} + \beta V(a(x-bc-d)) \quad a > 1, \ \alpha \in (-\infty, 1], \ \eta \geq 0$$

FOC

$$M(c+\eta)^{\alpha-1} = \beta ab V'(a(x-bc-d))$$

$$V'(x) = \beta a V'(a(x-bc-d)) = \frac{M}{b}(c+\eta)^{\alpha-1}$$

$$V'(a(x-bc-d)) = \frac{M}{b}(c(\bullet)+\eta)^{\alpha-1}$$

$$M(c+\eta)^{\alpha-1} = \beta a M(c(\bullet)+\eta)^{\alpha-1}$$

Let: $c = \lambda x + \mu$. Then FOC:

$$(\lambda x + \mu + \eta)^{\alpha-1} = \beta a[\lambda a x(1-\lambda b) + \mu(1-\lambda ba) - \lambda ad + \eta]^{\alpha-1}$$

$$(\lambda x + \mu + \eta) = (\beta a)^{\frac{1}{\alpha-1}}[\lambda a x(1-\lambda b) + \mu(1-\lambda ba) - \lambda ad + \eta]$$

Equating coeff for x:

$$1 = \beta^{\frac{1}{\alpha-1}} a^{\frac{\alpha}{\alpha-1}}(1-\lambda b)$$

$$Let \quad K = (\beta a^{\alpha})^{\frac{1}{1-\alpha}} \ : \ then \quad \lambda = \frac{1-K}{b}$$

Equate constant terms:

$$\mu + \eta = (\beta a)^{\frac{1}{\alpha-1}}[(1-\lambda ba)\mu - \lambda ad + \eta]$$

$$\mu\left[1-(a\beta)^{\frac{1}{\alpha-1}}(1-ab\lambda)\right] = \left((a\beta)^{\frac{1}{\alpha-1}} - 1\right)\eta - (a\beta)^{\frac{1}{\alpha-1}}\lambda ad$$

$$\mu = \frac{(a\beta)^{\frac{1}{\alpha-1}} - 1}{1-(a\beta)^{\frac{1}{\alpha-1}}(1-ab\lambda)}\eta - \frac{\lambda ad(a\beta)^{\frac{1}{\alpha-1}}}{1-(a\beta)^{\frac{1}{\alpha-1}}(1-ab\lambda)}$$

$$\mu = \frac{(1-Ka)\eta}{a-1} - \frac{(1-K)ad}{(a-1)b} \qquad \left(follows\ from\ b\lambda = 1-K\right)$$

Therefore:

$$c = \lambda x + \mu$$

$$\lambda = \frac{1-K}{b} = \frac{1-a^{\frac{\alpha}{1-\alpha}}\beta^{\frac{1}{1-\alpha}}}{b}$$

$$\mu = \frac{(1-Ka)\eta}{a-1} - \frac{\lambda ad}{a-1}$$

Value Function: $\alpha^{-1}MA(x+aQ)^{\alpha}$

$$\alpha^{-1}MA(x+aQ)^{\alpha} = \alpha^{-1}M(c+\eta)^{\alpha} + \beta\alpha^{-1}MA(a(x-bc-d)+aQ)^{\alpha}$$

Solve for A, Q. From: $V'(x) = \beta a V'(a(x-bc-d)) = \frac{M}{b}(c+\eta)^{\alpha-1}$

$$c+\eta = (A\beta ab)^{\frac{1}{\alpha-1}} a(x-bc-d+Q)$$

$$c+\eta = (Ab)^{\frac{1}{\alpha-1}} K^{-1}(x-bc-d+Q)$$

$$c\left(1+(Ab^{\alpha})^{\frac{1}{\alpha-1}}K^{-1}\right) = \left((Ab)^{\frac{1}{\alpha-1}}K^{-1}\right)x + \left((Ab)^{\frac{1}{\alpha-1}}K^{-1}\right)(Q-d) - \eta$$

$$c = \frac{\left((Ab)^{\frac{1}{\alpha-1}}K^{-1}\right)}{1+\left((Ab)^{\frac{1}{\alpha-1}}K^{-1}\right)}x + \frac{\left((Ab)^{\frac{1}{\alpha-1}}K^{-1}\right)(Q-d)-\eta}{1+\left((Ab)^{\frac{1}{\alpha-1}}K^{-1}\right)}$$

$$= \lambda x + \mu$$

Equating and solving, first for λ, and then for μ :

$$A = \left(\frac{\lambda K}{1-\lambda}\right)^{\alpha-1} b^{-1}$$

$$Q = \frac{\mu}{\lambda} + \frac{\eta(1-\lambda)}{\lambda} + d$$

$$V(x) = \frac{M}{\alpha} b^{-1} \left(\frac{(1-K)K}{b+K-1}\right)^{\alpha-1} \left[x + \left(\frac{b}{1-K}\right)\mu + a\left(\frac{b+K-1}{1-K}\right)\eta + ad\right]^{\alpha}$$

$$U(c) = \alpha^{-1} M (c+\eta)^{\alpha}$$

$$c = \lambda x + \mu \qquad \lambda = (1-K) b^{-1}$$

$$K = a^{\frac{\alpha}{1-\alpha}} \beta^{\frac{1}{1-\alpha}} \quad \mu = \frac{(1-Ka)}{a-1}\eta - \frac{\lambda a d}{a-1}$$

Special case of $\eta = d = 0.$ $(Solved\ for\ any\ \lambda)$

Note that this implies that $\mu = 0.$

Value Function: $V(x) = \alpha^{-1} M A x^{\alpha}$

$$\alpha^{-1} M A x^{\alpha} = \alpha^{-1} M c^{\alpha} + \beta \alpha^{-1} M A (a(x - bc))^{\alpha}$$

$$A x^{\alpha} = (\lambda x)^{\alpha} + \beta A (a x (1 - b\lambda))^{\alpha}$$

$$A = \lambda^{\alpha} + \beta A a^{\alpha} (1 - b\lambda)^{\alpha}$$

$$A = \frac{\lambda^{\alpha}}{(1 - \beta a^{\alpha} (1 - b\lambda)^{\alpha})}$$

$$V(x) = \alpha^{-1} M \left(\frac{\lambda^{\alpha}}{(1 - \beta a^{\alpha}(1 - b\lambda)^{\alpha})}\right) x^{\alpha}$$

Note that maximizing A wrt λ yields:

$$\begin{aligned}
\alpha\lambda^{\alpha-1}(1-\beta a^{\alpha}(1-b\lambda)^{\alpha})^{-1} &= \lambda^{\alpha}(1-\beta a^{\alpha}(1-b\lambda)^{\alpha})^{-2}\alpha\beta a^{\alpha}b(1-b\lambda)^{\alpha-1} \\
(1-\beta a^{\alpha}(1-b\lambda)^{\alpha}) &= \lambda b\beta a^{\alpha}(1-b\lambda)^{\alpha-1} \\
1 &= \beta a^{\alpha}(1-b\lambda)^{\alpha-1}(\lambda b+1-b\lambda) \\
1-\lambda b &= (\beta a^{\alpha})^{\frac{1}{\alpha-1}} \\
\lambda &= \left(1-(\beta a^{\alpha})^{\frac{1}{\alpha-1}}\right)b^{-1}
\end{aligned}$$

Cooperative solution with two agents, and equal consumptions

General case

$$V(x) = \frac{M}{\alpha}b^{-1}\left(\frac{(1-K)K}{b+K-1}\right)^{\alpha-1}\left[x+\left(\frac{ab}{1-K}\right)\mu+a\left(\frac{b+K-1}{1-K}\right)\eta+ad\right]^{\alpha}$$

$$V(x) = \underset{c}{Max}\ \alpha^{-1}M(c+\eta)^{\alpha}+\beta V(a(x-2c))$$

Note, $d = 0, b = 2$:

$$V(x) = \frac{\alpha^{-1}M}{2}\left(\frac{(1-K)K}{1+K}\right)^{\alpha-1}\left[x+\left(\frac{2a}{1-K}\right)\mu+a\left(\frac{1+K}{1-K}\right)\eta\right]^{\alpha}$$

$$\lambda = \frac{(1-K)}{2} = \frac{1-a^{\frac{\alpha}{1-\alpha}}\beta^{\frac{1}{1-\alpha}}}{2} \qquad \mu = \frac{(1-Ka)}{a-1}\eta$$

If in addition $\eta = 0$,

$$V(x) = \frac{1}{2}\alpha^{-1}M\left(\frac{(1-K)K}{1+K}\right)^{\alpha-1}x^{\alpha}$$

For the logarithmic utility case $(\alpha = \eta = 0)$:

$$\lambda = \frac{(1-\beta)}{2} \qquad \mu = 0$$

MNE-2 Players

$$V(x) = \underset{c}{Max} \quad \alpha^{-1}M(c+\eta)^{\alpha} + \beta V(\hat{a}(x - \lambda_2 x - \mu_2 - c))$$

$$V(x) = \underset{c}{Max} \quad \alpha^{-1}M(c+\eta)^{\alpha} + \beta V\left(\hat{a}(1-\lambda_2)\left(x - \frac{c}{1-\lambda_2} - \frac{\mu_2}{1-\lambda_2}\right)\right)$$

So let $a = \hat{a}(1-\lambda_2)$, $d = \frac{\mu_2}{1-\lambda_2}$, $b = \frac{1}{1-\lambda_2}$, $\boldsymbol{\kappa} = (\hat{a}(1-\lambda_2))^{\frac{\alpha}{1-\alpha}} \boldsymbol{\beta}^{\frac{1}{1-\alpha}}$

General case

$$V(x) = \frac{M}{\alpha} b^{-1}\left(\frac{(1-\kappa)\kappa}{b+\kappa-1}\right)^{\alpha-1}\left[x + \left(\frac{ab}{1-\kappa}\right)\mu + a\left(\frac{b+\kappa-1}{1-\kappa}\right)\eta + ad\right]^{\alpha}$$

$$V(x) = \frac{M}{\alpha}(1-\lambda_2)\left(\frac{(1-\kappa)\kappa}{(1-\lambda_2)^{-1}+\kappa-1}\right)^{\alpha-1}$$
$$\left(x + \left(\frac{a\mu}{(1-\kappa)(1-\lambda_2)}\right) + a\left(\frac{(1-\lambda_2)^{-1}+\kappa-1}{1-\kappa}\right)\eta + \frac{a\mu_2}{1-\lambda_2}\right)^{\alpha}$$

SYMMETRIC CASE: Solve for $\lambda_1 = \lambda_2 = \lambda$:

$$\lambda_1 = (1-\kappa)b^{-1} = \left(1 - (\beta(\hat{a}(1-\lambda_2))^{\alpha})^{\frac{1}{1-\alpha}}\right)(1-\lambda_2)$$

$$\lambda_1 = 1 - \lambda_2 - (\beta(\hat{a}^{\alpha}(1-\lambda_2)))^{\frac{1}{1-\alpha}}$$

$$\lambda = \frac{1 - (\beta\hat{a}^{\alpha}(1-\lambda))^{\frac{1}{1-\alpha}}}{2}$$

$$\mu = \frac{(1-\kappa a)\eta}{a-1} - \frac{(1-\kappa)a\mu_2}{(a-1)}$$

$$\mu = \frac{(1-\kappa a)\eta}{(a-1)+(1-\kappa)a}$$

If $\eta = 0$, $\mu = 0$ is a solution. Value function in this case:

$$V(x) = \alpha^{-1}M(1-\lambda_2)\left(\frac{(1-\kappa)\kappa}{(1-\lambda_2)^{-1}+\kappa-1}\right)^{\alpha-1}x^{\alpha}$$

THE LOGARITHMIC CASE: $\alpha = 0,\ \eta = 0$:

$$\lambda_1 = (1-\beta)(1-\lambda_2)$$

$$\frac{\lambda}{1-\lambda} = 1-\beta$$

$$\lambda = \frac{1-\beta}{2-\beta}$$

Defection with reversion to extreme threats

$$V_{DE}(x) = \underset{\tilde{c}}{Max}\ \frac{M}{\alpha}(\tilde{c}+\eta)^{\alpha} + \beta\frac{M}{\alpha}((.5)a(x-\lambda x-\mu-\tilde{c})+\eta)^{\alpha} + K$$

FOC:

$$(\tilde{c}+\eta)^{\alpha-1} = \beta(.5)a((.5)a(x-\lambda x-\mu-\tilde{c})+\eta)^{\alpha-1}$$

$$(\tilde{c}+\eta) = (\beta(.5)a)^{\frac{1}{\alpha-1}}((.5)a(x-\lambda x-\mu-\tilde{c})+\eta)$$

$$\tilde{c} = \left(1-(\beta(.5)a)^{\frac{1}{\alpha-1}}(.5)a\right)^{-1}(\beta(.5)a)^{\frac{1}{\alpha-1}}((.5)a(x(1-\lambda)-\mu)+\eta)-\eta$$

$$\tilde{c} = \lambda_D x + \mu_D$$

Value fn:

$$V_{DE}(x) = \frac{M}{\alpha}(\lambda_D x+\mu_D+\eta)^{\alpha} + \beta\frac{M}{\alpha}(.5a(x(1-\lambda)-\mu-\lambda_D x+\mu_D)+\eta)^{\alpha} + K$$

Simplify by assuming $\eta = 0$, $\mu = 0$. Then $\mu_D = 0$ as well.

$$V_{DE}(x) = \frac{M}{\alpha}(\lambda_D x)^\alpha + \beta\frac{M}{\alpha}((.5)a(x(1-\lambda)-\lambda_D x))^\alpha$$
$$= \frac{M}{\alpha}(A_D)x^\alpha$$

Defection with reversion to MNE

Take case with $\eta, \mu = 0$

$$V(x) = \alpha^{-1}M(1-\lambda_2)\left(\frac{(1-\kappa)\kappa}{(1-\lambda_2)^{-1}+\kappa-1}\right)^{\alpha-1} x^\alpha$$

$$V_{DM}(x) = \underset{\hat{c}}{Max} \quad \frac{M}{\alpha}\hat{c}^\alpha$$
$$+\beta\frac{M}{\alpha}\left(1-\hat{\lambda}\right)\left(\frac{(1-\kappa)\kappa}{\left(1-\hat{\lambda}\right)^{-1}+\kappa-1}\right)^{\alpha-1} (a(x(1-\lambda)-\hat{c}))^\alpha$$

FOC:

$$\hat{c}^{\alpha-1} = \alpha\beta\left(1-\hat{\lambda}\right)\left(\frac{(1-\kappa)\kappa}{\left(1-\hat{\lambda}\right)^{-1}+\kappa-1}\right)^{\alpha-1} (a(x(1-\lambda)-\hat{c}))^{\alpha-1}a$$
$$\hat{c} = \lambda_{MD}x$$

$$V_{DM}(x) = \frac{M}{\alpha}\lambda_{MD}x^\alpha$$
$$+\beta\frac{M}{\alpha}\left(1-\hat{\lambda}\right)\left(\frac{(1-\kappa)\kappa}{\left(1-\hat{\lambda}\right)^{-1}+\kappa-1}\right)^{\alpha-1} (a(x(1-\lambda)-\lambda_{MD}x))^\alpha$$
$$= \frac{M}{\alpha}A_{MD}x^\alpha$$

Equivalence between constant term in utility or in production

$$Max \sum_{t=0}^{\infty} \beta^t U(c)$$

$$\begin{aligned}
ST \quad k_{t+1} &= ak_t + b - c_t \\
Let \ w &= k + \frac{b}{a-1} \\
w_{t+1} &= ak_t + b + \frac{b}{a-1} - c_t \\
w_{t+1} &= ak_t + b\left(\frac{a}{a-1}\right) - c_t \\
w_{t+1} &= aw_t - c_t
\end{aligned}$$

$$Max \sum_{t=0}^{\infty} \beta^t \alpha^{-1} (c+b)^{\alpha}$$

$$\begin{aligned}
ST \quad k_{t+1} &= ak_t - c_t \\
Let \quad \bar{c} &= c - b
\end{aligned}$$

$$Max \sum_{t=0}^{\infty} \beta^t \alpha^{-1} (\bar{c})^{\alpha}$$

$$\begin{aligned}
ST \quad k_{t+1} &= ak_t - \bar{c}_t + b \\
w_{t+1} &= aw_t - \bar{c}_t \\
where \ w &= k + \frac{b}{a-1}
\end{aligned}$$

Union-Firm

Profits:

$$\underset{l_t \in [0,\bar{l}]}{Max} \sum_{t=0}^{\infty} al_t - w_t l_t - b(l_t - l_{t-1})^2 \qquad \#$$

Wage bill:

$$\underset{w_t \in [0,\infty)}{Max} \sum_{t=0}^{\infty} \beta^t w_t l_t \qquad \#$$

Stationary Strategies:

Union: $W:[0,\bar{l}] \to [0,\infty) : \; w_t = W(l_{t-1})$
Firm: $L : [0,\infty) \times [0,\bar{l}] \to [0,\bar{l}] : \; l_t = L(w_t, l_{t-1})$

(W, L) is an (sp) equilibrium if W maximizes (2) given (1) and L maximizes (1) given (2).

FOC:

$$a - w_t - 2b(l_t - l_{t-1}) = -\beta\left[2b(l_{t+1} - l_t) - l_{t+1}\frac{dW(l_t)}{dl_t}\right]$$

$$l_t + w_t \frac{dL(w_t, l_{t-1})}{dw_t} = \beta l_{t+1} \frac{dL(w_t, l_{t-1})}{dw_t} \cdot \frac{dL(w_{t+1}, l_t)}{dl_t} \div \frac{dL(w_{t+1}, l_t)}{dw_{t+1}}$$

Derivation:

$$V(l_{t-1}) = \underset{w_t}{Max} \quad w_t L(w_t, l_{t-1}) + \beta V(l_t(w_t, l_{t-1}))$$

$$0 = l_t + w_t \frac{dL(w_t, l_{t-1})}{dw_t} + \beta V'(l_t) \frac{dL(w_t, l_{t-1})}{dw_t}$$

$$\beta V'(l_t) = -\frac{l_t}{\frac{dL(w_t, l_{t-1})}{dw_t}} - w_t$$

$$V'(l_{t-1}) = w_t \frac{dL(w_t, l_{t-1})}{dl_{t-1}} + \beta V' \bullet \frac{dL(w_t, l_{t-1})}{dl_{t-1}}$$

$$= w_t \frac{dL(w_t, l_{t-1})}{dl_{t-1}} - \left(\frac{l_t}{\frac{dL(w_t, l_{t-1})}{dw_t}} + w_t \right) \frac{dL(w_t, l_{t-1})}{dl_{t-1}}$$

$$V'(l_t) = -\left(\frac{\frac{dL(w_{t+1}, l_t)}{dl_t}}{\frac{dL(w_{t+1}, l_t)}{dw_{t+1}}} \right) l_{t+1}$$

$$0 = l_t + w_t \frac{dL(w_t, l_{t-1})}{dw_t} - \beta \left(\frac{\frac{dL(w_{t+1}, l_t)}{dl_t}}{\frac{dL(w_{t+1}, l_t)}{dw_{t+1}}} \right) \frac{dL(w_t, l_{t-1})}{dw_t}$$

Conjecture:

$$l_{t+1} = k_0 + k_1 l_t + k_2 w_{t+1}$$
$$w_t = c_1 + c_2 l_t$$

Equilibrium solutions must hold for all (l_t, w_{t+1}). Use undetermined coefficients method in FOC to solve for $(k_0, k_1, k_2, c_{1,} c_2)$.

Value Functions

$$F^s(l_0) = b_0 + b_1 l_1 + b_2 l_0^2 = \underset{l_0 \in [0, \bar{l}]}{Max} \quad a l_1 + W(l_0) l_1 + b(l_1 - l_0) + \beta F^s(l_1)$$

$$U^s(l_0) = a_0 + a_1 l_1 + a_2 l_2 = \underset{w_1 \in [0, \infty)}{Max} \quad w_1 L(w_1, l_0) + \beta U^s(L(w_1, l_0))$$

Solve for $(a_0, a_1, a_2, b_0, b_1, b_2)$.

Dynamics:

$$l_{t+1} = k_0 + k_1 l_t + k_2(c_1 + c_2 l_t)$$
$$= k_0 + k_2 c_1 + (k_{1+} k_2 c_2) l_t$$

with steady state $\tilde{l} = \frac{k_0 + k_2 c_1}{1 - k_1 - k_2 c_2}$.

Cooperation:

Maximize equally weighted sum (say add profits plus wage bill, so wage bill cancels: otherwise all surplus goes either to firm or union):

$$S(l_0) = \underset{l_0 \in [0,\bar{l}]}{Max} \quad al - b(l_1 - l_0)^2 + \beta S(l_1) \qquad \#$$

Define wage giving all surplus to Union as $\bar{w}$:

$$\bar{w}\hat{l}_1 = \underset{l_0 \in [0,\bar{l}]}{Max} \quad al - b(l_1 - l_0)^2 + \beta S(l_1)$$

where $\hat{l}_1$ is the value of l that maximizes $al - b(l_1 - l_0)^2 + \beta S(l_1)$. Optimal solution for (3):

$$l_t = l_{t-1} + z \quad if\ l_{t-1} + z \le \bar{l}, \qquad z = \frac{a}{2b(1-\beta)}$$
$$= \bar{l} \qquad otherwise$$

We will denote value functions of a cooperative equilibrium as $(F^C(l_0, \hat{w}), U^C(l_0, \hat{w}))$.

History Dependent Strategies

$$H_t = \{(l_0, \ldots l_{t-1}; w_1, \ldots w_t) | l_i \in [0, \bar{l}], w_j \in [0, \infty); i = 0, \ldots . t-1; j = 1, \ldots t\}$$

Let $h_t \in H_t$.

Firm Strategy: $L : H_t \to : [0, \bar{l}]$:

$$L_t(h_t) = l_{t-1} + z \qquad if\ w_i = \hat{w}\ for\ all\ i \le t$$
$$= k_0 + k_1 l_{t-1} + k_2 w_t \quad otherwise$$

Union Strategy: $W_t : \{H_{t-1}, l_{t-1}\} \to [0, \infty)$

$$W(\{h_{t-1}, l_{t-1}\}) = \hat{w} \qquad if\ w_i = \hat{w}\ for\ all\ i \leq t-1$$
$$= c_1 + c_2 l_{t-1} \qquad otherwise$$

Deviation:

$$F^D(l_{t-1}) = \underset{l_t \in [0,\bar{l}]}{Max} \quad al_t - w_t l_t - b(l_t - l_{t-1})^2 + \beta F^S(l_t)$$
$$U^D(l_{t-1}) = U^S(l_{t-1})$$

For Cooperation to be enforcable to be enforcable,

$$F^C(l_t, \hat{w}) \geq F^D(l_{t-1}), U^C(l_{t-1}, \hat{w}) \geq U^S(l_{t-1})$$

The second always holds.

Switching Strategies

Suppose there exists an $l^* < \bar{l}$ and an agreed upon $\hat{w}$ such that $F^C(l_t, \hat{w}) \geq F^D(l_{t-1})\ for\ l \in [l^*, \bar{l}]$

Switching strategy:

Firm: When $l_{t-1} \in [l^*, \bar{l}]$ and $w_i = \hat{w}$ for $i \leq t$,

$$l_t = l_{t-1} + z \quad if\ l_{t-1} + z \leq \bar{l}, \qquad z = \frac{a}{2b(1-\beta)}$$
$$= \bar{l} \qquad otherwise$$

When $l_{t-1} \notin [l^*, \bar{l}]$

$$l_t = k_0 + k_1 l_{t-1} + k_2 w_t$$

Union: When $l_{t-1} \in [l^*, \bar{l}]$ and $w_i = \hat{w}$ for $i \leq t$,

$$W(h_t) = \hat{w}$$

When $l_{t-1} \notin [l^*, \bar{l}]$

$$W(h_t) = c_1 + c_2 l_{t-1}$$

If $\tilde{l} \geq l^*$ and $\tilde{l}$ is stable under stationary strategies, the above is an equilibrium and converges to the cooperative solution and $l_t \rightarrow \bar{l}$. Note this is not necessarily the best subgame perfect equilibrium. Why?

Also note, if $\tilde{l} < l^*$ and $\tilde{l}$ is stable under stationary strategies we get trapped in a low level equilibrium, and always remain with stationary strategies.