Indeterminacy and Sunspots in Macroeconomics

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1 Introduction

Modern macroeconomics is based on dynamic general equilibrium theory and for some time it has been known that, unlike static general equilibrium theory, in dynamic general equilibrium economies equilibria may be indeterminate.[1] Indeterminacy means that there may be an infinite number of equilibria, all very close to each other, and the existence of indeterminacy in a dynamic model has, in the past, been considered to be a weakness of a theory that should be avoided by careful modeling assumptions. In contrast, a recent literature has grown up in macroeconomics that exploits the existence of an indeterminate set of equilibria as a means of understanding macroeconomic data. This chapter surveys this literature and places it in the context of other recent developments in macroeconomics.

The literature on quantitative aspects of indeterminacy can be organized around three strands. First, there is work that uses models with indeterminate equilibria to explain the propagation mechanism of the business cycle. Second there is a group of papers that uses indeterminacy to explain the monetary transmission mechanism, specifically the fact that prices are sticky and third, there is work in growth theory that uses indeterminacy to understand why the per capita incomes of countries that are similar in their fundamentals nevertheless save and grow at different rates. In this survey we explain the ideas that underlie each of these strands; we discuss the mechanisms that lead to indeterminacy and we report on the current state of quantitative models of the business cycle. We pay particular attention to areas in which models with indeterminate equilibria might offer a significant improvement over a more conventional approach.

A closely related concept to that of indeterminacy is the idea of a sunspot equilibrium, an idea developed by Cass and Shell [1983], [1977], to refer to equilibrium allocations influenced by purely extrinsic belief shocks in general equilibrium models.[2] A sunspot equilibrium is one in which agents receive different allocations across states with identical fundamentals; that is, preferences, endowments and technology are the same but consumption and or production differs. Sunspot equilibria can often be constructed by randomizing over multiple equilibria of a general equilibrium model and models with indeterminacy are excellent candidates for the existence of sunspot equilibria since there are many equilibria over which to randomize. Sunspots cannot occur in finite general equilibrium models with complete markets since their existence would violate the first welfare theorem; risk averse agents will generally prefer an allocation that does not fluctuate to one that does. Examples of departures from the Arrow-Debreu structure that permit the existence of sunspots include 1) incomplete participation in insurance markets as in the overlapping generations model, 2) incomplete markets due to transactions costs or asymmetric information, 3) increasing returns to scale in the technology, 4) market imperfections associated with fixed costs, entry costs or external effects and 5) the use of money as a medium of exchange.

We have drawn attention to three strands of literature; business cycles, monetary transmission and economic growth. The literature that uses indeterminacy and sunspots to understand business cycles is more fully developed than the work on growth theory and models have been developed of both business cycles and the monetary transmission mechanism that provide quantitative explanations of economic data. These models exploit two ideas; first that indeterminacy may provide a rich source of propagation dynamics to an equilibrium model and second that sunspots may provide an alternative impulse to technology or taste shocks. In monetary models the dynamics of indeterminate equilibria have been exploited to explain how a purely nominal shock may have real effects in the short run without invoking artificial barriers to price adjustment.

In addition to its contribution to the theory of economic fluctuations, models of indeterminacy have been used in the literature on economic growth to explain different and sometimes divergent growth rates of countries and regions that start out with similar endowments and wealth levels. Despite the explosion of research in modern growth theory, many important questions remain unsettled and results are frequently not robust to alternative empirical specifications (see Levine and Renelt [1992]). The growth literature on indeterminacy highlights the possibility that economic fundamentals alone will not pin down the savings rates for different countries since countries with identical endowments and wealth levels may coordinate on different equilibrium savings rates that may be determined by cultural, social or historical considerations.

2 Why Should we Care?

The initial work on indeterminacy in general equilibrium models was often abstract and far removed from issues of economic policy. Part of our goal in this survey is to dispel the misconception that indeterminacy is an esoteric area that is unconnected with the core of macroeconomics. We will show that, if one accepts dynamic general equilibrium theory as an organizing principle, the possibility of indeterminacy is part of the package. Furthermore, indeterminate equilibria can illuminate a number of issues that are otherwise puzzles. Two issues that we will discuss in this section are 1) the role of beliefs in business fluctuations, and 2) the monetary transmission mechanism. In our concluding comments at the end of the chapter we will draw attention to some unanswered questions associated with the research agenda. These include the question of co-ordination of beliefs on a specific equilibrium and the way that an equilibrium is maintained.

2.1 Technical Aspects of Linear Models

In deterministic models of dynamic economies, indeterminacy implies the existence of many equilibrium paths for quantities and prices that can be indexed by specifying initial conditions for prices. In stationary stochastic contexts, the effect of initial conditions on the evolution of the economic variables fade away. Nevertheless the indeterminacy of equilibrium in these environments allows the introduction of exogenous shocks that are not based on fundamentals. Such shocks would be inconsistent with equilibrium if the rational expectations equilibrium were unique. However, this is no longer the case in the presence of indeterminacy. As long as the sunspot shocks follow a stochastic process that is consistent with the expectations of agents, equilibrium conditions can be satisfied, and sunspots will affect the evolution of real economic variables. Since the stochastic process for sunspots can typically be chosen from a wide class, there are many possible stationary rational expectations equilibria. The particular equilibrium that prevails depends upon the beliefs that agents use to forecast future values of prices and it is in this sense that sunspots "select" a stochastic equilibrium.

We begin by discussing some technical aspects of linear stochastic models in order to illustrate the content of indeterminacy for the applied econometrician. Our discussion centers on solution methods for linear models and illustrates the implications of indeterminacy for the methods that are used to formulate, simulate and estimate these models.[3] Later in the chapter we will discuss the class of behavioral models that give rise to approximate linear models. These behavioral models are typically derived from an infinite horizon maximizing problem solved by a representative agent, although there is no reason to maintain the representative agent assumption and similar linear models follow from a much larger class of dynamic general equilibrium models.[4]

We start with the assumption that we have already solved for the equilibrium of a dynamic model and that the non-stochastic version of this model contains a balanced growth path. Linearizing around this balanced growth path leads to a system of equations of the form:

$$y_t = Ay_{t-1} + BE_t[y_{t+1}] + Cx_t + u_t, \tag{2.1}$$

$$x_t = Dx_{t-1} + v_t, \tag{2.2}$$

where y is a vector of endogenous variables, x is a vector of policy variables, u and v are conformable vectors of stochastic shocks and A, B, C, and D are matrices of parameters that are found by taking first order Taylor series approximations to the functions that describe a non-stochastic version of the model around its balanced growth path. These equations consist of market clearing conditions, Euler equations and static first order conditions and a set of transversality conditions that impose boundedness conditions on the elements of y_t. We assume that policy is stationary, that is, the roots of D are all within the unit circle.

To find the rational expectations solution to this model, one can rewrite equations (2.1) and (2.2) as follows:

$$\tilde{A}z_t = \tilde{B}z_{t+1} + \tilde{C}e_{t+1}, \tag{2.3}$$

where

$$z_t \equiv \begin{bmatrix} y_t \\ y_{t-1} \\ x_{t-1} \end{bmatrix}, \quad \tilde{A} \equiv \begin{bmatrix} I & -A & 0 \\ I & 0 & 0 \\ 0 & 0 & D \end{bmatrix}, \quad \tilde{B} \equiv \begin{bmatrix} B & 0 & C \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{2.4}$$

$$e_{t+1} \equiv \begin{bmatrix} u_t \\ v_t \\ y_{t+1} - E_t[y_{t+1}] \end{bmatrix}, \quad \tilde{C} \equiv \begin{bmatrix} I & 0 & -B \\ 0 & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \tag{2.5}$$

Premultiplying by $\tilde{A}^{-1}$,using the notation $\Phi \equiv \tilde{A}^{-1}\tilde{B}$, $\Gamma \equiv \tilde{A}^{-1}\tilde{C}$, leads to equation (2.6):

$$z_t = \Phi z_{t+1} + \Gamma e_{t+1}. \tag{2.6}$$

Generally, one can invert equation (2.6) and write z_{t+1} as a function of

z_t, but since some of the roots of the matrix Φ lie outside the unit circle, this procedure does not typically allow one to construct the stochastic process for z_t that constitutes the rational expectations equilibrium. The problem is that arbitrary solutions to (2.6) fail to remain bounded and they violate the transversality conditions of one or more of the agents in the underlying equilibrium model. In other words, the system

$$z_{t+1} = \Phi^{-1} z_t - \Phi^{-1} \Gamma e_{t+1}, \tag{2.7}$$

is explosive. In determinate rational expectations models one can eliminate the effect of explosive roots of Φ^{-1} by placing restrictions on z_t.

Suppose we partition the vector z_t into two disjoint sets z_t^1, of dimension n_1, and z_t^2, of dimension n_2, where z_t^1 contains those variables that are predetermined at date t and z_t^2 those variables that are free to be chosen by the equilibrium conditions of the model. Let λ be the roots of Φ and partition λ into two sets λ^1 of dimension m_1 and λ^2, of dimension m_2, where λ^1 consists of the roots of Φ are outside the unit circle and λ^2, those that are inside the unit circle.[5] The condition for a determinate solution is that one has exactly as many non predetermined variables as non explosive roots of Φ, in other words, $n_2 = m_2$.

The solution to the determinate model is found by diagonalizing the matrix Φ and writing equation (2.7) as a system of scalar equations in the (possibly complex) variables $\tilde{z}_t$ where the $\tilde{z}$ are linear combinations of the z_t formed from the rows of the inverse matrix of eigenvectors of Φ. The elements of $\tilde{z}$ associated with stable roots of Φ are set to zero; in the case when $m_2 = n_2$ these elements provide exactly enough linear restrictions to exactly determine the behavior of the z_t. In the case when there are fewer stable roots of Φ than non-predetermined initial conditions there is the possibility of multiple indeterminate equilibria. We illustrate the importance of this issue for economics with two examples.

2.2 Indeterminacy and Propagation Mechanisms in Real Models of Business Cycles

The idea behind models of indeterminacy as a source of propagation of shocks can be best understood within the context of the Cass-Koopmans stochastic growth model. The equilibrium of this model can be described as the solution to a system of difference equations. If we allow for a productivity shock that enters the model as a disturbance to the production function this system has three state variables, consumption, capital and productivity:

$$\begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} = \Phi \begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} + \Gamma \begin{bmatrix} \tilde{e}_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix}. \tag{2.8}$$

The variables c and k are consumption and capital and s is productivity. The tilde's denote deviations from the balanced growth path, $\tilde{u}$ is the innovation to the productivity shock and $\tilde{e}$, defined as

$$\tilde{e}_{t+1} = \tilde{c}_{t+1} - E_t\left[\tilde{c}_{t+1}\right], \tag{2.9}$$

is the one step ahead forecast error of consumption. In the linearized model, there are two other endogenous variables, output and labor, that are described as a linear function of the "state variables" c, k and s.

Using the definitions from the previous section, k and s are predetermined and c is non-predetermined. When the model is derived from a maximizing model with a constant returns-to-scale technology one can show that the matrix Φ has two unstable roots (outside the unit circle) and one stable root (inside the unit circle). Since there is one non-predetermined variable, c, and one stable root of Φ one has a unique equilibrium; this equilibrium is found by eliminating the influence of the stable root of Φ. Since the roots of Φ are the inverses of the roots of Φ^{-1} this procedure eliminates the influence of the *unstable* root of Φ^{-1} in the equation:

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} = \Phi^{-1} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} - \Phi^{-1}\Gamma \begin{bmatrix} \tilde{e}_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix}, \tag{2.10}$$

by making $\tilde{c}$ a function of $\tilde{k}$ and $\tilde{s}$. In the special case when there are no shocks to productivity, $\tilde{s}$ is identically zero and the steady state of the bivariate system in $\tilde{c}$ and $\tilde{k}$ is a saddle point. In this case the unique solution makes $\tilde{c}$ a function of $\tilde{k}$ that places the system on the stable branch of the saddle. In the stochastic case $\tilde{c}$ depends not only on $\tilde{k}$ but also on $\tilde{s}$.

In the stochastic model, the expectational error $\tilde{e}_{t+1}$ is a function only of the innovation to the productivity shock $\tilde{u}_{t+1}$ and there is thus no independent role for errors in beliefs to influence outcomes. This was the original point of the rational expectations revolution; it is possible to show that, if there is a unique rational expectations equilibrium, expectations must be a unique function of *fundamentals*. When there is a unique equilibrium, it can be described as the solution to a second order stochastic difference equation of the form:

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} = \tilde{\Phi}^{-1} \begin{bmatrix} \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} + \tilde{\Gamma}\tilde{u}_{t+1} \tag{2.11}$$

with a set of side conditions,

$$\begin{bmatrix} \tilde{c}_t \\ \tilde{y}_t \\ \tilde{l}_t \end{bmatrix} = \tilde{C} \begin{bmatrix} \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} \tag{2.12}$$

that determine the values of the other variables of the system. The matrices $\tilde{\Phi}$ and $\tilde{\Gamma}$ are functions of the elements of Φ and Γ found by replacing $\tilde{c}$ with a function of $\tilde{k}$ and $\tilde{s}$. $\tilde{y}$ and $\tilde{l}$ represent output and employment and the elements of $\tilde{C}$ are found from the linearized first order conditions of the original equilibrium model.

The main idea of the recent indeterminacy literature is that small departures from the assumptions of the real business cycle model lead to big departures from its implications. Farmer and Guo [1994] take a variant of the Cass-Koopmans stochastic growth model, originally studied by Benhabib and Farmer [1994], in which the technology displays increasing returns.[6] Their model has a representation of the same form as equation (2.8) but, in contrast to the standard version of this model, in the Farmer-Guo version all the roots of the matrix Φ lie outside of the unit circle. It follows that all roots of Φ^{-1} are *inside* the unit circle and hence equation (2.10) describes a stationary process for an arbitrary series of iid shocks $\tilde{e}_t$. In the standard model the forecast errors $\tilde{e}_t$ are a function of the fundamental errors $\tilde{u}_t$. In contrast, in the increasing returns model, the forecast errors enter as independent shocks to the business cycle.

There are two important implications of models of indeterminacy for business cycle dynamics. The first is that animal spirits can represent an independent impulse to the business cycle even in a model in which all agents have rational expectations. The second is that models with indeterminacy display much richer propagation dynamics than models with a unique determinate equilibrium. In the model with increasing returns, for example, the matrix Φ may generate dynamics that lead to hump shaped responses to shocks, *wherever these shocks originate*. In the standard model, the dynamics of consumption, capital and GDP are dictated mainly by the assumed form of the dynamics of the shocks.[7]

2.3 Indeterminacy and Propagation Mechanisms in Monetary Models of Business Cycles

We have described how models with indeterminate equilibria can potentially be used to understand business cycles. A second area in which indeterminacy may prove important is in models of the monetary transmission mechanism. Once again, it is the propagation mechanism inherent in models with indeterminacy that sets these models apart. Early papers on the issue of indeterminacy and monetary propagation were set in the context of the two period overlapping generations model; papers in this literature include Geanakopolis and Polemarchakis [1986], Azariadis and Cooper [1985], Farmer and Woodford [1998],Farmer [1991], [1992] and Chiappori and Guesnerie [1993]. Later work has switched focus and more recently there has been an attempt to include money in infinite horizon models of money. Papers that exploit the existence of indeterminate monetary equilibria in an infinite horizon framework include Woodford [1986], [1988], Beaudry and Devereux [1993], Bennett [1997], Lee [1993], Matheny [1992], [1998], Matsuyama [1991b], Benhabib and Farmer [1996b], and Farmer [1997]. One of the key ideas in this literature is that indeterminacy can be used to understand the monetary transmission mechanism.

The equilibria of a simple monetary economy can often be characterized by a functional equation of the form:

$$m_t = E_t \left[G \left(m_{t+1}, \mu_{t+1}, u_t \right) \right] \tag{2.13}$$

where m_t represents the real value of monetary balances, μ_{t+1} is the ratio of the money supply at date $t+1$ to the money supply at date t, u_t is a fundamental shock with known probability distribution, E_t is the expectations operator and G is a known function. There exist examples of more complicated monetary rational expectations models that include additional state variables and allow for endogenous capital formation. These more complicated models rely on the same key insight as simpler models with a single state variable: in models with indeterminacy, prices may be predetermined one period in advance and yet all markets may clear at all dates and agents may have rational expectations.

Assuming that equation (2.13) has a steady state we may linearize around it to generate an equation that must be (approximately) satisfied by any sequence of real balances in a rational expectations equilibrium:

$$\tilde{m}_t = \alpha E_t \left[\tilde{m}_{t+1}\right] + \beta\mu_{t+1} + \gamma u_t \tag{2.14}$$

The variables $\tilde{m}$ and $\tilde{\mu}$ represent deviations from the non-stochastic steady state. In a standard monetary model the parameter α is between zero and one in absolute value. In standard models one solves equation (2.14) by iterating forwards to find the current value of real balances as a function of the rule governing the evolution of μ_t. In models with indeterminacy on the other hand, α may be greater than one and in this case there exist many solutions to this model of the form:

$$\tilde{m}_{t+1} = \frac{1}{\alpha}\tilde{m}_t - \frac{\beta}{\alpha}\mu_{t+1} - \frac{\gamma}{\alpha}u_t + \tilde{e}_{t+1} \tag{2.15}$$

where $\tilde{e}_{t+1}$ is an arbitrary iid sunspot sequence. Farmer and Woodford [1998] showed that one of these equilibria has the property that the price at date $t+1$ is known at date t; in this sense prices are "sticky" even though there is no artificial barrier to prevent them from adjusting each period.

The existence of a predetermined price equilibrium is significant because it offers the possibility of using equilibrium theory to understand one of the most difficult puzzles in monetary economics; the characteristics of the monetary transmission mechanism. In his classic essay "Of Money", published in the eighteenth century, David Hume described the empirical facts that at that time were known to characterize the aftermath of what today we would call an "unanticipated monetary injection". Following an addition of money to an economy we typically observe (1) a short run increase in real economic activity, (2) a fall in short rates of interest, and (3) an increase in the real value of money balances. In the short run, the price level does not respond. Over a longer period of time the interest rate increases to its initial level, real economic activity contracts and the full impact of the monetary increase is felt in prices. David Hume's observations were based on the effect of the importation of gold to Europe in the aftermath of the discovery of the "New World" but his observations have proven remarkably consistent with modern econometric evidence based on the analysis of vector autoregressions that allow us to construct estimates of impulse response functions.[8]

The sequence of events described in the previous paragraph constitutes a description of what we believe to be a consensus view of the facts concerning the "monetary transmission mechanism". But although these facts have been known for two hundred years we still have to reach a consensus theory that can account for them. The current leading contenders as explanations of the monetary transmission mechanism are some version of the "menu costs" model due to Akerlof and Yellen [1985] and Mankiw [1985], the contract approach of Taylor [1980], the staggered price setting model of Calvo [1983] or the closely related cost of adjustment models of Rotemberg [1982], [1996]. Each of these approaches has its merits. However, using standard approaches, it is often difficult to generate impulse response functions that resemble the data. Chari-Kehoe and McGrattan [1996] point out some problems of the staggered price setting approach in capturing monetary dynamics. The cost-of-adjustment model of Rotemberg [1996] with quadratic adjustment costs does a better job empirically, and new research in this area is likely to achieve further improvements in explaining propagation dynamics. In part this progress may arise from recognition that monetary models, with or without menu costs, staggered price setting, or informational problems that rationalize labor contracts, can contain a continuum of rational expectations equilibria (see for example section 5.4 below). By allowing the data to pick an equilibrium in which prices are slow to respond to new information, one may make considerable progress in reconciling equilibrium theory with the facts. This is the main message of the recent literature that we will review in Sections 4 and 5.2.

As with the work on indeterminacy and business cycles, the main criticisms of monetary models with indeterminacy have been leveled at the plausibility of the mechanisms that cause indeterminacy to arise. For example, one can show that for certain classes of fiscal policy, monetary overlapping generations models possess two steady states; one is determinate and one is indeterminate. Since these example are difficult to match with time series data they are open to the criticism that they have little relevance to real world economies. This criticism led to a set of papers based on the infinite horizon model in which money enters because of a cash-in-advance constraint, or as a result of money in the utility function or in the production function. Models in this class have proved harder to dismiss than two period overlapping generations models although they leave open a number of issues to which we will return in section 4. The main unanswered issues are: 1) how do agents co-ordinate on a particular equilibrium, 2) are models with indeterminacy possible for plausible parameter values and 3) what are the implications of these models for a theory of optimal monetary policy.

3 Indeterminacy in Real Models

In Section 3 we turn our attention the specific mechanisms that give rise to indeterminacy. Our goal is to provide a common framework to assess the mechanisms that have been discussed in the recent literature. In the class of models that we will focus on, the source of indeterminacy may be viewed as arising from a coordination problem and in this sense our survey provides an extension of the survey of static coordination problems, by Cooper and John [1988], to models in which there is an essential dynamic element to the equilibrium concept. We begin by providing a very simple example that contains the essential elements of the models discussed in recent literature and in the remaining parts of Section 3 we elaborate on the elements that are specific to each of a number of variants of this basic mechanism.

Consider a specific equilibrium path for prices and rates of return; we are going to illustrate how, beginning from one particular equilibrium path, it may be possible to construct another. Suppose that agents collectively change their expectations and they come to believe that the rate of return on an asset will increase. As a consequence of this belief, they begin to accumulate this asset at a faster rate and its price increases. Suppose the return on the asset indeed tends to increase with higher stocks, maybe because of the presence of increasing returns, or a mechanism that mimics increasing returns. Since the overall rate of return on assets must remain equal to an intertemporal rate of discount, maintenance of the new belief as an equilibrium path requires an expected depreciation in the price of the asset, or a capital loss to offset the initial increase in its rate of return. If this price decline is sufficient to contain the explosive accumulation of the asset, then the resulting new path is also an equilibrium. We can now repeat this argument starting with the new equilibrium path, and construct yet another equilibrium. Since there are infinitely many such paths, the original equilibrium is indeterminate. Some of the models that we will discuss work through a mechanism in which increasing returns to scale is an essential part of the argument. However, as a number of the models described in this survey will demonstrate, indeterminacy does not *necessarily* require increasing returns to scale.

3.1 A Framework for Comparing Different Models

Recent interest in models with increasing returns was inspired by the literature on endogenous growth initiated by Lucas [1988] and Romer [1990]. Early work by these authors showed how to make increasing returns consistent with an equilibrium growth model by building externalities or monopolistic competition into an otherwise standard dynamic general equilibrium model. The work of Hall [1988], [1990], and Caballero and Lyons [1992] provided a further impetus to the increasing returns agenda by suggesting that externalities might be important not only in generating growth but also as a business cycle propagation mechanism. Their work suggested that the degree of increasing returns, exhibited via external effects or markups, was significant in many sectors of the economy. Subsequently a number of authors notably Basu and Fernald [1995], [1997], Burnside, Eichenbaum and Rebelo [1995], and Burnside [1996] have scaled down the early estimates of Hall, bringing them closer to constant returns, and in certain cases even finding decreasing returns in some industries. Earlier theoretical models of indeterminacy, for example the model of Benhabib and Farmer [1994], had relied on large increasing returns in line with Hall's original estimates. Subsequent theoretical work however has substantially reduced the degree of increasing returns needed to generate indeterminacy, (as in Benhabib and Farmer [1996a]), and it now has become clear that in an economy with some small market imperfections, even a technology with constant marginal returns can generate indeterminacy.[9]

To illustrate the critical elements that generate indeterminacy, we begin with a simple equilibrium structure that can, with slight modifications, accommodate a number of different models. The standard model with infinitely-lived identical agents maximizing the discounted sum of instantaneous utilities given by $U(c) + V(1-L)$, where c is consumption and L is labor, gives rise to the following system of equations:

$$U'(c) = p \tag{3.1}$$

$$U'(c)\, w_0\left(k, L; \bar{k}, \bar{L}\right) = V'(1-L) \tag{3.2}$$

$$\dot{k} = y\left(k, L(p,k); \bar{k}, \bar{L}(p,k)\right) - gk - c \tag{3.3}$$

$$r = \frac{\dot{p}}{p} + \left(w_1\left(k, p; \bar{k}, \bar{L}(p,k)\right) - g\right) \tag{3.4}$$

Here w_0 is the marginal product of labor, at this point taken to be equal to the real wage, and w_1 is the rental rate on capital, equal to the marginal product of capital: the marginal products are with respect to private inputs, keeping $\bar{k}$ and $\bar{L}$ fixed. The depreciation rate is g, the discount rate is r, the shadow price of capital is p, and the production function is given by y. In equations (3.3) and (3.4), it is assumed that $L(k,p)$ has been obtained by solving equations (3.1) and (3.2). Equation (3.2) represents the labor market equilibrium, equation (3.3) equates net investment to capital accumulation, and equation (3.4) is the standard "Euler" equation requiring the equality of the return on the asset, (its net marginal product plus its shadow price appreciation), to the rate of discount.

The only non-standard feature of the model above is the inclusion of external effects, generated by the aggregate inputs $\bar{k}$ and $\bar{L}$ in the production function. Of course some deviation from the standard framework in the form of a market imperfection must be introduced to obtain indeterminacy, since the standard representative agent model has locally unique equilibria.

3.2 The One Sector Model with Increasing Returns

We start by investigating the one sector model of Benhabib and Farmer [1994], which demonstrates how indeterminacy can arise in a representative agent model with increasing returns.[10] Increasing returns in this model is reconciled with private optimization by introducing either external effects in production, or a monopolistically competitive market structure where firms face downward sloping demand curves. Benhabib and Farmer show that the simple one-sector model with external effects is identical, in the sense of giving rise to the same reduced form, to a model with monopolistic competition and constant markups.

To see how indeterminacy comes about in a model with external effects and increasing returns, consider starting from an equilibrium path, but let the agents believe that there is an equilibrium in which the shadow price of investment p is higher than its current value, and that future returns justify a higher level of investment. If agents act on this belief, the higher current price of investment reduces consumption and induces agents to divert GDP from consumption to investment. If there were no externalities, investment would increase, and the marginal product of capital would begin to decline as additional capital is accumulated. This decline would have to be offset by capital gains in the form of increases in the shadow price of capital in order to validate the belief of agents that higher rates of investment will yield the appropriate return. Trajectories of this sort for investment and prices may be maintained for a number of periods, but the resulting over-accumulation of capital and the exploding prices will violate transversality conditions: an agent will never get to consume enough in the future to justify the sacrifice of a higher rate of investment. In other words, an agent conjecturing such a path of prices and returns would be better off consuming, rather than accumulating additional capital at such a pace.

Consider now an alternative parameterization of this model in which externalities are sufficiently big to permit indeterminate equilibria. Once again, let agents conjecture that there exists an alternative equilibrium path, starting with a price of investment p, which is higher than the one in the current steady state equilibrium. The higher price will cause agents to divert GDP from consumption to investment but, if externalities are strong enough they will simultaneously increase their consumption of leisure. The increase in leisure will cause GDP to decline, and investment will eventually fall as well. Benhabib and Farmer show that this dynamic argument has a representation in terms of labor demand and supply curves, with strong increasing returns to the labor input, and where the labor demand curve slopes up more steeply than the labor supply curve. In this framework shifts of the curves generated by conjectured changes in the shadow price of capital can lead to the contractionary employment effects mentioned above.

As the marginal product of capital falls with the decline of labor, the shadow price of investment must appreciate to produce a capital gain, because in equilibrium the overall return on capital must be equal to the rate of discount. This reinforces the original impulse of a higher relative price for the capital good. The contraction of labor however causes GDP and investment to decline, and the capital stock starts to fall. The decline in the stock of capital reverses the process, because it decreases labor demand. Since the labor demand curve slopes up more steeply than the labor supply curve, a decrease in labor demand tends to increase employment. This is the critical element that gives rise to indeterminacy in the model. Higher employment and the low level of the capital stock both cause the marginal product of capital to increase, and intertemporal equilibrium now requires a depreciation, rather than appreciation, in the price of capital to equate the overall return to the discount rate. As the price of capital falls, the economy returns to its original steady state along this new equilibrium path.

The key to indeterminacy in this model then lies in the non-standard slopes of the labor demand and supply curves, which induce a perverse labor market response. This feature, which requires a high enough externality to induce increasing returns in the labor input by itself, is what makes the model empirically implausible.

3.3 The Two Sector Model with Increasing Returns

A more satisfactory specification leading to indeterminacy is given by Benhabib and Farmer [1996a]. They start with a two-sector model, but with identical production functions in the consumption and investment sectors. One might think that by making the production functions of the two sectors identical, the model would collapse to a one-sector economy. However, the two sector structure is preserved by the distinct external effects in each sector, each arising from their own sectoral outputs, rather than from the aggregate output. The model yields a linear production possibilities surface (ppf) from the private perspective, but one that is convex (to the origin) from the social perspective. To develop a parallel exercise conducted for the one sector case above, consider again starting at a steady state and increasing the (shadow) price of capital p. Since we have a two-sector model, equation (3.1) has to be modified to reflect the relative price of capital in terms of the consumption good. If we denote this price by q, we get:

$$qU'(c) = p \tag{3.5}$$

A higher q now raises consumption since, given the convexity of the ppf, an increase in consumption relative to investment will be associated with an increase in the relative price of capital. The impact effect on p is ambiguous because $U'(c)$ declines as well, but if the curvature of $U(c)$ is not too severe, p and q will change in the same direction. Benhabib and Farmer [1996a] use logarithmic utility of consumption in their model. Nevertheless, the reader may want to think of $U'(c)$ as constant in order to get a clearer picture of the logic of the argument.

When consumption increases, the supply curve of labor shifts to the left, and since the demand curve for labor slopes down in the two sector model, the result is a contraction in labor, and also in investment. As in the previous case, the decline in labor decreases the marginal product of capital. Maintaining intertemporal equilibrium now requires an appreciation in the (shadow) price of capital to keep the overall return equal to the discount rate, reinforcing the initial rise in q, and therefore in p. However, the decline of investment and of the capital stock must eventually raise the marginal product of capital, and reverse the appreciation of p: intertemporal equilibrium eventually requires a depreciation of the shadow price of capital. The process therefore is reversed, and the economy moves back towards the steady state, giving rise to an alternative equilibrium trajectory.

It is clear in this case that the elements responsible for the "stability" of the steady state, and therefore for "indeterminacy," are no longer the perverse slopes of demand and supply in the labor market, but the convexity of the ppf. Benhabib and Farmer [1996a] find that even a small output externality, resulting in increasing returns as low as 1.07, coupled with standard values for the other parameters, is sufficient to generate indeterminacy. Furthermore their calibration analysis incorporating iid sunspot shocks does quite well by the standards of the recent real business cycle analysis.

In spite of an apparent improvement over the one sector model, one may question the implication of a convex social ppf which implies that the sectoral aggregate supply curves slope down. This issue has been empirically investigated by Shea [1993]. He studies 26 manufacturing industries and finds that in 16 of them, the supply curves slope up. The question therefore arises as to whether indeterminacy requires increasing returns in all sectors. Recent results of Harrison [1996] on the two-sector model show that indeterminacy obtains for roughly the same parametrizations as in Benhabib and Farmer [1996a], with the exception that only the investment sector is assumed to have increasing returns. Furthermore, her estimates of increasing returns and externalities, obtained by allowing them to vary across sectors, indicate that they may be sufficiently high in the investment sectors to support indeterminate equilibria[11]. Perli [1994] obtains similar results by introducing a home production sector as in Benhabib, Rogerson and Wright [1991], with sector-specific external effects only in the market sector, and he obtains indeterminacy with small increasing returns. Furthermore the times series generated by his calibrated model are comparable to U.S. postwar series. In a recent paper Weder [1996] also shows that in a two-sector model indeterminacy can arise with mild increasing returns in the investment sector alone. His specification relies on imperfectly competitive product markets rather than external effects.

3.4 The Two Sector Model with Constant Marginal Returns

So far it would seem that some increasing returns may be necessary to generate indeterminacy. In a recent paper Benhabib and Nishimura [1996] demonstrate that social constant returns to scale, coupled with some small external effects that imply very mild decreasing returns at the private level, can also generate indeterminacy. In their framework there are private decreasing returns to scale and small external effects that give rise to *constant returns to scale* at the level of the aggregate economy.

In the two models we discussed in Sections 3.2 and 3.3 a rise in the shadow price of capital eventually induces the capital stock to fall and its marginal product to increase. Consequently, intertemporal equilibrium requires the price of capital to fall, reversing its original increase. This mechanism can be duplicated in a two-sector model without upward sloping labor demand curves or a convex ppf. The reason is that in models with more than one sector, the marginal product of capital depends not only on factor inputs, but also on the composition of output and on the relative factor intensities of the underlying technology.

We can express the technology of the two-sector economy in per capita variables as a transformation surface given by $c = T(y, k)$. Such a technology implies that $T_1 = -p$ and $T_2 = w_1$, where the subscripts of T indicate derivatives with respect to the appropriate argument. T_1 then is the slope of the ppf, while T_2 corresponds to the marginal product of capital in the production of the capital good.[12] Consider first a simple two-sector model *without externalities*, and production functions that differ across the consumption and investment sectors. For simplicity also assume that total labor is fixed, and that the utility function is linear in consumption, so that in terms of equation (3.5) we have $q = p$. If the production possibility frontier is strictly concave we can invert the relation $T_1(y, k) = -p$ to obtain the output of the investment good as $y = y(k, p)$. Equation (3.3) must now be modified as follows:

$$\dot{k} = y(k, p) - gk \tag{3.6}$$

Equations (3.4) and (3.6) now fully describe the dynamics of the system in (k, p). With external effects suppressed and a fixed labor supply, the local dynamics around the steady state will depend on the Jacobian matrix J :

$$J = \begin{bmatrix} \frac{\partial y}{\partial k} - g & \frac{\partial y}{\partial p} \\ 0 & -\frac{\partial w_1}{\partial p} + (r+g) \end{bmatrix}$$

Note that the lower left submatrix, $J_{21} \equiv \left[-\frac{\partial w_1}{\partial k}\right]$, is identically zero. This is because under constant (social) returns to scale relative factor prices uniquely determine prices as well as input coefficients. To demonstrate that equilibrium is determinate we must show that the roots of J have opposite sign; but, since J_{21} is zero, the roots of J are equal to the elements on the main diagonal, J_{11} and J_{22}. Determinacy is the assertion that these elements have opposite sign. Benhabib and Nishimura demonstrate that the signs of J_{11} and J_{22} are related to two familiar theorems in the international trade literature, the *Stolper–Samuelson theorem* and the *Rybczinski theorem*.

We deal here only with the case in which the investment good is labor intensive although the argument that we will present can be easily extended to the case when the investment good is capital intensive, by reversing the signs of the two inequalities that we will present. Consider first the Stolper–Samuelson theorem which asserts (in the labor intensive case) that a rise in p will decrease the rental price of capital, ω_1. In symbols this asserts that $\left[\frac{\partial \omega_1}{\partial p}\right] < -\left[\frac{\omega_1}{p}\right]$. Since, at the steady state, $\omega_1 = p(r+g)$, the Stolper–Samuelson theorem implies that the element J_{22} is positive. Now consider the Rybczinski theorem which asserts that if more capital is used in the investment goods sector, output of investment goods will rise less than proportionately. In symbols this is represented by the inequality, $\left[\frac{\partial y}{\partial k}\right] < -\left[\frac{y}{k}\right]$. Since, at the steady state, $y = gk$, the Rybczinski theorem implies that the element J_{11} is negative. It follows, *for the case when J_{21} is zero*, that the Stolper–Samuelson and Rybczinski theorems can be used to establish that equilibrium is determinate.

More generally, in multisector models J_{11} and J_{22} will be matrices rather than scalars, and with linear utility J_{21} will still be a zero matrix, so that the roots of J will be given by the roots of J_{11} and J_{22}. However, as shown in Benhabib and Nishimura [1996], the duality between the Stolper–Samuelson theorem and the Rybczinski theorem will imply that at least half of the roots of J will be positive, implying that the equilibrium is determinate[13].

What happens when there are external effects? Benhabib and Nishimura [1996] establish that in this case one can break the reciprocal relation between the Rybczinski and Stolper–Samuelson theorems. Output effects on investment are still governed by the logic of the Rybczinski theorem, but the Stolper–Samuelson theorem requires that costs of production equal prices. With external effects, markets are distorted and, the relevant factor intensities can be reversed. This reversal may break the saddle-point property (even when external effects are 'small') and generate a situation in which more than half of the roots of J are negative and there are multiple indeterminate equilibria. Benhabib and Nishimura [1996] calibrate and simulate a discrete time version of their model, with one consumption and two investment goods, which incorporates iid sunspot shocks and logarithmic utility. Their calibration can match the moments of GDP, consumption, aggregate investment and hours in US data as well as any other standard RBC model, and can generate impulse responses to technology shocks that resemble the hump-shaped impulse responses generated with vector-autoregressions on US data. (See section 8.1.2 below.)

3.5 Fixed Costs and the Role of Profits

The results sketched above suggest that introducing even small market imperfections into the standard infinitely-lived representative agent model can produce empirically plausible indeterminacy even under constant social returns. Increasing returns to scale at the level of the aggregate social production function are not necessary for indeterminacy. In order to generate this result, however, there must be decreasing returns at the level of private firms and an implication of this is that firms will earn positive profits. In the parameterized examples given by Benhabib and Nishimura [1996] these profits are quite small because the size of external effects, and therefore the degree of decreasing returns needed for indeterminacy are minimal. Nevertheless positive profits would invite entry, and unless the number of firms are fixed, a fixed cost of entry must be assumed to determine the number of firms along the equilibrium path. Such a market structure would exhibit increasing private marginal costs but constant social marginal costs, which is in line with current empirical work on this subject.

It seems therefore that models of indeterminacy based on market imperfections which drive a wedge between private and social returns must have some form of increasing returns, no matter how small, either in variable costs, or through a type of fixed cost that prevents entry in the face of positive profits.[14] The point is that while some small wedge between private and social returns is necessary for indeterminacy, this in no way requires decreasing marginal costs, or increasing marginal returns in production.

3.6 Models with Variable Markups

The use of *variety* in intermediate or consumption goods, coupled with a monopolistically competitive market structure, has been incorporated by a number of authors into the standard optimal growth model. Woodford [1987] first demonstrated that a monopolistically competitive market structure with free entry, coupled with variable markups, can lead to indeterminacy and to self-fulfilling sunspot equilibria. He constructs a model in which aggregate output, investment, as well as employment, are driven by expectations of aggregate demand. Furthermore, kinks in the demand curves faced by firms give rise to variable markups, allowing the quantity of labor supplied to adjust to the quantity demanded through changes of aggregate demand in the goods market. As noted by Woodford, this labor market structure with variable markups relaxes the rigid relation between wages and the marginal product of labor that is a feature of competitive market models, and avoids the implication that wages must move countercyclically in the absence of technology shocks.[15]

Two related approaches, using variable markups, are those of Gali [1994], [1996] and Rotemberg and Woodford [1992]. Gali develops a monopolistically competitive market structure in which firms markup price over marginal cost. He shows that changes in the composition of aggregate demand between investment and consumption may cause the markup to vary systematically over the business cycle and he uses the countercyclical markup to demonstrate the possibility that there may be many indeterminate equilibria. Rotemberg and Woodford develop a model of the business cycles based on implicit collusion among firms that strategically vary markups depending on the state of aggregate demand. Their model also gives rise to indeterminacy and although the exact mechanism is somewhat different from that in Gali or Benhabib and Farmer, the implications for business cycle data are similar as demonstrated in the recent paper by Schmitt-Grohé [1997]. We will return to this paper in section 8.1.1 in which we discuss empirical aspects of indeterminacy.

We can illustrate the mechanism that gives rise to indeterminacy in the models of Gali, and of Rotemberg and Woodford, in the context of our simplified model consisting of equations 3.1, 3.3 and 3.4. For simplicity we assume that labor is fixed, although variable labor would allow a smaller and less variable markup to generate indeterminacy. First we note that the return on (or user cost of) capital in equation 3.4 will have to be modified: it is not the marginal product but the marginal product divided by the markup:

$$r = \frac{\dot{p}}{p} + \left(\frac{w_1 \left(k, p; \bar{k}, \bar{L}(p, k)\right)}{\mu} - g \right) \tag{3.7}$$

Here, we simply assume that the markup μ is related to p, although in the works cited above this dependence of the variability of the markup is derived from more basic structural assumptions. The influence of a variable markup works in the following way. Starting at a steady state equilibrium, an increase in p raises the share of investment and leads to faster capital accumulation. The higher investment share also lowers the markup and raises $\left(\frac{w_1}{\mu}\right)$, even though w_1 declines. With r fixed, $\left(\frac{\dot{p}}{p}\right)$ then must decline, reversing the process and driving the economy back towards the steady state. This then is another equilibrium, implying that the initial value of p,and therefore the equilibrium trajectory is indeterminate. The same process would work if the markup depended directly on the capital stock, so that $\left(\frac{w_1}{\mu}\right)$ increased with a rise in the capital stock.

Gali [1994], [1996] produces variable markups by introducing different demand elasticities for investment and consumption goods. The average markup depends on the relative shares of consumption and investment in aggregate output. Gali assumes the elasticities are such that the markup is negatively related to investment share, and he presents evidence from US data to support his contention that this assumption is a reasonable first approximation to the facts. In the model of Rotemberg and Woodford [1992] the variability of μ results from a collusion arrangement between firms that share the market and earn a stream of profits in an implicit collusion arrangement. The equilibrium conditions imply that the markup μ depends on the ratio of the values of the firms to aggregate output. Implicit collusion among firms requires countercyclical markups to maintain discipline, and to prevent deviation from the collusive arrangement. One of the important implications of models with variable markups, as shown in the paper by Schmitt-Grohé [1997], is that indeterminacy in these models can occur with a lower degree of increasing returns to scale than in the case of constant markups. We return to this idea in Section 8.1.1.

4 Indeterminacy in Monetary Models

In this section we discuss indeterminacy in models that include the real value of money balances as an argument of the utility function or the in production function, following the work of Patinkin [1956]. Brock [1974] is the first to formally discuss the possibility of indeterminacy associated with self-fulfilling hyperinflationary and deflationary equilibria, as well as conditions that rule out such equilibria, in a model of an infinitely lived representative agent that has real balances in the utility function. Since Brock's paper, much of the literature on indeterminacy in monetary models has focused on self-fulfilling inflations and deflations, rather than the possibility of a continuum of equilibria converging to a steady state, possibly because the former can occur under relatively weak conditions. Hyperinflationary equilibria can be ruled out if it is assumed that it is prohibitively costly for the economy to operate at low levels of real balances, or close to a barter economy. For certain classes of monetary policies, speculative deflations can also be ruled out by restrictions on preferences. Obstfelt and Rogoff [1983] provide conditions under which such hyperinflationary and deflationary equilibria can be ruled out. An extensive overview of indeterminacy in a cash-in-advance or a cash good-credit good model is given by Woodford [1994]. He analyzes the conditions for the existence of hyperinflationary and deflationary equilibria as well as equilibria described by a continuum of paths that converge to a steady state and which can give rise to stationary sunspot equilibria.[16]

One of the first to note the possibility of indeterminacy in the form of a continuum of equilibria converging to a steady state was Calvo [1979], and there have been many related papers since. Wilson [1979] gives some of the early examples of indeterminate equilibria in the context of a cash-in-advance model. Taylor [1977] has one of the first discussions of indeterminacy in a monetary model where he proposes a selection principle that picks the equilibrium solution exhibiting the minimum variance. McCallum [1983] provides a survey of early literature on monetary indeterminacy and also proposes a 'minimal state variable solution' to select an equilibrium. Rather than try to survey additions to this vast body of work we concentrate instead on recent papers that have exploited indeterminacy to address and study empirical features of the business cycle. Our point of departure then is the recent interest in calibrated versions of indeterminate models and in particular the idea that indeterminacy can be a feature that allows us additional freedom to explain properties of economic fluctuations that are otherwise difficult to understand.

4.1 Monetary Models with One State Variable and Fixed Labor Supply

We begin our discussion with a class of models in which money plays the role of facilitating exchange and in which all other aspects of the model are stripped down to a bare minimum. These are models in which real money balances is the sole state variable. By focusing on the simplest possible monetary model we will highlight an idea that holds in more general examples; indeterminacy is most easily obtained in monetary models when changes in the stock of real balances have large effects on output. These effects can come from including money in the utility function as in Calvo [1979], money in the production function as in Benhabib and Farmer [1996b], or from a cash-in-advance constraint as in the calibrated monetary models of Cooley and Hansen [1989] [1991].

To relate monetary models to our discussion of indeterminacy in real economies, suppose that output depends on real balances: $y = y(m)$, with y increasing and strictly concave.[17] We assume that money is injected into the economy in equal lump-sum transfers to all agents, and that nominal balances grow at the rate σ. Then the net rate of return on holding money is $y'(m) - \pi = y'(m) - \left(\sigma - \frac{\dot{m}}{m}\right)$ where π is inflation, and $\pi \equiv \left(\sigma - \frac{\dot{m}}{m}\right)$in equilibrium. If we assume that money is the only asset, then $\dot{c} = y'(m)\dot{m}$. Now using equations (3.1) and (3.4), we obtain:

$$\dot{m}\left(\frac{1}{m} + \frac{U'(c)\,y'(m)}{U''(c)}\right) = \left(r + \sigma - y'(m)\right) \tag{4.1}$$

If we define the elasticities

$$\varepsilon_c = -\frac{U''(c)\,c}{U'(c)}, \qquad \varepsilon_m = \frac{y'(m)\,m}{y(m)} \tag{4.2}$$

then equation (4.1) becomes:

$$\dot{m} = \frac{m\left(r + \sigma - y'(m)\right)}{1 - \varepsilon_c \varepsilon_m} \tag{4.3}$$

Equation (4.3) is a differential equation that must be obeyed by paths for real balances that are consistent with rational expectations equilibrium. The model has an equilibrium in which real balances are constant and equal to m^*. This steady state equilibrium is defined as the solution to the equation:

$$r + \sigma = y'(m^*).$$

Since equilibria can be described as functions of a single state variable, there will exist a set of indeterminate equilibria if the steady state in equation (4.3) is locally stable: this requires the right-hand side of the equation to be decreasing in m at m^*.[18] Since $m(r + \sigma - y'(m))$ is increasing in m at m^*, a sufficient condition for indeterminacy is that money is sufficiently productive, or alternatively put, that ε_m is large enough so that $\varepsilon_m \varepsilon_c > 1$.[19]

The intuition for this result is straightforward. When money is *not* sufficiently productive, an increase in nominal balances or alternatively an initial low price level that corresponds to real balances higher than m^*, generates an excess supply of money that spills onto the goods market: prices have to jump so that output, $y(m^*)$, and real balances, m^*, remain fixed at their steady state values. If prices did not jump to restore the steady state values, real balances would rise, the return on money $y'(m)$ would decline, and the higher money balances would be held only if some further deflation were expected. If such deflationary expectations were confirmed, real balances would eventually explode, and unless certain assumptions are made to rule out such hyperinflationary equilibria, we would still have indeterminacy, even though the steady state level of real balances is unstable.

On the other hand if *money were sufficiently productive*, an increase in real balances at m^* would increase output and create a net *excess demand* for money, rather than an excess supply. More formally, the higher real balances would raise consumption and reduce the marginal utility of current consumption so much that the agents would want to hold higher levels of the monetary asset.[20] Therefore an increase in nominal balances, or a low initial price that places real balances above m^*, would generate a net excess demand for money which would have to be offset by expected inflation in order to restore equilibrium. Inflation would then drive real balances back down to their steady state value without requiring an instantaneous jump in prices. Thus the steady state is stable, with a continuum of trajectories of real balances converging to it.

4.2 Money in the Utility Function and the Production Function

The model that we discussed in Section 4.1 included money as an argument in the production function. In this section we will show how models of this kind can be extended to include money as an argument of production *and* utility functions. This modification will be useful in our discussion of policy rules in Section 5.3 where money in the utility function allows us to demonstrate that indeterminacy can arise in broader range of cases than in simple production function model of Section 4.1.

We model the utility function with the specification $U(c, m)$ where U is increasing and concave in c and m.When money enters both the production and utility functions, equation (4.3) becomes

$$\dot{m} = \frac{m\left(r + \sigma - y'(m) - \frac{U_m(c,m)}{U_c(c,m)}\right)}{1 - \varepsilon_c \varepsilon_m - \varepsilon_{cm}} \tag{4.4}$$

where $U(c, m) = U(y(m), m)$ and the cross partial term

$$\varepsilon_{cm} = \frac{-mU_{cm}}{U_c} \tag{4.5}$$

plays an important role in the way the model behaves. This term measures the effect of holding extra real balances on the marginal utility of consumption. The term $\frac{U_m}{U_c}$,given by the expression

$$\frac{d\left(\frac{U_m}{U_c}\right)}{dm} = \frac{1}{U_c}\left(\left(U_{mc} - \frac{U_m}{U_c}U_{cc}\right)y' + \left(U_{mm} - \frac{U_m}{U_c}U_{cm}\right)\right)$$

is also important in determining whether equilibrium is determinate since if $\left(\frac{U_m}{U_c}\right)$is decreasing in m the steady state will be locally stable, and therefore indeterminate whenever $1 - \varepsilon_c \varepsilon_m - \varepsilon_{cm} < 0$. It seems reasonable to assume that neither consumption nor money are inferior goods, $\left(-U_{mc} + \left(\frac{U_m}{U_c}\right)U_{cc}\right) <$ 0 and $\left(U_{mm} - \frac{U_m}{U_c}U_{cm}\right) < 0$. But this is not enough to determine whether $\left(\frac{U_m}{U_c}\right)$is increasing or decreasing in m.

It might seem that the discussion in this section is of little empirical relevance. Perhaps utility functions that allow for peculiar cross partials can be easily ruled out by data and we should restrict attention to logarithmic functions or at least to utility functions that are separable in consumption and real balances. Unfortunately, this argument negates the very reason for including money in the utility function in the first place since it is precisely changes in the marginal utility of transacting that one might expect to characterize a monetary economy. In Section 4.4 we discuss a calibrated model used by Farmer [1997] in which he demonstrates that models with indeterminacy can be used to mimic impulse response functions in the data in addition to capturing the more salient features of velocity and the rate of interest in the US data. Farmer includes money in the utility function and he chooses in a sum of weighted CES utility functions that allows the term $\frac{d\left(\frac{U_m}{U_c}\right)}{dm}$ to have either sign. It is precisely this flexibility that allows the model to capture the empirical features that we will describe in Section 4.4.

4.3 Monetary Models with one State Variable and a Variable Labor Supply

In this section we will show how the extension of monetary models to allow for a second factor of production, labor, can increase the ability of these models to describe data by generating indeterminacy for a more plausible range of the parameter space. Models of this kind still have a single state variable since one can show that, in equilibrium, hours worked are a function of real balances. As in the case of models with a fixed supply of labor, indeterminacy is most likely to occur when money has a big effect on output.

There is a growing literature using cash-in advance or money in the utility function approaches to calibrated models of money that finds a unique rational expectations equilibrium; examples include Cooley and Hansen [1989], [1991] and related literature. One reason why calibrated models of money may appear to have a unique determinate equilibrium is that these models often use simple functional forms that allow for a single parameter to capture the magnitude of the importance of money. Recent work by Benhabib and Farmer [1996b] and Farmer [1997], demonstrates that indeterminacy may occur in a monetary model *for realistically calibrated parameter values* by modeling the role of money with more flexible functional forms that nest the cash-in-advance model as a special case. For example, in Benhabib and Farmer [1996b], output is produced using labor, and the service of money:

$$y = y(m, l). \tag{4.6}$$

If one makes the assumption that the technology is Cobb-Douglas, there is a single parameter that captures the effect of money; the elasticity of output with respect to real balances. This parameter can be directly measured in the same way that one measures the elasticity of output with respect to labor, through the share of resources used by the firm in transacting. This leads to the calibrated measure:

$$_m = \frac{my'(m,l)}{y(m,l)} = \frac{im}{y(m,l)} \tag{4.7}$$

where i is the opportunity cost of holding money and the left hand side of equation (4.7) is the elasticity of real balances. Since the opportunity cost of holding money cannot be much more than 2% and the velocity of circulation, y/m is around 5 in post war data, the elasticity of money in production must be small, less than half of one percent. This kind of magnitude is not enough to generate big effects.

Suppose, however, that money is highly complementary with other factors. In this case Benhabib and Farmer [1996b] show that indeterminacy may hold in a monetary model with an otherwise standard constant returns to scale technology. They use a technology of the form

$$y = (am^{\varepsilon} + l^{\varepsilon})^{1/\varepsilon}$$

which collapses to a Leontief technology as ε approaches $-\infty$ and to a Cobb-Douglas technology as ε approaches 0. The Leontief (or cash-in-advance) technology is rejected by the data since it can be shown to imply that the interest elasticity of the demand-for-money should be zero. The Cobb-Douglas function is also rejected since it would imply an elasticity of substitution of minus one. In the data, recent studies[21] find that the interest elasticity of the demand for money is close to -0.5 and one can use this fact to calibrate the value of ε.

Models that are calibrated to capture both money's share of output *and* the elasticity of money demand can be calibrated to display *indeterminate equilibria.* The reason why indeterminacy is more easily obtained in this case is that, in equilibrium, there is a relationship between real balances and labor demand that is found by solving first order conditions in the labor market. If one solves for labor demand as a function of real balances (call this function $l(m)$) using this condition and substitutes the result back into the production function one arrives at the equation:

$$y = y\left(m, l\left(m\right)\right).$$

Calibrations of the production function using equation (4.7) lead to the conclusion that the elasticity of *y with respect to its first argument* is small. However, although the direct effect of money on output may be small, the indirect effect through the fact that labor and real balances are increased together, the elasticity of *y with respect to its second argument,* may be large. Benhabib and Farmer [1996b] exploit the fact that their parameterization leads to indeterminacy to match a number features of the monetary propagation mechanism.

4.4 Monetary Models with Several State Variables

The Benhabib and Farmer explanation of monetary dynamics works by picking an equilibrium in which the price is predetermined one period in advance and hence an increase in the nominal quantity of money causes an increase in *real balances* and employment. Beaudry and Devereux [1993] and Farmer [1997] build on this idea by building money into versions of a real business cycle economy. The paper by Beaudry and Devereux adds money to a structure in which there is already a real indeterminacy because of increasing returns to scale. The work by Farmer adds money into the utility function and has a production sector that is identical to the standard real business cycle model. Both sets of authors calibrate their economies to fit the broad features of the US economy (both real and monetary) and both models perform as well or better than a standard RBC model at replicating the second moments of US time series on consumption, investment, capital, GDP and employment.

The following discussion is based on Farmer [1997] who allows for a fairly general specification of utility of the form

$$U = U\left(C, m, l\right)$$

where C is consumption, m is real balances and l is labor supply. In the spirit of the real business cycle models of King Plosser and Rebelo, Farmer argues that one should restrict attention to utility functions that allow for growth to be balanced and he shows that this implies that utility must be homogenous of degree ρ (a real number less than one) in m and C. The class of functions used in the paper is of the form:

$$U = \frac{X(C,m)^{1-\rho}}{1-\rho} - W(C,m)^{1-\rho} V(l) \tag{4.8}$$

where X and W are CES aggregators and V is an increasing convex function that measures the disutility of working. The following discussion is based on the special case of this utility function:

$$U = \frac{C^{1-\rho}}{1-\rho} - m^{1-\rho} V(l), \qquad \rho > 1.$$

The production side of the model is a standard RBC economy in which output is produced with the technology

$$Y = F(K,l)\, S$$

F is Cobb Douglas and S is an autocorrelated productivity shock.

Farmer considers two kinds of monetary policies. Policies in which there is an interest rate rule of the kind,

$$i_t = \bar{\imath}$$

and money growth rules of the kind

$$M_t = \mu M_{t-1}$$

where i is the nominal rate of interest, M is the nominal quantity of money and μ is the money growth factor.

In the case when the monetary authority fixes the money growth rate in advance, the model can be described by a four variable difference equation of the form:[22]

$$\begin{bmatrix} \mu_{t+1} \\ C_{t+1} \\ K_{t+1} \\ m_{t+1} \end{bmatrix} = A \begin{bmatrix} \mu_t \\ C_t \\ K_t \\ m_t \end{bmatrix} + B \begin{bmatrix} u^1_{t+1} \\ u^2_{t+1} \\ e^1_{t+1} \\ e^2_{t+1} \end{bmatrix}$$

where u^1 and u^2 are fundamental shocks and e^1 and e^2 are sunspot shocks. Unlike the sunspot models that we have discussed so far, Farmer allows for multiple shocks to both sunspots *and* fundamentals and he calibrates the magnitude of the shocks by estimating the variance co-variance matrix of the residuals from a four variable vector autoregression on US data.

Indeterminacy in this model can be understood by appealing to the Benhabib and Farmer's [1994] results on the real model with increasing returns. Consider the case in which the Central Bank pegs the nominal interest rate. It is well known that this policy rule leads to price level indeterminacy. What Farmer shows is that for utility functions in the class described by equation 4.8 there may also be a *real* indeterminacy.

Optimal decisions in Farmer's model are characterized by three Euler equations, one for capital, one for money and one for bonds and one static first order condition describing the labor market. One may combine the Euler equations for money and bonds to yield a second static first order condition:

$$\tilde{m} = \tilde{c} + \frac{1+\chi}{\rho}\tilde{l} - \frac{1}{\rho}\tilde{I}, \tag{4.9}$$

where the variables $\tilde{c}$, $\tilde{m}$, and $\tilde{l}$, are the logarithms of consumption, real balances and labor supply. This equation plays the role of the "demand for money" in this model. The labor market equations can also be broken down into demand and supply of labor equations as in the real model discussed in section 3.2. These demand and supply equations are given below:

$$(1-\rho)\,\tilde{m} + \rho\tilde{c} + \chi\tilde{l} = \tilde{\omega}, \tag{4.10}$$

$$(1-\alpha)\,\tilde{k} + (\alpha-1)\,\tilde{l} = \tilde{\omega}, \tag{4.11}$$

where $\tilde{\omega}$ is the log of the real wage and $\tilde{k}$ is the log of capital. Equation 4.11 is a "labor demand" equation. If we were to graph the real wage against labor demanded and supplied it would be represented by a downward sloping line, shifted by changes in the capital stock. Equation 4.10 is a "labor supply equation". On the same graph, it would be represented by an upward sloping line that was shifted by changes in consumption or changes in real balances. The key to understanding indeterminacy in the monetary model is to notice that one can replace $\tilde{m}$ in 4.10 by a function of $\tilde{c}$, $\tilde{l}$. and $\tilde{I}$, from the money market equilibrium condition, equation 4.9. This leads to the hybrid equation:

$$\tilde{c} - \frac{1-\rho}{\rho}\tilde{I} + \left(\frac{1+\chi}{\rho} - 1\right)\tilde{l} = \tilde{\omega}. \tag{4.12}$$

In the real model in continuous time Benhabib and Farmer show that indeterminacy occurs when the labor demand and supply curves cross with the wrong slopes. This occurs in their model as a result of increasing returns to scale in production that causes the labor demand curve to slope up. If one eliminates real balances from the labor market equations, using equation 4.10, the resulting model has *exactly* the same structure as the real model of Benhabib-Farmer, with the additional twist that the interest rate enters as an exogenous variable. Equation 4.12 is a compound equation that combines the labor supply curve and the money demand equation; this plays the same role as the standard labor demand equation in the real model. Notice that this hybrid "labor supply curve" *slopes down* whenever $(1+\chi)/\rho$ is less than 1. Using the Benhabib Farmer indeterminacy condition, it follows that the monetary model has a *real* indeterminacy whenever the "labor supply" curve slopes down more steeply than the labor demand curve; for reasonable calibrations this occurs when χ is small (elastic labor supply) α equals 2/3 (labor's share of national income) and ρ is bigger than 1.5.

5 Indeterminacy and Policy Feedback

So far we have discussed models in which indeterminacy arises in the context of models with a government sector; but we have allowed only government policies that are determined by simple rules such as fixed money growth rates or fixed government debt. In this section we will examine indeterminacy that may arise as a consequence of more complicated government policies that allow for feedback from the private sector to future values of fiscal or monetary policy variables.

5.1 Fiscal Policy Feedback

We begin with a class of models in which there are "fiscal increasing returns," first discussed and elaborated on by Blanchard and Summers [1987]. In the simplest formulation of such a model an increase in the capital stock can increase the post-tax return on capital, because it expands the tax base and reduces the tax rate. If G is the constant real government expenditures and $G = \tau f(k)$, where τ is the tax on capital and $f(k)$ is income, we can obtain the analogue of equation (3.4) as:

$$r = \frac{\dot{p}}{p} + \left(f'(k) \left(1 - \frac{G}{f(k)} \right) - g \right) \tag{5.1}$$

If the after tax return $f'(k)\left(1 - \frac{G}{f(k)}\right)$is increasing in k, a shift in p will raise investment and the capital stock, as well as the return to capital, so that equation (5.1) will be satisfied only if $\left(\frac{\dot{p}}{p}\right)$ falls. This reverses the original rise in p and moves the system back toward the steady state, generating another equilibrium path. In fact, as shown in Velasco [1996], such a system has two steady state values for k, corresponding to a high and a low tax rate, with the low tax steady state representing a saddlepoint. Note that the term $\left(1 - \frac{G}{f(k)}\right)$ is analogous to the reciprocal of a markup that varies inversely with the stock of capital k.

Two related papers are those of Guo and Lansing [1996] and Schmitt-Grohé and Uribe [1997]. Guo and Lansing explicitly compare the welfare properties of alternative fiscal policies in a model with increasing returns in the production sector. Their focus is on the ability to Pareto rank alternative equilibria with an eye to asking if models of indeterminacy might eventually be used to conduct welfare analysis, and to design optimal fiscal policies to select the best equilibrium.. The model of Schmitt-Grohé-Uribe includes labor and capital taxes, and generates two steady states by fixing government revenues and requiring the tax rate to be determined endogenously. Their model does not rely on explicit increasing returns to generate indeterminacy, although the labor market effects are similar to those of Benhabib and Farmer [1994] with upward sloping labor demand curves. The mechanism that operates in their paper works through increases in employment that decrease equilibrium tax rates, and raise the after tax return on labor. The tax rates at the indeterminate steady state are below those that maximize the revenue on the Laffer curve. Schmitt-Grohé and Uribe provide a calibration of their model to fit the US data and show that a successful calibration requires an elastic labor supply and a labor tax rate above the share of capital in aggregate income. They introduce a non-taxed home production sector, which allows indeterminacy under realistic tax rates and labor supply elasticities.

5.2 Monetary Policy Feedback

The monetary models discussed so far assume no feedback from the private economy to government behavior. In practice, however, central banks typically react to the private sector and the existence of central bank reaction functions has led to the development of a literature in which it is the central bank that is itself responsible for indeterminacy. Many of the early monetary models simply assumed that the path of the money supply is an exogenous process determined by the central bank. In practice, central banks do not control a monetary aggregate directly. For example, in the US the Federal Reserve system manipulates non-borrowed reserves on a day to day basis in an attempt to peg an interest rate (the Federal Funds rate) at a level that is revised periodically in light of economic conditions.

Why does much of the literature assume that the central bank controls the money supply when in practice interest rate control is more common? One reason is that, as pointed out by Sargent and Wallace [1975], interest rate rules lead to price level indeterminacy and until recently, most authors have avoided building models with indeterminate equilibria because it was not known how to match up models of this kind with data. Recently there has been more interest in the design of central bank operating rules and this has led to a revival of interest in indeterminacy and its implications in calibrated monetary models.[23]

One of the first to derive an indeterminate equilibrium from a central bank reaction function is Black [1974] . He assumes that the central bank responds, at time t, to the inflation rate between times $t-1$ and t, decreasing (increasing) real money balances if this inflation rate is positive (negative). In the absence of a central bank reaction of this kind, higher inflation would be required to sustain equilibrium in response to an initial (upward) departure of the initial price from its unique equilibrium level. When the central bank follows a contractionary reactive policy, inflation is no longer necessary to sustain equilibrium. If the monetary policy response is sufficiently strong, prices must decline to offset the expected further contraction of nominal balances, reversing the inflation, and returning the system to its steady state level of real balances. Therefore deviations of real balances from steady state levels are reversed, and the initial price level is indeterminate.

More recently Leeper [1991], and Schmitt-Grohé and Uribe [1997] have studied similar models where the monetary policy rule ties the nominal rate of interest to past inflation. One way to interpret the policy rule in models of this class is to assume that current inflation is forecast by past inflation. If one assumes that marginal utility, and endowments are constant, and that the utility function is separable in real balances and consumption, Leeper's model can be characterized by a discrete Euler equation of the form:

$$\frac{P_{t+1}}{P_t} = \beta i_{t+1}. \tag{5.2}$$

In this equation β is the discount factor and i_{t+1} is the nominal interest rate, representing the payout at time $t+1$ to an investment in t and P_t is the price level at time t. If we assume a simplified feedback rule, with the constants a, and γ,

$$i_{t+1} = a\frac{P_t}{P_{t-1}} + \gamma \tag{5.3}$$

and combine this equation with (5.2) to obtain a first order difference equation in the inflation rate, one can show that there exists an indeterminate set of equilibria if $a\beta < 1$. If on the other hand the nominal rate responds to the contemporaneous inflation, or in a stochastic model to expected inflation, so that

$$i_{t+1} = \alpha E\left(\frac{P_{t+1}}{P_t}\right) + \gamma,$$

real indeterminacy disappears and the inflation rate is pinned down. The price level however is indeterminate because the interest rate rule can now accommodate any level of nominal balances consistent with price expectations, as noted by Sargent and Wallace [1975] for policies that peg the nominal interest rate. Woodford [1996], [1991] has argued that the price level can nevertheless be pinned down to eliminate nominal indeterminacy if we introduce departures from "Ricardian equivalence," that is if the government does not follow policies that are solvent under all possible equilibria. Woodford's distinction between Ricardian and non-Ricardian regimes is similar to Leeper's [1991] distinction between *active* and *passive* monetary policies. In the case of a non-Ricardian policy it is the requirement that private agents have confidence in the solvency of the government budget rule that 'selects' an equilibrium. In this case Woodford argues that the price level is determined by fiscal, rather than monetary, policy.[24] (See also Sims [1997].)

The discussion of Leeper's analysis makes clear that the presence and nature of indeterminacy is influenced by the existence of lags in the implementation of policy. In particular, with separability of consumption and real balances in the utility function, real indeterminacy would be ruled out in a continuous-time formulation, unless delays were explicitly introduced into the policy rule. The continuous-time formulation of monetary dynamics under interest rate policy rules given in the next subsection below also illustrates this point. On the other hand, even with separability in the utility function and interest rules where the nominal rate responds to contemporaneous inflation, the slightest price stickiness can convert price level indeterminacy into real indeterminacy, as shown later in the next subsection, in the discussion of the sticky price model of Calvo [1983].

5.3 Interest Rate Rules and Indeterminacy

In general, policy rules that tie the interest rate to current inflation, or in a stochastic model to expected inflation, imply growth rates for nominal balances that are not constant. By backing out the implied growth rates of nominal balances, we can study equilibria of monetary models of the type studied in section 4. To illustrate how such monetary feedback policies can generate indeterminacies, we will use a simple continuous time model, based on work by Benhabib-Schmitt and Uribe [1998].

We begin by describing the structure of the private economy. For simplicity we assume that there is no production, and that a representative agent receives a constant, nonstorable endowment e at each moment in time. The agent carries wealth from one period to the next by holding bonds, B, and money real balances, M. We define the real holdings of these two assets as $m = \frac{M}{P}$ and $b = \frac{B}{P}$, where P is the price level and we let a refer to total real assets so that at each instant $a = b + m$. Money balances do not pay interest but bonds pay a nominal interest which we denote by i. The flow budget constraint of an agent is given by

$$\dot{a} = a\left(i - \pi\right) - im + e + T - c,$$

where $\pi = \frac{\dot{P}}{P}$ is the inflation rate and T are lump-sum transfers or taxes.

Now we turn to the structure of preferences. We denote the utility function of the agent by $U\left(c, m\right)$, which we assume to be increasing and concave in consumption and real balances and we assume that the agent maximizes the discounted sum of utilities over an infinite horizon, with discount rate ρ.

The first order conditions for this problem yield the equation

$$U_c\left(c,m\right)=p$$

(this is the monetary analog of (3.1) given in section 3.1), the portfolio condition which equates the return on bonds to the marginal benefit of holding money:

$$i=\left(\frac{U_m}{U_c}\right), \tag{5.4}$$

and the following Euler equation which is the analog of equation (3.4):

$$\frac{\dot{p}}{p}=\rho+\pi-\frac{U_m}{U_c} \tag{5.5}$$

Since endowments are constant, market clearing in goods requires that $c=e$, so that $\dot{c}=0$. Totally differentiating $U_c\left(c,m\right)=p$, and noting that $\dot{c}$ is zero, we have $U_{cm}\dot{m}=\dot{p}$. If we substitute this into equation 5.5, and use the money market identity

$$\pi\equiv\left(\sigma-\frac{\dot{m}}{m}\right), \tag{5.6}$$

where σ is the growth rate of nominal money balances, equation (5.5) becomes

$$\frac{\dot{p}}{p}=-\varepsilon_{cm}\frac{\dot{m}}{m}=\left(\rho+\sigma-\frac{\dot{m}}{m}-\frac{U_m}{U_c}\right) \tag{5.7}$$

where $\varepsilon_{cm}=\left(\frac{-U_{cm}m}{U_c}\right)$.

To discuss policy we use a continuous time version of the same rule given by equation (5.3). Notice that unlike the case when the central bank picks σ, we must find an expression for σ as a function of the interest rate by solving for σ from the policy rule. To accomplish this task we write the following representation of the monetary policy rule

$$i=R+\alpha\left(\pi-\left(R-\rho\right)\right)$$

where R and α are constants. If we use the definition of inflation (5.6) and the first order condition (5.4);

$$i=\frac{U_m}{U_c}=R+\alpha\left(\pi-\left(R-\rho\right)\right)=R+\alpha\left(\sigma-\frac{\dot{m}}{m}-\left(R-\rho\right)\right) \tag{5.8}$$

we can find an expression for the money growth rate σ:

$$\sigma = \frac{\dot{m}}{m} + \alpha^{-1}\left(\frac{U_m}{U_c}\right) - a^{-1}R + R - \rho \tag{5.9}$$

Finally, by substituting this expression into equation (5.7) we obtain the following differential equation that characterizes time paths for real balances that are consistent with equilibrium in the economy with interest rate feedback[25][26]:

$$\dot{m} = (\varepsilon_{cm})^{-1} m \left(\frac{U_m}{U_c} - R\right) \alpha^{-1} (\alpha - 1) \tag{5.10}$$

Once again it seems reasonable to suppose that money is a normal good. This implies that $\left(\frac{U_m}{U_c}\right)$ will be decreasing in m, which implies that the nominal rate i and the demand for real balances m are inversely related. This model has a unique steady state, defined by the level of real balances, $\hat{m}$, for which $\left(\frac{U_m}{U_c}[\hat{m}]\right) = R$. Further, the differential equation 5.10 will be stable if ε_{cm} and $(\alpha - 1)$ are of the same sign. Since α measures the sensitivity of the central bank to the inflation rate in setting he nominal interest rate, it follows that, depending on the sign of ε_{cm},there may be multiple equilibria with interest rate rules that are either sensitive $(\alpha > 1)$ or insensitive $(\alpha < 1)$ to the inflation rate.[27]

The mechanism at work here depends on the feedback rule: for example a rise in real balances causes the nominal rate, which in equilibrium must be equal to $\left(\frac{U_m}{U_c}\right)$,to fall. This induces a tighter monetary policy that reigns inflation in. Therefore, even if $\left(\frac{U_m}{U_c}\right)$ declines, the net return to holding money may either increase or decrease, depending on the strength of the central bank response to the nominal rate. The other channel through which the demand for money is affected is through the effect of money on the marginal utility of goods, as discussed in section 4.1: depending on the sign of ε_{cm}, the demand for money may increase or decrease with a rise in real balances. Therefore both α and ε_{cm} play a role in determining the nature of the dynamics of m and the stability of the steady state.

The results in this section also cover a cash-in-advance economy as a special case. A cash-in-advance model is equivalent to having money in the utility function where consumption and money are combined with a CES aggregator, which in the limit becomes a Leontief production function as the elasticity of substitution goes to zero. Since in such a case $\varepsilon_{cm} < 0$, indeterminacy with the interest rate rule used above and a cash-in-advance constraint is only possible if $\alpha < 1$[28].

5.4 Monetary Models and Sticky Prices due to Frictions in the Trading Process

In this section we will discuss the role of interest rate rules in *generating* indeterminacy under "sticky" prices. Recently Woodford [1996], Chari, Kehoe and McGrattan [1996], Gali, Gertler and Clarida [1997], and Kiley [1997], among others, have studied models with sticky prices, based on variants of a staggered price setting model originally due to Calvo [1983] and Taylor [1980]. These papers study monetary policies that target the nominal interest rate as a function of past or current inflations, and each of them has noted the possibility of indeterminacy in models in which staggered price setting is assumed to be part of the environment rather than part of the equilibrium concept. One approach to modelling sticky prices due to Calvo [1983] is to specify that firms can change their prices at random intervals, but with a continuum of firms, a fixed fraction of them can do so at each instant. The firms set their prices fully expecting that their price will remain fixed over a random interval while some of the other firms will change their price, and aggregate demand will also vary. This structure may be interpreted as one of monopolistic competition with firms facing downward sloping demand curves which depend on aggregate demand and on the prices of other firms.

The following example is based on the Calvo model. We assume that money enters the utility function and we write the Euler equation for consumption as:

$$\dot{c} = \frac{c}{\varepsilon_c}\left(\frac{U_m}{U_c} - r - \pi\right) \tag{5.11}$$

where the portfolio condition again implies that $i = \left(\frac{U_m}{U_c}\right)$. Substituting the policy rule for the nominal interest rate given by the first equality in equation (5.8) into equation (5.11), we can rewrite the Euler equation as:

$$\dot{c} = \frac{c}{\varepsilon_c}\left(\left(a-1\right)\left(\pi - \left(R - r\right)\right)\right) \tag{5.12}$$

Under sticky prices the inflation rate π is governed by the dynamics of staggered prices which leads to the following equation describing the rate of change of inflation:

$$\dot{\pi} = b\left(q - c\right) \tag{5.13}$$

Here q and b are a constants, with q representing a capacity level associated with full employment: $(q - c)$ may be interpreted as excess aggregate demand[29].

Equations (5.13) and (5.12) constitute a system of differential equations in (c, π) ,where neither c nor π are predetermined variables and the local dynamics of these equations depend on the Jacobian of the system evaluated at the steady state. If $a < 1$ the steady state is indeterminate since the Jacobian of the linearized dynamics around the steady state has one negative root. If $a > 1$, the relevant roots are imaginary with zero real part, and the stability properties of the steady state depend on higher order terms of the Taylor expansion in the linearization.[30]

The novelty of the class of models with staggered price setting is that indeterminacy may arise for reasons that are *independent* of other mechanisms, in the sense that real indeterminacy may disappear if one removes staggered price setting. In our earlier formulation with flexible prices and a nominal interest rate feedback rule, real indeterminacy was only possible if money entered the utility function in a non-separable manner. But with Calvo–style price setters, real indeterminacy may occur even with a separable utility function. It follows that real indeterminacy in this case is attributable directly to the monopolistically competitive price setting mechanism that we introduced to model sticky prices[31]. One way to interpret these results is to note that price level indeterminacy that occurs under interest rate rules and flexible prices with separable preferences, turns into real indeterminacy as soon as we introduce some degree of price stickiness. This is in contrast to our earlier discussion of indeterminacy in monetary models in which sticky prices implement one of the possible set of equilibria. In contrast, in the staggered price setting literature it is the sticky prices that *cause* indeterminacy rather than the other way around.

6 Indeterminacy and Models of Endogenous Growth

Our discussion so far has centered on models of business cycles. Another important area in which indeterminacy plays role is economic growth. Recently, Levine and Renelt [1992] demonstrated the lack of robustness of many of the empirical results explaining the differences in the growth rate of countries by institutional, and policy differences, and by differences in their rates of factor accumulation, initial wealth and income distribution. The presence of indeterminacies offers an additional and complementary explanation of why countries that have very similar endowments and fundamentals, nevertheless save and grow at different rates.[32]

The recent literature on endogenous growth, initiated by Lucas [1988] and Romer [1990] contain elements of market imperfections that can be shown to generate indeterminacies under reasonable parametrizations. In contrast to the business cycle literature however, in models of endogenous growth it is the balanced growth path that is indeterminate, rather than the steady state level of GDP. The distinctive feature of endogenous growth models is their production technology which allows proportional growth in some accumulated assets like human or physical capital, or the stock of research and development. The fact that the technology allows for linear growth implies that there must exist increasing returns at the social level to overcome diminishing returns at the private level. It is a small step from here to generate indeterminacy through complementarities between the factors of production.

An interesting feature of endogenous models is their ability to generate multiple balanced growth paths in conjunction with indeterminacy. We can illustrate how multiple balanced growth paths and indeterminacy can arise in such models with small modifications to the simple structure of the equations (3.1), (3.2), (3.3) and (3.4). We will rely on a simple illustrative structure of production that is linear in an accumulated asset, and with sufficiently strong external effects from the labor input. Our endogenous growth model will have a balanced growth path, along which the ratio of the asset to consumption will be constant. If we denote the asset by k,we will have $k = sc$,where c is consumption and s is a constant. For simplicity lets assume that the utility of consumption is logarithmic, and that the production function is of the Cobb-Douglas form, $y = k^{\alpha}\bar{k}^{(1-\alpha)}L^{\beta}$ where $\bar{k}$ represents an external effect. Consider the endogenous growth version of equation (3.4), where we have replaced p by c using equation (3.1):

$$\frac{\dot{c}}{c} = \left(w_1\left(L\right) - \left(r+g\right)\right) \tag{6.1}$$

Note that since y is linear in k, w_1 only depends on L, and is given by $w_1 = \alpha L^{\beta}$. We can also write equation (3.3) for the goods market equilibrium as

$$\frac{\dot{k}}{k} = a(L) - s - g \tag{6.2}$$

where $a(L)$ is the average product of capital and only a function of L because y is linear in $k: \ a(L) = L^{\beta}$. Since s is a constant along the balanced growth path, the difference between the right hand sides of (6.1) and (6.2) must be zero[33]:

$$a(L) - s - w_1(L) + r = s - r + a(L)(1-\alpha) = 0 \tag{6.3}$$

The second equality follows because the marginal and average products of capital, $a(L)$ and $y_1(L)$, are proportional, and in our Cobb-Douglas example their difference is $a(L)(1-\alpha)$. We can also express s as a function of L by using the labor market equilibrium condition given by equation (3.2):

$$s = \frac{c}{k} = \frac{m(L)}{V'(1-L)} = v(L) \tag{6.4}$$

Here $m(L)$ is the marginal product of labor divided by k. Substituting this expression into (6.3) we have:

$$a(L)(1-\alpha) - v(L) = r \tag{6.5}$$

Equation (6.5) can have one, two or no solutions corresponding to the balanced growth paths, depending on the parameters of the model. The right-hand side of equation (6.5) is monotonic in L if $v(L)$ is decreasing, but if $v(L)$ is increasing, there may be two balanced growth paths. An increasing $v(L)$ however is only possible if the marginal product of labor is increasing, and this requires a significant labor externality. This is precisely what happens in the endogenous growth version of the model in Benhabib and Farmer [1994], when the labor externalities are high enough. One of the balanced growth paths is determinate, while other is indeterminate.

A more extensive analysis of a related mechanism in the Lucas model [1988], with small external effects confined only to the research sector, is given in Benhabib and Perli [1994] (see also Xie [1994]). They show that multiple balanced growth paths and indeterminacy can appear for reasonable parametrizations of the model.[34] A similar mechanism produces indeterminacy and multiple balanced growth paths in Romer's model [1990], as analyzed by Benhabib, Perli and Xie [1994] and by Evans, Honkapohja and Romer [1996]. Evans, Honkapohja and Romer [1996] also study a modification of the Romer model by introducing adjustment costs that generate a nonlinear production possibility curve between the consumption and investment sectors. Their model has three balanced growth paths, two of which are stable under a learning mechanism. Introducing sunspots induces jumps across the two stable (indeterminate) balanced growth paths, and generates fluctuations in the growth rate. Such regime switching equilibria giving rise to sunspot fluctuations in the growth rate are also studied, both theoretically and empirically, in Christiano and Harrison [1996]. As in Benhabib and Perli [1994], they observe that indeterminacy can arise even if the balanced growth paths are locally determinate, because rates of investment can be chosen to place the economy on either one of them.

Another specification generating indeterminacy is given in the endogenous growth model of Gali and Zilliboti [1995]. They use a model with monopolistic competition, coupled with fixed costs and entry. Markups are inversely related to entry and to the capital stock, so that raw returns can increase in k. This model is an endogenous growth version of the variable markup model of Gali [1994]. It gives rise to two balanced growth paths, one with zero growth representing a corner solution, and the other one with a positive growth rate. Furthermore there is a range of initial conditions for capital in which the equilibrium trajectory is indeterminate, and may converge to either of the balanced growth paths depending on the initial choice of consumption.

7 Some Related Work

So far the framework presented by equations (3.1)-(3.4) assumed that the preferences were standard, and in particular that the discount rate was constant. We may however allow the discount rate to be affected by some social norm, proxied for example by the value of aggregate consumption. If preferences and the discount rate are subject to such external effects, it is clear from equation (3.4) that they can substitute for external effects and increasing returns in technology. A higher price of investment may lead to a higher capital stock, and may well decrease the marginal returns to capital. If the discount rate declines as well however, increasing price appreciations in the price of the capital good may be unnecessary to sustain equilibrium. The price of the investment good may well decline, and move back towards its stationary equilibrium value, generating indeterminacy. Such a mechanism is explored in detail in a recent paper by Drugeon [1996]. In general, endogenous preferences coupled with some market imperfections are likely to provide a basis for multiple equilibria and indeterminacy.

An alternative route to indeterminacy may be through increasing returns not in the production function, but in the utility function. In such a setup there must be sufficient discounting of the future to assure that utilities remain finite in equilibrium. In a recent paper Cazavillan [1996] studies indeterminacy in such a model, where public goods financed by taxes enter the (constant returns to scale) production function. Since the public goods are productive, they create externalities because agents take the tax rate as given. The result is an endogenous growth structure with an indeterminate balanced growth path.

Indeterminacy can also arise from variations in capacity utilization if utilization rates co-move with labor, as would be the case if intensified utilization accelerates capital depreciation. This possibility has recently been shown by Wen [1998]. In his model a shift in production towards investment will raise the capital stock, but an associated increase in labor will cause the marginal product of capital to increase rather than decrease, very much like the model of Benhabib and Farmer [1994]. The reason for the expansion in labor however is not an upward sloping demand curve for labor due to external effects, but a rightward shift in the labor demand curve due to increased capacity utilization. Wen calibrates his model to US data and finds that indeterminacy can provide a remarkably good match to the data with mild increasing returns in the order of 0.1.

Guo and Sturzenegger [1994] study the application of indeterminacy to the study of international consumption data. The RBC model has trouble with the fact that consumption across countries is predicted to be perfectly correlated under simple variants of the international RBC model with complete markets. But in practice the correlation between consumption across countries is quite low. The Guo and Sturzenegger explanation drives business cycles with sunspots as in the single country model of Farmer and Guo [1994], but they assume that agents are unable to perfectly insure across countries. Their calibrated model does a fairly good job of explaining the cross country data and is one of the first applications of empirical models of indeterminacy to international data sets.

We should note that we have not touched upon the literature that deals with indeterminacy in overlapping generations models or in finite markets with incomplete participation or market imperfections. Some recent overviews of these topics can be found in Balasko, Cass and Shell [1995] or Bisin [1996], among others.

8 Empirical Aspects of Models with Indeterminacy

In Section 2 we mentioned two areas in which models with indeterminate equilibria might potentially improve upon existing models of the business cycle. The first is that of propagation dynamics and the second is related to monetary features of business cycles. In this section we elaborate on the claim that indeterminacy might be a fruitful research direction by surveying known results in which some progress has been made on each of these issues.

8.1 Real Models and Propagation Dynamics

The real business cycle literature represented a major departure from the Keynesian models that preceded it. On the theoretical front RBC theorists argued that the correct way forward for macroeconomics is some version of dynamic general equilibrium theory. On the empirical front they argued that the standards for what should be considered a successful description of the data should be considerably relaxed from the requirements imposed by time series econometricians. Following the approach initiated by Kydland and Prescott [1990], much of the RBC literature dispenses with attempts to study the low frequency components of time series by passing data (both actual and simulated) through a filter that leaves only high frequency components.[35] If simulated data from an artificial model can replicate a few of the moments of the data from an actual economy then RBC economists argue that the model is a successful description of the real world.

There is much to disagree with in the RBC methodology. It has nevertheless had the effect of providing a unified framework for comparing and evaluating alternative economic theories. In this section of the survey we will turn our attention to calibrated models of indeterminacy that have used the RBC methodology to provide competing explanations of business cycle phenomena. These models all build on some simple variant of a representative agent economy and the variables they describe includes consumption, investment, GDP and employment as a subset. It is therefore possible to ask how their predictions compare with those of the benchmark model[36].

8.1.1 One Sector Models

In Section 2.2 we pointed out that the one sector real business cycle model, driven by productivity shocks, has a representation as a difference equation in three state variables. We reproduce this equation below:

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} = \Phi^{-1} \begin{bmatrix} \tilde{c}_{t} \\ \tilde{k}_{t} \\ \tilde{s}_{t} \end{bmatrix} - \Phi^{-1}\Gamma \begin{bmatrix} \tilde{e}_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix}. \tag{8.1}$$

The variables $\tilde{c}, \tilde{k}$, and $\tilde{s}$ represent deviations of consumption, capital and the productivity shock from their balanced growth paths; $\tilde{e}$ is a belief shock and $\tilde{u}$ is an innovation to the productivity shock. The variables k_{t+1} and s_{t+1} are determined at date t but c_{t+1} is free to be determined at date $t+1$ by the equilibrium conditions of the model. If the matrix Φ^{-1} has three roots inside the unit circle then it is possible to construct equilibria in which the business cycle is driven purely by iid sunspot errors (the variable e_{t+1}) and the artificial data constructed in this way can be compared with actual data in the same way that one matches RBC models by comparing moments. This idea was exploited by Farmer and Guo [1994] who pointed out that there are some dimensions in which the sunspot model can perform better than models driven by fundamentals. We return to this idea shortly.

To get a better idea for how an array of sunspot models compare with each other, with the RBC model and with the data, Schmitt Grohé analyses four different models all of which are calibrated in a similar way and all of which have a representation of the kind illustrated in equation (8.1). The models that she studies are 1) a model similar to that of Gali [1994] in which changes in the Composition of Aggregate Demand (the CAD model) allow the markup to be countercyclical; 2) a model based on Rotemberg and Woodford [1982] in which markups may again be countercyclical but in this case the variability of the markup follows from Implicit Collusion (the IC model); 3) a model with increasing returns and decreasing marginal costs (the IR model) and finally 4) a model with externalities (the EXT model) based on the work of Farmer and Guo [1994]. The main question addressed by her work is "For what values of the parameters can the matrix Φ^{-1} in equation (8.1) have three roots all inside the unit circle?" This is an interesting question in light of the results of Farmer and Guo [1994] since, when all of the roots of Φ^{-1} are inside the unit circle, one can generate artificial time series for consumption, investment, hours and GDP by simulating sequences of variables using the equation:

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \end{bmatrix} = \Phi_{11}^{-1} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \end{bmatrix} - \left[\Phi^{-1}\Gamma\right]_{11} \left[\tilde{e}_{t+1}\right]. \tag{8.2}$$

where Φ_{11}^{-1}, and $[\Phi^{-1}\Gamma]_{11}$ are the 2×2 upper left blocks of Φ^{-1} and $[\Phi^{-1}\Gamma]$. The formulation of the model in equation (8.2) is one in which equilibrium business cycles in which all agents are fully rational are driven *purely by sunspots.*

Schmitt-Grohé [1997] simulates series of artificial data for all four types of one sector model. In each case she calibrates the baseline parameters as in a standard RBC model, laid out in work by King Plosser and Rebelo [1987a], and she sets the increasing returns, externality and markup elasticity parameters in a way that minimizes the degree of aggregate increasing returns but still allows equilibria to be indeterminate.

Table 1 reproduces sections a, b and c of Table 6, page 136 in Schmitt-Grohé [1997]. The first two columns of this table are reproduced from King Plosser Rebelo [1987a] and they illustrate the dimensions on which the RBC model is often evaluated. The column labeled "RBC model" contains statistics generated by simulations of a 'standard' RBC model in the source of business cycle dynamics is a highly persistent productivity shock. Column 1, for comparison, is the U.S. data. Columns 3 through 6, are statistics generated by Schmitt-Grohé in which each of the four sunspot models are used to simulate data but, in contrast to the RBC model, each of these columns simulates data that is generated by a pure sunspot shock. The main point of this table is to illustrate that by the standards of the calibration literature, the models driven purely by sunspots perform about as well. This in itself is an interesting observation because although early work on sunspot models had demonstrated that sunspots could exist, much of this literature had little or no connection with data.

Earlier in this survey we drew attention to two aspects in which sunspot models with indeterminate equilibria are different from standard models with a unique equilibrium. The first was that models with sunspots can generate an alternative source of the impulse to the business cycle and it is this claim, that sunspots may be a primary impulse, that is evaluated in Table 1. A second, and perhaps more interesting feature of models with indeterminacy, is that they offer an alternative explanation of propagation dynamics.

To evaluate this claim, Farmer and Guo generate a set of impulse response functions from three different models and they compare these impulse response functions with those from US data. The impulse response functions to innovations in output for US data are derived from a vector autoregression of output, employment, consumption and investment with a linear time trend and five lags, over the period 1954.1 to 1991.3. The three models are a standard RBC economy (the same calibration as the RBC economy in table 1) and two different models with externalities. One of these models is calibrated as in work by Baxter and King [1991] who introduce externalities but calibrate these externalities in a way that is *not* large enough to generate indeterminacy. The second is a calibration *with* indeterminacy in line with the EXT model discussed by Schmitt-Grohé.

Figure 1 compares the impulse responses in each of these three models with the impulse response to a set of shocks in the US data. Notice, in particular, the dynamic pattern of investment in the data and compare it with models 1 and 2 in Figure 1. The impulse responses for US data shows clear evidence of a cyclical response pattern whereas models 1 and 2 (the RBC model and the externality model *without* indeterminacy) both show monotonic convergence patterns. Farmer and Guo point out that monotonic convergence in the RBC economy follows from the fact that, although the dynamics in k and s are two dimensional, there is no feedback in the dynamics of the system from the productivity shock s to the capital stock k. The system

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} = A \begin{bmatrix} \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} + B\tilde{u}_{t+1}$$

that characterizes the RBC dynamics has monotonic impulse response functions because the matrix A is upper triangular and it necessarily has two real roots. The equation

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{bmatrix} = \bar{A} \begin{bmatrix} \tilde{k}_t & \\ \tilde{c}_t & + \bar{B}e_{t+1} \end{bmatrix},$$

on the other hand, that characterizes the dynamics of the sunspot models incorporates feedback both from c_t to k_t and vice versa hence the matrix $\bar{A}$ that determines the properties of the impulse response functions in this case *can* have a complex roots. It is this feature that Farmer and Guo exploit to generate the features of the dynamic responses illustrated in Figure 1.

Although the one sector models discussed above do a relatively good job of describing data they rely on large markups and imperfections to generate indeterminacy, which may not be empirically plausible. Schmitt-Grohé [1997] concludes that while "...the relative volatility, autocorrelation, and contemporaneous correlation properties of macroeconomic aggregates predicted by each of the endogenous business cycle models are broadly consistent with those actually observed in the US data...the degree of market power or returns to scale required for the existence of expectation driven business cycles lies in the upper range of available empirical estimates.." The more recent models of Perli [1994], Schmitt and Uribe [1997] and Wen [1998], which modify the one-sector model by introducing home production, taxes, and variable capacity utilization, are also successful in their calibration analysis but they avoid the high degree of increasing returns to scale required by Benhabib and Farmer [1994] to generate indeterminacy. In the next section we discuss the empirical performance of multi-sector models which do not rely on large market distortions or external effects to generate indeterminacy.

8.1.2 Two-Sector Models

In this section we discuss a class of two-sector models that are able to generate indeterminate equilibria for much lower degrees of returns to scale or market imperfections than the one sector models discussed above. However this the lower increasing returns makes it much harder to obtain procyclical consumption by relying exclusively on sunspot shocks and ruling out technology shocks. Indeed one of the most successful features of the RBC models is their ability to deliver procyclical consumption and to avoid countercyclical wages implied by the neoclassical model without shocks to technology. The discussion below centers on the issue of procyclical consumption in calibrated multi-sectors models that require little or no increasing returns to generate indeterminate equilibria.

We begin our discussion with the calibrated two sector model of Benhabib and Farmer [1996a], discussed in section 3.3. This model can generate indeterminate equilibria for sector specific external effects that are significantly milder than those needed for models with a one sector technology; one obtains indeterminacy for returns to scale in the consumption and investment sectors of about 1.07, when one assumes that net of external effects, the firms face production technologies that exhibit constant returns. But although 1.07 will generate indeterminacy, it is not enough to successfully match the various moments of US macroeconomic data at least in the case when business cycles are solely driven by sunspots. Successfully matching the data requires returns to scale of around 1.2. Unlike the earlier model of Benhabib and Farmer [1994], returns to scale of 1.2 does not imply an upward sloping labor demand curve, but it still remains high in light of recent empirical work by Basu and Fernald [1997] and others.

The main reason that a high externality is needed for a reasonable calibration is to assure that consumption is procyclical when the only stochastic shocks in the model are sunspots. We can easily illustrate this point, following the discussion in Benhabib and Farmer [1996a]. Let $U'(C)$ be the marginal utility of consumption, $V'(-L)$ be the marginal utility of leisure and $MPL(L)$ the marginal product of labor where for simplicity we ignore the dependence of MPL on capital. The first order condition for the choice of labor in a standard one-sector model takes the form: $U'(C)\,MPL(L) = V'(-L)$. Suppose that employment increases spontaneously in this model, as would be the case if "sunspots" were the dominant source of fluctuations. In this case the increase in L would decrease MPL and increase $V'(L)$: equality for the first order condition for the labor will be restored only if C were to fall and $U'(C)$ to rise. In other words, pure sunspot fluctuations will cause consumption to be countercyclical. In the following discussion we identify several channels that might break this link.

(1) The first possibility is that demand and or supply curves may have non-standard slopes. If the marginal product of labor, MPL, is increasing in L, which gives an upward sloping labor demand, or if $V'(L)$ is decreasing in L, which gives a downward sloping labor supply curve, then an increase in L may be associated with an increase in C. When we estimate a model that involves this first order condition, the procyclical consumption in the data may well force the estimated parameters to imply an upward sloping demand, a downward sloping supply, or both this, for example, is exactly what Farmer and Guo [1994] find when they estimate a one sector model. The existence of an upward sloping demand curve for labor requires externalities or monopolistic competition, but a downward sloping supply curve can occur even when utility functions are concave. For example, an alternative specification of utility that permits procyclical consumption would replace $U'(C)$ and $V'(-L)$ with $U_1(C,L)$ and $U_2(C,L)$. This non-separability may allow the labor supply curve to slope down even in the absence of externalities. However, one may show that a downward sloping labor supply curve also implies that consumption is an inferior good.

(2) A second way in which one may reintroduce procyclical consumption follows from work on monopolistic competition. In this setting the relevant variable for the first order condition for labor is not MPL, but MPL adjusted for the markup. If the markup is constant the conclusions that follow from the first order condition are unchanged, but if the markup is countercyclical, then procyclical consumption can be rescued, as is the models of Rotemberg and Woodford [1992] or Gali [1994]

(3) All of the above discussion is concerned with the difficulty of explaining procyclical consumption in models in which all shocks arise from sunspots as for example, in Farmer and Guo [1994]. Procyclical consumption is easier to obtain with technology shocks since in this case output may rise sufficiently to allow both investment and consumption to increase in response to a positive shock, even though labor may move out of the production of consumption goods to the production of investment goods. Indeterminacy would still remain, so that given the capital stock and the realization of the technology shock, investment and consumption would not be uniquely determined. In other words, even if one thinks that technology shocks provide the impulse to the business cycle indeterminacy still has a considerable amount to add to the story by providing a plausible explanation of an endogenous propagation mechanism. Benhabib and Farmer [1996a] pursue this last route in their calibration, and with the help of increasing returns of 1.2, obtain a correlation of 0.54 between consumption and GDP. The same calibration with sunspot shocks alone gives a correlation of 0.32, which is still positive due to movements in the capital stock, but low relative to this correlation in US data. Lowering the external effects so that returns to scale are only of the order 1.11 yields countercyclical consumption[37].

(4) An alternative approach is to introduce a naturally countercyclical sector that will feed labor into the economy during booms, and absorb labor during recessions. The "home" sector will serve that purpose, even in the absence of technology shocks, and will deliver procyclical consumption as well as procyclical employment in the consumption sector. In such a setup ignoring the home sector and the movements of labor between home and market may indeed make it seem as if leisure is inferior. A calibrated model of indeterminacy and sunspots along such lines is given by Perli [1994].

The model of Benhabib and Farmer [1996a], as discussed in section 3.3, relies on identical technologies in the two sectors, which nevertheless give rise to a non-linear production possibilities frontier because of sector specific externalities. In a multi-sector model without identical technologies, the marginal products of labor and the capital goods depend not only on factor stocks, but on the composition of output, which is endogenous. As pointed out in section 3.4, this may allow the marginal product of a capital good to increase in response to an increase in its stock and give rise to indeterminacy, even tough we have constant marginal costs in the production technology. Furthermore, this may also alleviate the difficulty of obtaining procyclical consumption because the marginal product of labor now depends not just on L, but on the composition of output.

8.1.3 Multi-sector Models

Benhabib and Nishimura [1996] calibrate a three sector model under constant social returns using a standard RBC parametrization. The presence of external effects coupled with constant social returns result in private decreasing returns, and necessitates some fixed costs to prevent entry. These however can be taken to be small the external effects are also small, implying private decreasing returns of the order 0.93 in each sector. Utility is assumed logarithmic in consumption and separable between leisure and consumption, and is parametrized to imply a labor supply elasticity of 5. The production functions are Cobb-Douglas, and the quarterly discount rate is taken as 0.11. The model allows for *iid* sunspot shocks, as well as technology shocks driven by a first order autoregressive process with standard persistence parameters. Table 2 below gives the moments of simulated data, with numbers in parentheses corresponding to US quarterly data[38]: In the table above, investment corresponds to its aggregated value, evaluated at the current relative prices of the two investment goods. GNP contains consumption plus investment, with the price of the consumption good normalized to unity each period. The impulse responses, generated by the linearized dynamics of the system around the steady state, are driven by positive real roots within the unit circle, and resemble the hump-shaped impulse responses generated with vector-autoregressions on US data.

Figure 2 shows the impulse responses for consumption, investment and GNP, generated by an aggregate productivity shock impacting the three sectors simultaneously. The aggregative shock leads to a surge of investment, initially at the expense of consumption. Again we find that this feature, that is the initial negative response of consumption to the aggregative technology shock, typically arises for standard RBC calibrations of multi-sector models whether or not they have any external effects or exhibit indeterminate equilibria. GNP also drops by a small amount when the shock hits, but rises immediately afterward as investment surges, and then subsides, generating the hump-shaped response associated with the data. Another feature, shared with calibrated multisector models without external effects or market distortions that have determinate equilibria, is that prices and outputs of investment goods tend to be more volatile than the aggregated value of investment, with some sectors even exhibiting countercyclical behavior (see for example Benhabib, Perli and Plutarchos [1997]. These counterfactual observations about calibrated multi-sector models in the context of a determinate economy has led Huffman and Wynne [1996] to introduce adjustment costs for the sectoral reallocations of factors of production. It seems then that, with or without sunspots and multiple equilibria, the multisector real business cycle models solve some of the empirical issues encountered in simpler one-sector models, but also introduce empirical complications of their own[39].

8.2 Monetary Models and the Monetary Transmission Mechanism

A second area in which calibrated monetary models are registering some progress is in describing the dynamics of the monetary transmission mechanism. Recall that the model described by Farmer [1997] has a representation as a difference equation of the form

$$\begin{bmatrix} \mu_{t+1} \\ C_{t+1} \\ K_{t+1} \\ m_{t+1} \end{bmatrix} = A \begin{bmatrix} \mu_t \\ C_t \\ K_t \\ m_t \end{bmatrix} + B \begin{bmatrix} u^1_{t+1} \\ u^2_{t+1} \\ e^1_{t+1} \\ e^2_{t+1} \end{bmatrix} \tag{8.3}$$

where μ is the money growth rate, C is consumption, K is capital and m is real balances. The variables u^1 and u^2 are fundamental shocks and e^1 and e^2 are sunspots shocks. The model has two variables, μ_{t+1} and K_{t+1} that are determined at date t and two variables, m_{t+1} and C_{t+1} that are free to be determined by the equilibrium behavior of agents in the model. The condition for there to be a unique rational expectations equilibrium is that two of the three roots of the matrix A are inside, and two roots are outside, the unit circle. There are two possible dimensions for sunspots to influence the equilibrium of this model depending on whether three or four of these roots are stable. Farmer shows that it is relatively easy to choose calibrated values of the parameters in a way that makes all four of these roots lie within the unit circle and, in this case, he shows that one is free to pick stationary iid white noise processes for each of the two sunspot variables, e^1 and e^2. He then goes on to show that the variance-covariance matrix of the vector $\{u^1, u^2, e^1, e^2\}$ can be estimated from the residuals of a vector autoregression on US data. [40]

The important point from this discussion, is that it suggests an empirical approach to the resolution of indeterminacy. If agents live in world that is well described by a model in which equilibrium is indeterminate; these individuals must still act. To make a decision it is necessary to form an expectation of what will happen; the fact that there are many possible expectations that might be fulfilled is not a problem for the agent. He need only pick one of them. Farmer argues that there is some coordination mechanism that causes agents to act in a particular way and that this mechanism can be represented by a fixed expectation function. The econometrician will observe the outcome of this expectation function. Lets suppose that agents have solved the coordination problem and that they coordinate on *the same* equilibrium from one year to the next. This implies that the response to a given constellation of fundamental shocks will have the same probability structure in each year. In terms of the monetary VAR described by equation 8.3, the particular equilibrium in which we find ourselves will show up in the covariance matrix of the residuals of the VAR. Each of the possible sunspot equilibria will result in a *different* joint covariance structure of the sunspot shocks with the fundamentals. We expand on this point in the following section in which we address some criticisms that have been leveled against models with indeterminacy as descriptions of data.

9 Some Criticisms of the Use of Models with Indeterminate Equilibria to Describe Data

In this section we evaluate and address a number of concerns that have been raised by critics of models of indeterminacy and of the use of indeterminacy to explin economic data. We begin with the issue of how an equilibrium is chosen in a model where many things can happen.

9.1 Equilibrium Selection

In any model with multiple equilibria one must address the issue of how an equilibrium comes about; this is true of finite general equilibrium models with multiple determinate equilibria and it is, a fortiori, true of dynamic models with indeterminate equilibria. In dynamic models one thinks of the economy as evolving in a sequence of periods. In each period, agents form forecasts of future prices and they condition their excess demand functions on these forecasts. In an economy with a finite number n of commodities each period and a finite number m of agents with time separable preferences one can write the equilibrium of the economy as the solution to a set of equations that set excess demand functions equal to zero:

$$Z_t \left(p_t, p_{t+1}^{1E} \left[s_{t+1} \right], \ldots p_{t+1}^{mE} \left[s_{t+1} \right], W_{t+1}^{1E}, \ldots W_{t+1}^{mE}; s_t \right) = 0 \tag{9.1}$$

where Z_t is the n dimensional vector of excess demands, s_t is the state of nature, p_t is the n dimensional vector of prices, $p_{t+1}^{iE} \left[s_{t+1} \right]$ is the i'th agents belief of the value of the price vector at date $t + 1$in state of nature $s_{t+1}$and W_{t+1}^{iE} is the i'th agent's belief of the value of his wealth. Wealth must be forecast since it depends on the present value of future prices in all possible realizations of states.

Rational expectations is the assumption that all agents know future state dependent prices and can therefore correctly forecast their future wealth. When a model has a unique equilibrium, the set of excess demand functions at each date has a unique solution for p_t when each expected price vector is replaced by the actual price vector in that state and when the wealth of each agent is computed accordingly. When a model has an indeterminate set of equilibria, there are many solutions to these equations: excess demand functions alone, reflecting preferences and technology, are insufficient to pin down an equilibrium. In either case, the equilibrium assumption does not address the problem of how rational expectations comes about.

Most work on rational expectations models begins with the assumption that there is a representative agent thereby drastically reducing the complexity of the problem. The usual justification for rational expectations is to appeal to the assumption that the world is stationary, and to argue that in a stationary environment agents would eventually come to learn the unique set of state dependent prices. There is a body of work on out of equilibrium learning that begins by conjecturing that there exists a learning rule used by agents to forecast the future. The main result of this literature is to show that rules of this kind can select an equilibrium. Initially, some authors conjectured that 'plausible' learning rules would always select a determinate equilibrium but this has proved not to be the case. Woodford [1990], for example, has shown that a simple learning rule can converge to a sunspot equilibrium and Duffy [1994] has demonstrated that learning rules can converge to one of a set of indeterminate equilibria. Grandmont [1994] puts forward the view that the problem is so complex that agents are unlikely ever to learn how to behave in a rational expectations equilibrium.

For a more detailed exposition of the issues concerning equilibrium selection in models with endogenous cycles and sunspot equilibria is the reader is referred to the survey by Guesnerie and Woodford [1992].

9.2 Equilibrium Forecast Functions

A separate, but related question, is how a given equilibrium is maintained. It is all very well to assume that agents know future prices, but how do they behave in any given state? One possibility, is that agents use a forecast rule that maps from current and past observable variables to future values of state dependent prices. Consider a model of the form:

$$x_t^D = ap_t + bE_t\left[p_{t+1}\right], \tag{9.2}$$

$$x_t^S = s_{t,}. \tag{9.3}$$

where x_t^D is aggregate demand, s_t is aggregate supply which we take to be an iid sequence of random variables with mean zero and bounded support and p_t is the log price. Equating demand and supply leads to a functional equation that must be satisfied by stochastic processes for p_t that are candidate equilibria :

$$p_t = \frac{1}{a}s_t - \frac{b}{a}E_t\left[p_{t+1}\right]. \tag{9.4}$$

There are two cases to consider. If $\left|\frac{b}{a}\right| < 1$ then there is a locally unique equilibrium given by the equation:

$$p_t = \frac{1}{a}s_t. \tag{9.5}$$

This is the case of a unique determinate equilibrium. But if $\left|\frac{b}{a}\right| > 1$ then there is a set of equilibria of the form:

$$p_{t+1} = \frac{1}{b}s_t - \frac{a}{b}p_t + e_{t+1} \tag{9.6}$$

where e_{t+1} is an arbitrary iid sunspot shock with zero conditional mean. In the determinate case, agents can forecast using the equilibrium function (9.5). Plugging this function into (9.2) leads to the expectation:

$$E_t\left[p_{t+1}\right] = E_t\left[\frac{s_{t+1}}{a}\right] = 0 \tag{9.7}$$

which implies that demand is given by the function:

$$x_t^D = ap_t. \tag{9.8}$$

As the current price varies, market demand varies with current price according to equation (9.8). A Walrasian auctioneer, calling out prices, would find a unique price, $p_t = \frac{s_t}{a}$ at which demand equals supply.

In the indeterminate case it is not so obvious how an equilibrium could be maintained. Suppose that agents forecast the future using the equilibrium pricing rule, equation (9.6). Substituting this rule back into equation (9.2) leads to the demand function:

$$x_t^D = s_t,$$

which is identical to supply for all possible values of p_t. Equation (9.6) cannot be used to forecast the future price since if agents were to use the equilibrium price function demand equals supply for *any* possible price. But although equation, (9.6) cannot be used to forecast, there *is* a rule that can. Suppose that agents use only lagged information to forecast the future price. In particular, suppose that they use rule:

$$p_{t+1} = \frac{1}{b}s_t + e_{t+1} - \frac{a}{b^2}s_{t-1} - \frac{a}{b}e_t + \frac{a^2}{b^2}p_{t-1} \tag{9.9}$$

which is obtained by lagging (9.6) by one period. We will refer to this rule as a *forecast function.* Using equation (9.9) they can compute an expectation of the price in period $t+1:$

$$E\left[p_{t+1}\right]=\frac{1}{b}s_t-\frac{a}{b^2}s_{t-1}-\frac{a}{b}e_t+\frac{a^2}{b^2}p_{t-1}. \tag{9.10}$$

Plugging this expectation back into equation (9.2) leads to the demand function:

$$x_t^D=ap_t+s_t-\frac{a}{b}s_{t-1}-ae_t+\frac{a^2}{b}p_{t-1} \tag{9.11}$$

that shows how current demand varies with price if agents use (9.9) to forecast. Equating demand to supply, it follows that the current price will be determined by the equation

$$p_t=\frac{1}{b}s_{t-1}+e_t-\frac{a}{b}p_{t-1} \tag{9.12}$$

which is the equilibrium pricing rule that we introduced in equation (9.6).

Lets recapitulate what we have said. We have shown that if agents forecast the future price using the forecast function (9.9) then the actual price will be described by the stochastic difference equation in equation (9.12). Since the forecast function was obtained by lagging the actual pricing rule, the forecast function is rational. To verify this one can substitute the equilibrium price rule (9.12) into the forecast function. Furthermore, the sequence of error terms e_t is arbitrary. We have shown that there are arbitrary forecast functions each of which can support a different rational expectations equilibrium.[41]

9.3 Does Indeterminacy Have Observable Implications?

Some authors have been concerned that models with indeterminate equilibria may not be useful models since, it might be thought, that anything can happen. This argument is false. In fact, models with indeterminate equilibria place relatively strong restrictions on the moments of data once one closes these models by specifying a process that determines the formation of beliefs. For example, consider the Farmer-Guo version of the RBC model with increasing returns. We showed earlier that this model is described by a set of equations of the form:

$$\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} = \Phi^{-1} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} - \Phi^{-1}\Gamma \begin{bmatrix} \tilde{e}_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix}. \tag{9.13}$$

It is true that if one allows the sequence of forecast errors $\tilde{e}_t$ to be arbitrary that this model allows additional freedom to describe the data.[42] But once one specifies a stationary stochastic process for the joint determination of sunspots and fundamentals, this model places strong restrictions on the joint process determining the evolution of the state variables. Indeed, Aiyagari [1995] has argued that these restrictions are falsified in data and this criticism of the Benhabib-Farmer [1994] model is in part responsible for the research agenda on two sector models that we described above. Although models of indeterminacy do place restrictions on data, these restrictions are often less severe than the standard real business cycle model. Indeed, it is the fact that some of the restrictions of the standard model are often rejected by the data that is one of the prime motivations for considering a wider class of economies.

10 Conclusion

The central theme of this chapter is that the standard infinite horizon model, modified to incorporate some mild market imperfection, often supports an indeterminate set of equilibria. When the non-stochastic version of a model has an indeterminate set of equilibria, variants of the model that explicitly incorporate uncertainty will typically support a continuum of stationary rational expectations equilibria some of which may be driven by sunspots. In this sense the property, that the equilibria of finite Arrow-Debreu economies are determinate, is fragile.

An implication of this argument is that minor perturbations of the (Hamiltonian) structure of a representative agent model allows self-fulfilling expectations to have a significant influence on the dynamics of prices and output. Furthermore, the economic mechanisms which give rise to such perturbations are varied, and the investigation of these mechanisms is a fruitful one, since it can potentially account for features of the time series data that are otherwise difficult to understand.

The models that we have discussed in this survey may lead to the development of a rich theory of economic policy. In some situations, as in models involving monetary policies with feedback rules, sunspots may exist under some policy regimes but not under others. In other instances, as in models where coordinating on higher investment rates lead to Pareto superior outcomes, the kind of policies needed to achieve such coordination may be quite complex, and even difficult to implement. The important consideration however is not so much to find policies that eliminate the possibility of multiple or sunspot equilibria, but to design policies that will select and implement the best possible equilibrium. Even if it is not possible to design policies that will select the best equilibrium, or to completely eliminate sunspot equilibria, the models that we have described in this survey may enable us to design Pareto improving policy rules. The argument that equilibria are indeterminate may be wrong; but interventionist policy arguments couched in this language are at least capable of comparison with their non-interventionists counterparts. If a dialogue is to be developed between those who favor active intervention and those who do not, it is important that the two groups speak the same language. Dynamic general equilibrium theory, allowing for indeterminacies, is exactly the kind of vehicle that is required to further communication in this debate.

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a. Relative standard deviation: std(x)/std(output)						
	U.S. data	RBC model	CAD model	IC model	IR model	EXT model
Output	1.00	1.00	1.00	1.00	1.00	1.00
Consumption	0.69	0.64	0.35	0.39	0.82	0.91
Investment	1.35	2.31	3.36	3.41	2.32	1.82
Hours	0.52	0.48	0.71	0.70	0.43	0.32
Real Wage	1.14	0.69	0.42	0.44	0.83	0.91

b. Autocorrelation coefficient AR(1)						
	U.S. data	RBC model	CAD model	IC model	IR model	EXT model
Output	0.96	0.93	0.89	0.71	0.60	0.81
Consumption	0.98	0.99	0.98	0.98	1.00	1.00
Investment	0.93	0.88	0.88	0.66	-0.08	0.16
Hours	0.52	0.86	0.88	0.66	-0.24	-0.12
Real Wage	0.97	0.98	0.94	0.88	0.97	0.99

c. Contemporaneous Correlation with Output						
	U.S. data	RBC model	CAD model	IC model	IR model	EXT model
Consumption	0.85	0.82	0.65	0.58	0.84	0.92
Investment	0.60	0.92	0.97	0.96	0.82	0.82
Hours	0.07	0.79	0.85	0.86	0.56	0.42
Real Wage	0.76	0.90	0.91	0.86	0.90	0.95

TABLE 1

	GNP	*CONSUMPTION*	*INVESTMENT*	*LABOR*
RELATIVE ST. DEV.	1.00	0.74 (0.73)	3.32 (3.20)	0.70 (1.16)
CORR. WITH GNP	1.00	0.53 (0.76)	0.83 (0.90)	0.71 (0.86)
AR1 COEFF.	0.93 (0.90)	0.97 (0.84)	0.92 (0.76)	0.80 (0.90)

TABLE 2

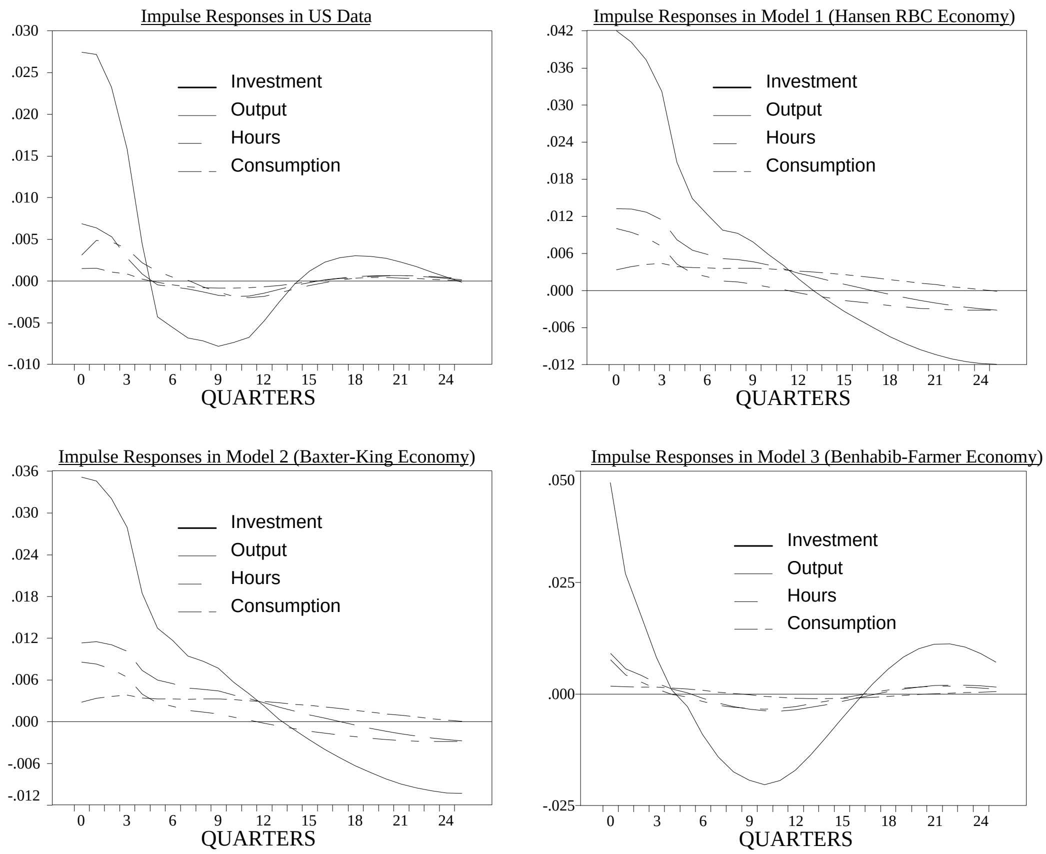

Figure 1: Three Simulated Impulse Responses Compared with US Data

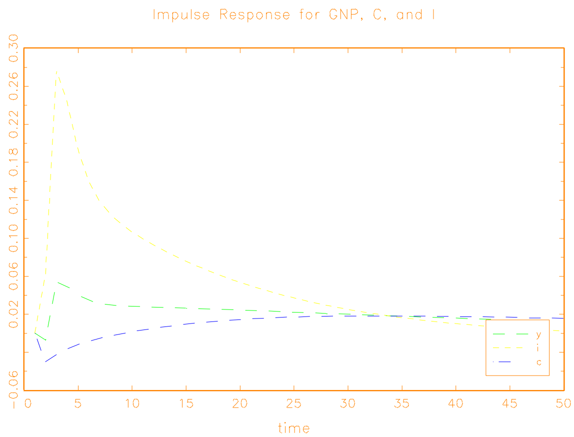
Impulse Response for GNP, C, and I
-0.06
0.02
0.06
0.10
0.14
0.18
0.22
0.26
0.30
0
5
10
15
20
25
30
35
40
45
50
time
y
i
c

Figure 2

–bg

Notes

[1]Gale[1974] first demonstrated that indeterminacy occurs in Samuelson's "consumption-loans" model and Calvo [1978] was one of the first to discuss the issue in this context. Kehoe and Levine [1985] have an excellent discussion of the conditions under which indeterminacy can and cannot occur in infinite horizon general equilibrium economies.

[2]Azariadis [1981] was the first published paper to show that sunspots may be responsible for business cycles although he uses the term self-fulfilling prophecies, originally coined by Robert K. Merton [1948]. Woodford [1986], [1988] further demonstrated how sunspots could be relevant to understanding macroeconomic fluctuations. Howitt and Macafee [1992] use the term 'animal spirits' (popularized by Keynes in the General Theory) to refer to the same concept. It is perhaps unfortunate that these terms are now closely connected. Jevons, for example, who worked on sunspots in the 19th century, did not intend that 'sunspots' should refer to extrinsic uncertainty; instead he believed that there was a real link between the sunspot cycle, the weather and the agricultural sector of the US economy. Similarly, Keynes did not use animal spirits to mean self-fulfilling beliefs; instead his view of uncertainty was closer to Frank Knight's concept of an event for which there is too little information to make a frequentist statement about probabilities.

[3]For a more complete discussion of solution methods in linear models with and without indeterminacies the reader is referred to Farmer [1993], King, Plosser and Rebelo [1987a] or

Blanchard and Kahn [1980]. For an alternative, excellent, treatent of sunspots in a variety of models the reader is referred to the survey by Chiappori, Geoffard and Guesnerie [1992].

[4]Kehoe and Levine [1985] show that infinite horizon models with a finite number of agents behave very much like the finite commodity model. The key distinction (originally pointed out by Shell [1971]) is between competitive models with a finite number of infinite lived agents, in which there is generically a finite odd number of equilibria, and those with an overlapping generations structure in which there is a double infinity of goods and agents.

[5]There is an important knife-edge special case in which one or more roots are exactly on the unit circle which we shall not explore in detail.

[6]The model used by Farmer and Guo is the one explored by Benhabib and Farmer [1994], although similar results would follow from the models of Gali [1994], or Rotemberg and Woodford [1992].

[7]See, for example, the paper by Cogley and Nason [1995] who point out the discrepancies between the dynamic predictions of the RBC model and the much richer dynamics apparent in US data.

[8]There is a huge array of work that studies the empirical characteristics of monetary impulse response functions in the US (e.g. Sims [1980], [1989]. One of the constant features of this data is the fact that the price level is slow to respond to purely nominal disturbances; that is, prices are "sticky". Nor does the characteristic of a slow price response seem to be peculiar to the US as demonstrated by Sims [1992] who compares the US, the UK, France

Germany and Japan.

[9]A separate and related branch of the literature that we do not have space to cover in this survey has shown that search externalities also give rise to indeterminacy. Examples of papers that illustrate this possibility are those by Howitt and McAffee [1988] and Boldrin, Kiyotaki and Wright [1993].

[10]See also Boldrin and Rusticini [1994] and Chamley [1993].

[11]Basu and Fernald [1997] do indeed find that there is heterogeneity of returns to scale across industries.

[12]That $T_2 = w$ is not immediate, but follows from efficiency conditions and envelope theorems.

[13]In some cases, more than half of the roots of J can have positive real parts. This situation however is associated with the presence of optimal cycles and not indeterminacy. See Benhabib and Nishimura [1979].

[14]For a more detailed statement of this argument see the papers by Gali [1994], and Gali and Zilibotti [1995].

[15]This issue is also related to the comovements of consumption and output, and is discussed further below in section 8.1.2.

[16]See also the symposium on "Determinacy of equlibrium under alternative policy regimes" in *Economic Theory*, volume 4, no. 3, 1994. There is also an extensive emipircal

literature initiated by Flood and Garber [1980] that tests for the existence of speculative bubbles.

[17]One way to derive a model of money in the production function is by assuming that firms face transcation costs, $T(q,m)$where q is gross output and m is real balamces. In this case net output is given by $y(m) = q - T(q,m)$. If one specifies T as a Leontief function then the model reduces to a cash in advance specification. One could also replace q with consumption c in T to derive a variant of a model with money in the utility function.

[18]It is easy to show that over the range of m for which $y'(m) > 0$, higher values of initial m lead to higher welfare because m and output will remain higher at all points in time.

[19]Note that if the optimal quantity of money rule is implemented so that $\sigma = -r$, at the steady state we will have $y'(m) = 0$ and $\varepsilon_m = 0$. Once a level of m is attained at which $y'(m) = 0$, it is natural to think that this will continue to hold for higher levels of m because even if money balances cannot cut transaction costs any further, they cannot hurt either. In such a case we will have a continuum of steady state equilibria with real balances satisfying $y'(m) = 0$, as discussed in Woodford [1987].

[20]Note here that this effect is the result not only of the high marginal productivity of money, reflected in ε_m, but also of the rate at which marginal utility declines, reflected in ε_c. This is clear from the condition of indeterminacy which requires the product $\varepsilon_m \varepsilon_c$ to be large.

[21]See, for example, Hoffman, Rasche and Tieslau [1995].

[22]The variables in Farmer [1997] are divided by a growing productivity term to deal with non-stationarities in the data. For brevity we omit this refinement in our discusion.

[23]To do justice to the literature on central bank policy rules would require a separate survey and the reader is merely referred to two recent papers in the area Taylor [1996] and Svensson [1996] and the literature cited therein. Also see the conference issue of the *Journal of Monetary Economics* (1997) no 5. that collects together a number of related papers on the issue of 'Rules and Discretion in Monetary Policy.'

[24]See also footnotes 5.3 and 5.4 below.

[25]Note that as in the discussion of Leeper's discrete time-model [1991], if the utility function is separable in consumption and money so that $\varepsilon_{cm} = 0$, there is no real indeterminacy when the nominal interest rate is set as a function of current exected inflation: equation (5.10) determines only the level of m.

[26]The model easily generalizes to the case where money is productive as well, if we replace e with $y(m)$, with $y'(m) \geq 0$. In that case equation 5.4 becomes $i = \left(\frac{U_m}{U_c}\right) + y'(m)$. Details af the analysis are straightforward and are left to the reader.

[27]Requiring the discounted value of government debt to remain asymptotically finite, as in Woodford [1996], eliminates price level indeterminacy but not the real indeterminacies discussed in this section. Benhabib, Schmitt and Uribe [1998] show that under indeterminacy, any one of the equilibrium trajectories of the real variables will have the discounted vaue of government debt remain asymptotically finite for an appropriate choice of the initial price level. See however footnote 5.4 for the case with sticky prices.

[28]See Benhabib, Schmitt and Uribe [1998].

[29]For a discussion of the relation between this equation and the standard Phillips curve see Calvo [1983].

[30]Benhabib, Schmitt and Uribe [1998] also discuss a sticky price model based on Rotemberg [1996], where agents optimally choose how much to adjust their prices at each instant. They show that indeterminacy obtains for $a < 1$ just as in Calvo's model, but that it also obtains for $a > 1$ under some conditions. In the latter case the steady state can has two stable roots rather than one, so the stable manifold is of dimension two. Benhabib-Schmitt and Uribe [1998] show that requiring the discounted value of government debt to remain asymptotically finite, as in Woodford [1996], restricts initial conditions so the dimension of the restricted stable manifold is reduced to one: this however still implies real indeterminacy since neither c and π are predetermined variables. Furthermore they also show that when $a > 1$, the steady state may be totally unstable with two positive roots, in which case indeterminacy takes the form of the stability of a limit cycle rather than of the steady state.

[31]As noted by Kiley [1997], sticky prices have the effect of increasing the responsiveness of output to monetary shocks, and in this sense they are "productive."

[32]For a study of the empirical relevance of indeterminacy in explaining economic growth, see Benhabib and Gali [1994].

[33]This approach of equating two growth rates to (graphically) characterize multiple steady states is also taken by Evans Honkapohja and Romer [1996]. To generate sufficiently

high returns that justify the higher investment and growth rates, the authors rely on production complementarities giving rise to external effects from the introduction of new intermediate goods, rather than from postulating increasing returns to the labor input.

[34]The conditions for indeterminacy in the Lucas model are less stringent than those presented above because it is a two-sector model. In fact the two-sector structure allows for indeterminacy with a fixed labor supply just like the two sector model discussed in section 3.1, but requires the utility of consumption not to exhibit too much curvature. This same feature arises in the two-sector model above, and it is the reason for introducing a third sector in Benhabib and Nishimura[1996].

[35]Hodrick and Prescott advocated this approach in their widely circulated discussion paper [1980]. Although the HP filter is widely used in the literature it has also been widely criticized since the filter itself can alter the covariance properties of the filtered data in ways that may introduce spurious cycles.

[36]For an interesting perspective on this issue, see Kamihigashi [1996]

[37]For returns to scale around 1.1 the impulse response functions are driven by positive real roots within the unit circle, whereas for returns of 1.2, the roots are again complex, yielding oscillatory impulse responses.

[38]The statistics for US are from HP filtered postwar data, and are in line with standard ones in the *RBC* literature. They differ from the US statistics given by Schmitt-Grohe [1997] who relies on unfiltered statistics reported in King Plosser and Rebelo [1987a].

[39]In a recent paper Weder [1998] introduces a model three sectors consisting of separate investment, consumption and durable consumption goods with variable average markups to address some of the empirical issues that arise in calibrating multisector models.

[40]Farmer calibrates his model and reports impulse response functions that appear to match well with US data. These impulse reponse functions exploit the indeterminacy of equilibrium to generate price responses to monetary shocks that mimic those that we observe in the data. In a private communication, Kiril Sossounov has pointed out to us that there is a computational error in the program used to generate the impulse response functions reported in Farmer's paper. For this reason, we have not reproduced them in this survey. The basic point of the paper, that including money in the utility function can lead to indeterminacy, is correct. But it is an open question as to whether indeterminacy occurs for a range of the parameter space that can mimic the low share of resources used through holding money.

[41]For a generalization of this argument to a higher dimensional linear model see Matheny [1996].

[42]On the other hand, even with arbitrary forecast errors for sunspot shocks, without technology shocks it would not be possible to match the procyclicality of consumption in the data for the reasons cited in section 8.1.2.