Indeterminacy and Sunspots with Constant Returns [1]

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Abstract

We show that indeterminacy can easily arise in multisector models that have constant variable returns to scale and very small market imperfections. This is in sharp contrast to models that require increasing returns to generate indeterminacy, and which have been criticized on the basis of recent empirical estimates indicating that returns to scale are roughly constant, and that market imperfections are small. We also show that we can calibrate our constant returns model with sunspots, using standard parametrizations to produce a close match to the moments of aggregate consumption, investment, output and employment in US data.
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1 Introduction

Recently there has been a renewed interest in the possibility of indeterminacy and sunspots, or alternatively put, in the existence of a continuum of equilibria that arises in dynamic economies with some market imperfections.[1] Much of the research in this area has been concerned with the empirical plausibility of indeterminacy in markets with external effects or with monopolistic competition, and which exhibit some degree of increasing returns. While the early results on indeterminacy relied on relatively large increasing returns and high markups, more recently Benhabib and Farmer [10] showed that indeterminacy can also occur in two-sector models with small sector-specific external effects and very mild increasing returns.[2] Nevertheless, a number of empirical researchers, refining the earlier findings of Hall [23], [24] on disaggregated US data, have concluded that returns to scale seem to be roughly constant, if not decreasing.[3] While one can argue whether the degree of increasing returns required for indeterminacy in Benhabib and Farmer [10] falls within the standard errors of these recent empirical estimates, one may also ask whether increasing returns are at all needed for indeterminacy to arise in a plausible manner. The purpose of this paper is to give a negative answer this question, and to show how indeterminacy can occur in a standard growth model with constant social returns, decreasing private returns, small or negligible external effects, and standard parameter values that are typically used in the literature on business cycles. Furthermore we will show that it is possible to realistically calibrate such a model and to obtain a reasonably good match to the moments of aggregate US data.

Indeterminacy or multiple equilibria emerges in dynamic models with small market distortions as a type of coordination problem. Roughly speaking, what is needed for indeterminacy is a mechanism such that, starting from an arbitrary equilibrium, if all agents were to simultaneously increase their investment in an asset, the rate of return on the asset would tend increase, and in turn set off relative price changes that would drive the economy back towards a stationary equilibrium. One such simple mechanism in one-sector models is increasing returns, typically sustained in a market context via external effects or monopolistic competition (see also the footnote above). In a multisector model however, the rates of return and marginal products depend not only on stocks of assets, but also on the composition of output across sectors. Increasing the production and the stock of a capital asset, say due to an increase in its price, may well increase its rate of return. It is possible therefore to have constant aggregate returns in all sectors at the social level, and to still obtain indeterminacy if there are minor or even negligible external effects in some of the sectors. A more detailed intuition for indeterminacy is given at the end of section (2) in the case of a simple two-sector model.

Constant social returns coupled with small external effects implies that some sectors must have a small degree of decreasing returns at the private level. This is in contrast to models of indeterminacy with social increasing, but private constant returns to scale. An implication of decreasing private returns is of course positive profits. In the parameterized examples given in the sections below, these profits will be quite small because the size of external effects, and therefore the degree of decreasing returns needed for indeterminacy, will also be small. Nevertheless positive profits would invite entry, and unless the number of firms are fixed, a fixed cost of entry must be assumed to determine the number of firms along the equilibrium. Such a market structure would then exhibit increasing private marginal costs but constant social marginal costs, which is in line current empirical work on this subject (see the footnote 3 above). It seems therefore that models of indeterminacy based on market imperfections which drive a wedge between private and social returns must have some form of increasing returns, no matter how small, either in variable costs as in some of the earlier models of indeterminacy, or through a type of fixed cost that prevents entry in the face of positive profits. (See also Gali [21], and Gali and Zilliboti [22].) The point is that while some small wedge between private and social returns is necessary for indeterminacy, this in no way requires decreasing marginal costs, or increasing marginal returns in production.

For reasons also given at the end of section (2) indeterminacy can arise in a constant returns two-sector economy only if the utility of consumption is close to linear. In order to calibrate the model with standard parameters for production and preferences we need a three sector model. Section 3 presents such a model in a continuous time framework. In section (3.2) we show that this model easily gives rise to indeterminacy with standard parametrizations for utility functions, labor supply elasticities, discount and depreciation rates, and factor shares. Much of the derivations are relegated to Appendix I.

In section (4) we present the stochastic, discrete-time version of our model and we calibrate it. We construct some simple sunspot equilibria and show that we can easily find standard parametrizations of our Cobb-Douglas technology and preferences to reasonably match the various moments of US data. The fuller derivations for this case are given in Appendix II.

2 The Two Sector Model

2.1 Basic structure

We model an economy having an infinitely-lived representative agent with instantaneous utility given by

$$U(c) = (1-\sigma)^{-1} c^{(1-\sigma)} - (1+v)^{-1} L^{(1+v)} \qquad \sigma,\, v \geq 0$$

where c is consumption, L is labor supply, v^{-1} is the labor supply elasticity and σ is the intertemporal elasticity of substitution in consumption. For simplicity of exposition we will start with a two-sector rather than an n-sector Cobb-Douglas production technology with a consumption good c, and investment goods, x. The agent's optimization problem will be given by:

$$Max \int_0^\infty \left(U(q_c L_c^{\alpha_0} K_c^{\alpha_1} \overline{L_c^{a_0} K_c^{a_1}}) - (1+v)^{-1} L^{(1+v)} \right) e^{-(r-g)t} dt \tag{1}$$

with respect to K_{xc}, L_c, K_{xx}, L_x and subject to

$$x = q_x L_x^{\beta_0} K_x^{\beta_1} \overline{L_x^{b_0} K_x^{b_1}} \tag{2}$$

$$c = q_c L_c^{\alpha_0} K_c^{\alpha_1} \overline{L_c^{a_0} K_c^{a_1}} \tag{3}$$

$$\frac{dk}{dt} = x - gk$$

$$K_x + K_c = k; \quad ; \quad L_x + L_c = L \tag{4}$$

with initial stock of k given. The components of the production functions, $\overline{L_c^{b_0} K_x^{b_1}}$ for x, and $\overline{L_c^{a_0} K_c^{a_1}}$ for c, represent output effects that are external, and are viewed as functions of time by the agent.

We can write the Hamiltonian as follows:

$$\begin{aligned} H &= U\left(q_c L_c^{\alpha_0} K_c^{\alpha_1} \overline{L_c^{a_0} K_c^{a_1}}\right) - (1+v)^{-1} L^{(1+v)} \\ &\quad + \bar{p}\left(q_x L_x^{\beta_0} K_x^{\beta_1} \overline{L_x^{b_0} K_x^{b_1}} - gk\right) \\ &\quad + \bar{w}_0 (L - L_x - L_c) + \bar{w}(k - K_x - K_c) \end{aligned}$$

Here $\bar{p}$, $\bar{w}_0$ and $\bar{w}$ are the Lagrange multipliers which will represent the utility prices of the capital goods x and y,the rental rates of capital goods and the wage rate of labor, all in terms of the price of the consumption good c. The first order conditions are with respect to K_c, L_c, K_x, L_x yield:

$$\bar{w}_0 = U'\alpha_0 q_c L_c^{\alpha_0+a_0-1} K_c^{\alpha_1+a_1} = \bar{p}\beta_0 q_x L_x^{\beta_0+b_0-1} K_x^{\beta_1+b_1}$$

$$\bar{w} = U'\alpha_1 q_c L_c^{\alpha_0+a_0} K_c^{\alpha_1+a_1-1} = \bar{p}\beta_1 q_x L_x^{\beta_0+b_0} K_x^{\beta_1+b_1-1}$$

If we define

$$\bar{w}_0 = U'w_0; \quad \bar{w} = U'w; \qquad \bar{p} = U'p$$

then the first order conditions become:

$$w_0 = \alpha_0 q_c L_c^{\alpha_0+a_0-1} K_c^{\alpha_1+a_1} = p\beta_0 q_x L_x^{\beta_0+b_0-1} K_x^{\beta_1+b_1} \tag{5}$$

$$w = \alpha_1 q_c L_c^{\alpha_0+a_0} K_c^{\alpha_1+a_1-1} = p\beta_1 q_x L_x^{\beta_0+b_0} K_x^{\beta_1+b_1-1} \tag{6}$$

The first order conditions with respect to L , after combining with the others, gives the labor market equilibrium condition:

$$c^{(1-\sigma)}\alpha_0 L_c^{-1} = L^v \tag{7}$$

If we assume constant returns at the social level, we have:

$$a_0 + \alpha_1 + a_0 + a_1 = \beta_0 + \beta_1 + b_0 + b_1 = 1$$

The equations of motion for the system are given by:

$$\left(\frac{dk}{dt}\right) = x - g\left(k\right) \tag{8}$$

$$\left(\frac{d\left(U'p\right)}{dt}\right) = U'(c)\left(rp - w\right) \tag{9}$$

Evaluated at the steady state, where quantities and prices are stationary, equation (9) can be written as

$$\frac{dp}{dt} \quad = \quad rp - w(p,k) - p\frac{U''(c)\frac{dc}{dt}}{U'(c)} \tag{10}$$

$$
\begin{aligned}
&= rp - w(p,k) - p\frac{U''(c)\left[\frac{\partial c}{\partial p}\frac{dp}{dt} + \frac{\partial c}{\partial k}\frac{dk}{dt}\right]}{U'(c)} \\
&= \left[1 + p\left(\frac{U''(c)}{U'(c)}\right)\left(\frac{\partial c}{\partial p}\right)\right]^{-1}\left[rp - w(p,k) - p\frac{U''(c)\left[\frac{\partial c}{\partial k}\frac{dk}{dt}\right]}{U'(c)}\right] \\
&= \left[1 - \sigma\left(\frac{p}{c}\right)\left(\frac{\partial c}{\partial p}\right)\right]^{-1} \\
&\quad \cdot \left[rp - w(p,k) + \sigma\left(\frac{p}{c}\right)\left(\frac{\partial c}{\partial k}\right)(x(p,k) - gk)\right]
\end{aligned}
$$

where $\sigma = \left(\frac{-U''(c)c}{U'(c)}\right)$.With logarithmic utility of consumption, we have of course, $\sigma = 1$. The first order conditions given by equations (5), (6), (7), and the equations of motion given by (8) and (10) completely describe the system.

2.2 Two-Sector Dynamics

The Jacobian matrix $[J]$ for the differential equations (8) and (10) is given by:

$$
[J] = \begin{bmatrix} \frac{\partial x}{\partial k} - g & \frac{\partial x}{\partial p} \\ \sigma E^{-1}\left(\frac{p}{c}\right)\left(\frac{\partial c}{\partial k}\right)\left(\frac{\partial x}{\partial k} - g\right) & E^{-1}\left[\left(-\frac{\partial w}{\partial p} + r\right) + \sigma\left(\frac{p}{c}\right)\left(\frac{\partial c}{\partial k}\right)\left(\frac{\partial x}{\partial p}\right)\right] \end{bmatrix} \tag{11}
$$

where $E = \left[1 - \sigma\left(\frac{p}{c}\right)\left(\frac{\partial c}{\partial p}\right)\right]$. Note that E can be written as one minus the product of two elasticities:

$$
E = (1 - \sigma\varepsilon_{cp}) \tag{12}
$$

where

$$
\varepsilon_{cp} \equiv \left(\frac{\hat{c}}{\hat{p}}\right) = \left(\frac{p}{c}\frac{\partial c}{\partial p}\right)
$$

If we multiply the first row of $[J]$ by $-\sigma E^{-1}p$ and add it to the second, we get a matrix with an unchanged determinant. We have:

$$DET\left[J\right]=\left(\frac{\partial x}{\partial k}-g\right)\left(-\frac{\partial w}{\partial p}+r\right)\left[1-\sigma\varepsilon_{cp}\right]^{-1}$$

If the utility of consumption is linear, then $\sigma=0$,and it is easy to see that the roots of the matrix $[J]$ become $\left(\frac{\partial x}{\partial k}-g\right)$ and $\left(-\frac{\partial w}{\partial p}+r\right)$. In Appendix I we show, from equation (82), that:

$$\begin{aligned}\left(\frac{\partial x}{\partial k}\right) &= \frac{r\alpha_0\left(1+\left(\frac{L}{L_c}\right)\left(\frac{\sigma}{v}\right)\right)}{\left(\beta_1\left(1-a_0-a_1\right)-\alpha_1\left(1-b_0-b_1\right)+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)} \\ &= \frac{r\alpha_0\left(1+\left(\frac{L}{L_c}\right)\left(\frac{\sigma}{v}\right)\right)}{\left(\beta_1\alpha_0-\alpha_1\beta_0+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)}\end{aligned}$$

If $\sigma=0$,we have:

$$\left(\frac{\partial x}{\partial k}-g\right)=\frac{r\alpha_0}{\left(\beta_1\alpha_0-\alpha_1\beta_0\right)}-g$$

Similarly, from equation (59) in Appendix I we have:

$$\begin{aligned}\left(-\frac{\partial w}{\partial p}+r\right) &= r\left(-\frac{\alpha_0+a_0}{\alpha_0+a_0-\beta_0-b_0}+1\right) \\ &= \left(\frac{r\left(\beta_0+b_0\right)}{\beta_0+b_0-\alpha_0-a_0}\right) \\ &= \left(\frac{r\left(\beta_0+b_0\right)}{\left(\alpha_1+a_1\right)\left(\beta_0+b_0\right)-\left(\alpha_0+a_0\right)\left(\beta_1+b_1\right)}\right)\end{aligned}$$

The last step above follows from multiplying (α_0+a_0) in the denominator by $(\beta_0+\beta_1+b_0+b_1,)$ which under constant returns equals one, similarly multiplying (β_0+b_0) by $(\alpha_0+\alpha_1+a_1+a_0)$, and cancelling to simplify the denominator. It is easily shown that comparing the ratios of Cobb-Douglas exponents of the production functions amounts to comparing factor intensities, since the ratios of exponents determine input ratios. These ratios can be defined either with or without the external effects entering the exponents. We may therefore say that the capital good is labor intensive ***from the private perspective*** if $(\beta_1\alpha_0-\alpha_1\beta_0<0)$, but that it is capital intensive ***from the social perspective*** if $((\alpha_1+a_1)(\beta_0+b_0)-(\alpha_0+a_0)(\beta_1+b_1)<0)$. The expressions above allow us to state the following simple result:

Proposition 1 *In the two-sector model with $\sigma = 0$, if the capital good is labor intensive* ***from the private perspective,*** *but capital intensive* ***from the social perspective****, that is if* $(\beta_1\alpha_0 - \alpha_1\beta_0 < 0)$ *but* $(\alpha_1 + a_1)(\beta_0 + b_0) - (\alpha_0 + a_0)(\beta_1 + b_1) < 0$, *then the steady state is indeterminate.*

A simple example illustrates the possibility of indeterminacy in the two-sector model, for $\sigma = 0$, any $r > 0$, $g \geq 0$, and only a small externality of the capital good in the production of the consumption good. Let:

$$\begin{aligned} \beta_0 &= 0.34\ ;\ \ b_0 = 0.00;\ \ \beta_1 = 0.66;\ \ b_1 = 0.0 \\ \alpha_0 &= 0.30;\ \ \ a_0 = 0.05\ \ \ \alpha_1 = 0.65;\ \ a_1 = 0.0 \end{aligned}$$

Then we have

$$\beta_1\alpha_0 - \alpha_1\beta_0 < 0$$

$$(\alpha_1 + a_1)(\beta_0 + b_0) - (\alpha_0 + a_0)(\beta_1 + b_1) < 0$$

and therefore both roots of $[J]$ are negative. Note also that without some external effects both of the above conditions cannot hold simultaneously. It is clear nevertheless that examples satisfying the above conditions for indeterminacy can be constructed with arbitrarily small external effects.

To establish the intuition behind this result we note the following. Without external effects, the sign of $\left(\frac{\partial w}{\partial p}\right)$depends on the sign of $(\alpha_0\beta_1 - \alpha_1\beta_0)$, which represents the factor intensity difference between the two goods. This dependence on factor intensities is in fact nothing but an expression of the Stolper-Samuelson theorem. Similarly, without external effects, the sign of term $\left(\frac{\partial x}{\partial k}\right)$also depends on the sign of $(\alpha_0\beta_1 - \alpha_1\beta_0)$, and reflects the Rybczynski theorem. We note from the Rybczynski theorem that this effect of stocks on outputs will, at constant prices, be more than proportional, and since at a steady state $x = gk$, it will be strong enough to overwhelm the term g[4]. It should be clear then that without external effects we have $\left(\frac{\partial w}{\partial p}\right) = \left(\frac{\partial x}{\partial k}\right)$, so that the roots of $[J]$ will be of opposite sign. The example of indeterminacy above works precisely because, through external effects, it destroys the duality between the Stolper-Samuelson and Rybczynski effects. Since input coefficients are determined by factor prices, a change in aggregate inputs with prices fixed requires an adjustment of output levels to maintain full employment. The adjustment must reflect the structure of input coefficient matrix, as implied by the Rybczynski Theorem. When there are no externalities the same is true, via Shepard's Lemma, for the effect of input prices on outputs, and this reflects the Stolper-Samuelson Theorem. However with market distortions true costs are not being minimized, and Shepard's Lemma no longer holds, breaking the reciprocal relation between the Rybczynski and Stolper-Samuelson effects[5]. We will make use of this point to provide a heuristic explanation of our indeterminacy result.

To understand the intuition for this indeterminacy result consider first a simple one sector model. Starting from an arbitrary equilibrium, consider another one with a higher rate of investment. A higher investment rate results in higher stocks and, if there are no increasing returns, in a lower marginal return to capital. The only way that this can be an equilibrium is if the other component of the return, the price (or shadow price) appreciation of capital, offsets the decline in the marginal product and justifies the increased holding of such stocks. This appreciating relative price induces a higher production of the capital good, that is a higher rate of investment. The result is a further decline in the marginal product of capital, which then requires an even higher price appreciation to justify the holding of the higher stocks. Transversality conditions rule out such an equilibrium. If there are increasing returns however, incorporated into the model through some market imperfections, the higher stock levels increase rather than decrease the marginal product of capital, and this higher return justifies the holding of the higher stocks without requiring explosive price appreciations and violating transversality conditions. Such increasing returns to capital are generally introduced indirectly. In Benhabib and Farmer [6] increasing returns to capital is the result of changes induced in labor supply due to the reallocation of production in favor of investment and capital accumulation. In Gali [21], Rotemberg and Woodford [28] or Schmitt-Grohé **??**, it is the result of countercyclical markups.

In a two-sector model another mechanism leading to indeterminacy becomes operational. The return to capital now depends on the composition of output as well as the level of the stock. Let us first consider the case without external effects. Take the case where the capital good is capital intensive, and again starting from an equilibrium consider an increase in the rate of investment above the level of its initial equilibrium, induced by an instantaneous increase in the relative price of the investment good. An increase in the stock of capital at constant prices would, from the Rybczynski theorem, lead to a more than proportional rise in its output . From the Stolper-Samuelson theorem the initial price rise leads to an increase in rate of return of capital given by w, and to maintain the equality of the overall return to capital and the discount rate, the price of the investment good must decline. However this is not enough to check the Rycbzynski effect: the increasing capital stock leads to further expansions of investment output despite the retreat of prices towards the steady state levels, and investment output becomes explosive.[6]

To get indeterminacy without relying on increasing returns there must be a mechanism to nullify the duality between the Rybczynski and Stolper-Samuelson theorems. This is precisely what happens in the two-sector model above in the presence of external effects, and is illustrated by Proposition 1. When the investment good is labor intensive from the private perspective, an increase in the capital stock decreases its output at constant prices through the Rybczynski effect. This checks the output side. Stolper-Samuelson theorem however operates through the "social" factor intensities, and the investment good is capital intensive from the social perspective. The initial rise in its price causes an increase in one of the components of its return, w, and requires a price decline to maintain the overall return to capital equal to the discount rate. This offsets the initial rise in the relative price of the investment good and prices also reverse direction toward the steady state. Therefore in the two sector model indeterminacy requires the destruction of the duality between the Rybczynski and Stolper-Samuelson effects through the introduction of market imperfections.[7]

Why then do we have to resort to a three-sector model to generate examples of indeterminacy that are empirically plausible? The problem in the two sector model arises because, when we consider constructing an alternative equilibrium with a higher investment rate we must initially curtail consumption. If there is some curvature on the utility function, the desire to smooth consumption over time can overwhelm the effects described above. (A formal demonstration of this, in terms of the roots of $[J]$, is tedious, but is available from the authors on request.) When a third non-consumption good is introduced however, indeterminacy can arise from compositional changes in outputs, without severely affecting the output of the consumption. Therefore, with a third sector it becomes possible to construct examples of indeterminacy with $\sigma \geq 1$, whereas in the two-sector model indeterminacy seems to hold for values of σ in a narrow range above 0.[8] For a realistic parametrization and calibration of indeterminacy therefore we must turn to a three-sector model.

Before focussing on the three sector model, it may be useful to briefly compare our results to the other two-sector models in the literature. The model of Benhabib and Farmer [10] uses a two sector model with sector specific externalities, but the production functions in the two sectors are identical so that compositional changes in production can affect returns only because of increasing returns in the form of sector specific external effects. The model of Gali [21] combines a setup of monopolistic competition with variable markups. Output is divided into a consumption and an investment good, and the composition of this division affects average markups and profits because the monopolistic competitors face demand curves that have different slopes for the consumption and the investment goods. The magnitude of average markups required however is large (see Schmitt-Grohé [29]. Gali's model is related to a model of Rotemberg and Woodford [28], which also analyzed by Schmitt-Grohé [29].The Rotemberg-Woodford model has a variable markup that depends on aggregate economic activity, rather than a composition effect as in Gali [21].

3 The Three-Sector Model

3.1 The Basic Structure

We again model an economy having an infinitely-lived representative agent with instantaneous utility given by

$$U(c) = (1-\sigma)^{-1} c^{(1-\sigma)} - (1+v)^{-1} L^{(1+v)} \qquad \sigma,\, v \geq 0$$

where c is consumption, L is labor supply, v^{-1} is the labor supply elasticity and σ is the intertemporal elasticity of substitution in consumption. For simplicity of exposition we construct a three-sector rather than an n-sector Cobb-Douglas production technology with a consumption good c, and two investment goods, x and y. The agent's optimization problem is given by:

$$Max \int_0^\infty \left(U(q_c L_c^{\alpha_0} K_{xc}^{\alpha_1} K_{yc}^{\alpha_2} \overline{L_c^{a_0} K_{xc}^{a_1} K_{yc}^{a_2}}) - (1+v)^{-1} L^{(1+v)} \right) e^{-(r-g)t} dt \quad (13)$$

with respect to K_c, L_c, K_y, L_y and subject to

$$x = q_x L_x^{\beta_0} K_{xx}^{\beta_1} K_{yx}^{\beta_2} \overline{L_x^{b_0} K_{xx}^{b_1} K_{yx}^{b_2}} \tag{14}$$

$$y = q_y L_y^{\gamma_0} K_{xy}^{\gamma_1} K_{yy}^{\gamma_2} \overline{L_y^{c_0} K_{xy}^{c_1} K_{yy}^{c_2}} \tag{15}$$

$$c = q_c L_c^{\alpha_0} K_{xc}^{\alpha_1} K_{yc}^{\alpha_2} \overline{L_c^{a_0} K_{xc}^{a_1} K_{yc}^{a_2}} \tag{16}$$

$$\frac{dk_x}{dt} = x - gk_x$$

$$\frac{dk_y}{dt} = y - gk_y$$

$$K_{xx} + K_{xy} + K_{xc} = k_x; \quad K_{yx} + K_{yy} + K_{yc} = k_y; \quad L_x + L_y + L_c = L \tag{17}$$

with initial stocks of k_x and k_y given. The components of the production functions, $\overline{L_x^{b_0} K_{xx}^{b_1} K_{yx}^{b_2}}$ for x, $\overline{L_y^{c_0} K_{xy}^{c_1} K_{yy}^{c_2}}$ for y, and $\overline{L_c^{a_0} K_{xc}^{a_1} K_{yc}^{a_2}}$ for c, represent output effects that are external and are viewed as functions of time by the agent.

We can write the Hamiltonian as follows:

$$\begin{aligned} H &= U\left(q_c L_c^{\alpha_0} K_{xc}^{\alpha_1} K_{yc}^{\alpha_2} \overline{L_c^{a_0} K_{xc}^{a_1} K_{yc}^{a_2}}\right) - (1+v)^{-1} L^{(1+v)} \\ &\quad + \bar{p}_x \left(q_x L_x^{\beta_0} K_{xx}^{\beta_1} K_{yx}^{\beta_2} \overline{L_x^{b_0} K_{xx}^{b_1} K_{yx}^{b_2}} - gk_x\right) \\ &\quad + \bar{p}_y \left(q_y L_y^{\gamma_0} K_{xy}^{\gamma_1} K_{yy}^{\gamma_2} \overline{L_y^{c_0} K_{xy}^{c_1} K_{yy}^{c_2}} - gk_y\right) \\ &\quad + \bar{w}_0 \left(L - L_x - L_y - L_c\right) + \bar{w}_x (k_x - K_{xx} - K_{xy} - K_{xc}) \\ &\quad + \bar{w}_y (k_y - K_{yx} - K_{yy} - K_{yc}) \end{aligned}$$

Here $\bar{p}_x$, $\bar{p}_y$, $\bar{w}_0$, $\bar{w}_x$ and $\bar{w}_y$ are the Lagrange multipliers which will represent the utility prices of the capital goods x and y,the rental rates of capital goods and the wage rate of labor, all in terms of the price of the consumption good c. The first order conditions are with respect to the inputs are:

$$\begin{aligned} \bar{w}_0 &= U' \alpha_0 q_c L_c^{\alpha_0 + a_0 - 1} K_{xc}^{\alpha_1 + a_1} K_{yc}^{\alpha_2 + a_2} \\ &= \bar{p}_x \beta_0 q_x L_x^{\beta_0 + b_0 - 1} K_{xx}^{\beta_1 + b_1} K_{yx}^{\beta_2 + b_2} \end{aligned}$$

$$
\begin{aligned}
&= \bar{p}_y \gamma_0 q_y L_y^{\gamma_0+c_0-1} K_{xy}^{\gamma_1+c_1} K_{yy}^{\gamma_2+c_2}
\end{aligned}
$$

$$
\begin{aligned}
\bar{w}_x &= U' \alpha_1 q_c L_c^{\alpha_0+a_0} K_{xc}^{\alpha_1+a_1-1} K_{yc}^{\alpha_2+a_2} \\
&= \bar{p}_x \beta_1 q_x L_x^{\beta_0+b_0} K_{xx}^{\beta_1+b_1-1} K_{yx}^{\beta_2+b_2} \\
&= \bar{p}_y \gamma_1 q_y L_y^{\gamma_0+c_0} K_{xy}^{\gamma_1+c_1-1} K_{yy}^{\gamma_2+c_2}
\end{aligned}
$$

$$
\begin{aligned}
\bar{w}_y &= U' \alpha_2 q_c L_c^{\alpha_0+a_0} K_{xc}^{\alpha_1+a_1} K_{yc}^{\alpha_2+a_2-1} \\
&= \bar{p}_x \beta_2 q_x L_x^{\beta_0+b_0} K_{xx}^{\beta_1+b_1} K_{yx}^{\beta_2+b_2-1} \\
&= \bar{p}_y \gamma_2 q_y L_y^{\gamma_0+c_0} K_{xy}^{\gamma_1+c_1} K_{yy}^{\gamma_2+c_2-1}
\end{aligned}
$$

If we define

$$
\bar{w}_0 = U' w_0; \quad \bar{w}_x = U' w_x; \quad \bar{w}_y = U' w_y; \quad \bar{p}_x = U' p_x; \quad \bar{p}_y = U' p_y
$$

then the first order conditions become:

$$
\begin{aligned}
w_0 &= \alpha_0 q_c L_c^{\alpha_0+a_0-1} K_{xc}^{\alpha_1+a_1} K_{yc}^{\alpha_2+a_2} \\
&= p_x \beta_0 q_x L_x^{\beta_0+b_0-1} K_{xx}^{\beta_1+b_1} K_{yx}^{\beta_2+b_2} \\
&= p_y \gamma_0 q_y L_y^{\gamma_0+c_0-1} K_{xy}^{\gamma_1+c_1} K_{yy}^{\gamma_2+c_2}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
w_x &= \alpha_1 q_c L_c^{\alpha_0+a_0} K_{xc}^{\alpha_1+a_1-1} K_{yc}^{\alpha_2+a_2} \\
&= p_x \beta_1 q_x L_x^{\beta_0+b_0} K_{xx}^{\beta_1+b_1-1} K_{yx}^{\beta_2+b_2} \\
&= p_y \gamma_1 q_y L_y^{\gamma_0+c_0} K_{xy}^{\gamma_1+c_1-1} K_{yy}^{\gamma_2+c_2}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
w_y &= \alpha_2 q_c L_c^{\alpha_0+a_0} K_{xc}^{\alpha_1+a_1} K_{yc}^{\alpha_2+a_2-1} \\
&= p_x \beta_2 q_x L_x^{\beta_0+b_0} K_{xx}^{\beta_1+b_1} K_{yx}^{\beta_2+b_2-1} \\
&= p_y \gamma_2 q_y L_y^{\gamma_0+c_0} K_{xy}^{\gamma_1+c_1} K_{yy}^{\gamma_2+c_2-1}
\end{aligned} \tag{20}
$$

The first order conditions with respect to L , after combining with the others, gives the labor market equilibrium condition:

$$
c^{(1-\sigma)} \alpha_0 L_c^{-1} = L^v \tag{21}
$$

If we assume constant returns at the social level, we have:

$$a_0+\alpha_1+\alpha_2+a_0+a_1+a_2=\beta_0+\beta_1+\beta_2+b_0+b_1+b_2=\gamma_0+\gamma_1+\gamma_2+c_0+c_1+c_2=1$$

The four equations of motion for the system are given by:

$$\begin{pmatrix} \frac{dk_x}{dt} \\ \frac{dk_y}{dt} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - g \begin{pmatrix} k_x \\ k_y \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} \frac{d(U'p_x)}{dt} \\ \frac{d(U'p_y)}{dt} \end{pmatrix} = rU'(c) \begin{pmatrix} p_x \\ p_y \end{pmatrix} - U'(c) \begin{pmatrix} w_x \\ w_y \end{pmatrix} \tag{23}$$

To simplify equation (23) we define the following two matrices:

$$\left[\frac{\partial c}{\partial p}\right] = \begin{bmatrix} \frac{\partial c}{\partial p_x} & \frac{\partial c}{\partial p_y} \\ \frac{\partial c}{\partial p_x} & \frac{\partial c}{\partial p_y} \end{bmatrix}$$

$$\left[\frac{\partial c}{\partial k}\right] = \begin{bmatrix} \frac{\partial c}{\partial k_x} & \frac{\partial c}{\partial k_y} \\ \frac{\partial c}{\partial k_x} & \frac{\partial c}{\partial k_y} \end{bmatrix}$$

Evaluated at the steady state, where quantities and prices are stationary, equation (23) can be written as

$$\begin{aligned} & \begin{pmatrix} \frac{dp_x}{dt} \\ \frac{dp_y}{dt} \end{pmatrix} \\ = \; & r \begin{pmatrix} p_x \\ p_y \end{pmatrix} - \begin{pmatrix} w_x \\ w_y \end{pmatrix} - \left(\frac{U''(c)c}{U'(c)}\right) \begin{bmatrix} \frac{p_x}{c} & 0 \\ 0 & \frac{p_y}{c} \end{bmatrix} \left(\left[\frac{\partial c}{\partial p}\right] \begin{pmatrix} \frac{dp_x}{dt} \\ \frac{dp_y}{dt} \end{pmatrix} + \left[\frac{\partial c}{\partial k}\right] \begin{pmatrix} \frac{dk_x}{dt} \\ \frac{dk_y}{dt} \end{pmatrix} \right) \\ = \; & \left[I - \sigma \begin{bmatrix} \frac{p_x}{c} & 0 \\ 0 & \frac{p_y}{c} \end{bmatrix} \left[\frac{\partial c}{\partial p}\right] \right]^{-1} \\ & \cdot \left(r \begin{pmatrix} p_x \\ p_y \end{pmatrix} - \begin{pmatrix} w_x \\ w_y \end{pmatrix} + \sigma \begin{bmatrix} \frac{p_x}{c} & 0 \\ 0 & \frac{p_y}{c} \end{bmatrix} \left[\frac{\partial c}{\partial k}\right] \begin{pmatrix} (x - gk_x) \\ (y - gk_y) \end{pmatrix} \right) \end{aligned} \tag{24}$$

where $\sigma = \left(\frac{-U''(c)c}{U'(c)}\right)$.With logarithmic utility of consumption, we have of course, $\sigma = 1$. The first order conditions given by equations (18), (19), (20), (21), and the equations of motion given by (22) and (24) completely describe the system.

3.2 Three Sector Dynamics

We now linearize dynamical system given by equations (22) and (24), we evaluate the associated Jacobian, $[JN]$, at the steady state. Let

$$[JN] = \begin{bmatrix} \left[\frac{\partial \dot{k}}{\partial k}\right] & \left[\frac{\partial \dot{k}}{\partial p}\right] \\ \left[\frac{\partial \dot{p}}{\partial k}\right] & \left[\frac{\partial \dot{p}}{\partial p}\right] \end{bmatrix}$$

where

$$\left[\frac{\partial \dot{k}}{\partial k}\right] = [[YK] - gI],$$

$$\left[\frac{\partial \dot{k}}{\partial p}\right] = [YP],$$

$$\left[\frac{\partial \dot{p}}{\partial k}\right] = [D]^{-1} \sigma \begin{bmatrix} \frac{p_x}{c} & 0 \\ 0 & \frac{p_y}{c} \end{bmatrix} \left[\frac{\partial c}{\partial k}\right] [[YK] - gI],$$

$$\left[\frac{\partial \dot{p}}{\partial p}\right] = [D]^{-1} \left[\sigma \begin{bmatrix} \frac{p_x}{c} & 0 \\ 0 & \frac{p_y}{c} \end{bmatrix} \left[\frac{\partial c}{\partial k}\right] [YP] + [rI - [WP]],\right]$$

and where

$$[D] = \left[I - \sigma \begin{bmatrix} \frac{p_x}{c} & 0 \\ 0 & \frac{p_y}{c} \end{bmatrix} \left[\frac{\partial c}{\partial p}\right]\right] \tag{25}$$

$$[WP] = \begin{bmatrix} \left[\frac{\partial w_x}{\partial p_x}\right] & \left[\frac{\partial w_x}{\partial p_y}\right] \\ \left[\frac{\partial w_y}{\partial p_x}\right] & \left[\frac{\partial w_y}{\partial p_y}\right] \end{bmatrix}, \tag{26}$$

$$[YK] = \begin{bmatrix} \left[\frac{\partial x}{\partial k_x}\right] & \left[\frac{\partial x}{\partial k_y}\right] \\ \left[\frac{\partial y}{\partial k_x}\right] & \left[\frac{\partial y}{\partial k_y}\right] \end{bmatrix}, \tag{27}$$

$$[YP] = \begin{bmatrix} \left[\frac{\partial x}{\partial p_x}\right] & \left[\frac{\partial x}{\partial p_y}\right] \\ \left[\frac{\partial y}{\partial p_x}\right] & \left[\frac{\partial y}{\partial p_y}\right] \end{bmatrix}, \tag{28}$$

$$\left[\frac{\partial c}{\partial p}\right] = \begin{bmatrix} \frac{\partial c}{\partial p_x} & \frac{\partial c}{\partial p_y} \\ \frac{\partial c}{\partial p_x} & \frac{\partial c}{\partial p_y} \end{bmatrix}, \tag{29}$$

$$\left[\frac{\partial c}{\partial k}\right] = \begin{bmatrix} \frac{\partial c}{\partial k_x} & \frac{\partial c}{\partial k_y} \\ \frac{\partial c}{\partial k_x} & \frac{\partial c}{\partial k_y} \end{bmatrix} \tag{30}$$

In the appendix we show how the elements of the matrices $[WP]$, $[YK]$, $[YP]$, $\left[\frac{\partial c}{\partial p}\right]$ and $\left[\frac{\partial c}{\partial k}\right]$ can be evaluated using the steady state output elasticities and the steady state values of the prices and quantities, all of which can be expressed in terms of the parameters of the economy. It is therefore possible to evaluate the roots of the Jacobian $[JN]$ at the steady state and check for indeterminacy, that is to check for parameter values that yield more than two roots of $[JN]$ with negative real parts. The production parameters given below easily generate indeterminacy for our three sector economy, where the discount rate is $r = 0.05$,the population growth rate is $g = 0.01$,the intertemporal elasticity of substitution in consumption is $\sigma = 1$(which implies logarithmic utility in consumption), and the inverse labor supply elasticity is $v = 1$:

Parameters for the consumption good c and investment goods x and y:

$$\begin{aligned} q_c &= 1,\ \alpha_0 = 0.66;\ \alpha_1 = 0.24,\ \alpha_2 = 0.1;\ a_0 = 0.00;\ a_1 = 0.00,\ a_2 = 0.00 \\ q_x &= 1,\ \beta_0 = 0.64;\ \beta_1 = 0.20,\ \beta_2 = 0.1;\ b_0 = 0.00;\ b_1 = 0.06,\ b_2 = 0.00 \\ q_y &= 1,\ \gamma_0 = 0.61;\ \gamma_1 = 0.23,\ \gamma_2 = 0.1;\ c_0 = 0.00;\ c_1 = 0.00,\ c_2 = 0.06 \end{aligned}$$

Roots of the Jacobian $[JN]$:

$-10.8767,\ -0.0599,\ -0.8579,\ 0.1138$

Clearly, the parameters above are entirely standard, and the external effects, which are present only for the capital input x in the production of x, and for the capital input y in the production of y, both of which are set to 0.06, are extremely small. The production functions of the three goods differ only slightly, with labor share in each one of them, given by α_0, β_0 and γ_0, all at roughly equal to $\frac{2}{3}$. There are constant returns to scale at the social level and that at the private level the agents face very slight diminishing returns to scale in producing x and y. Let us emphasize that the degree of perceived private decreasing returns in x and y is indeed negligibly small: while from the social perspective the Cobb-Douglas exponents add up to 1, from the private perspective they add up to 0.94 for both x and y. Furthermore, indeterminacy seems very robust: small variations in σ, v, r, g or in the production parameters do not change the values by of the roots much, or change their sign pattern. Eliminating the external effects completely however does eliminate indeterminacy, as expected, because the private and social optimum coincide in that case.[9]

4 The Stochastic Discrete Time Model and Calibration

The discrete time problem can be defined as:

$$\begin{aligned} V(k_x, k_y, z) &= Max \left(\frac{1}{1-\sigma}\right) \left(z_c q_c L_c^{\alpha_0} K_{xc}^{\alpha_1} K_{yc}^{\alpha_2} \overline{L_c^{a_0} K_{xc}^{a_1} K_{yc}^{a_2}}\right)^{(1-\sigma)} \\ & - (1+v)^{-1} L^{(1+v)} + \rho E V((1-g_x)k_x + x, (1-g_y) k_y + y, z') \end{aligned}$$

$$x = z_x q_x L_x^{\beta_0} K_{xx}^{\beta_1} K_{yx}^{\beta_2} \overline{L_c^{b_0} K_{xx}^{b_1} K_{yx}^{b_2}} \tag{31}$$

$$y = z_y q_y L_y^{\gamma_0} K_{xy}^{\gamma_1} K_{yy}^{\gamma_2} \overline{L_y^{c_0} K_{xy}^{c_1} K_{yy}^{c_2}} \tag{32}$$

where $\rho = (1+(r-g))^{-1}$is the discount factor, g_x , g_y are depreciation rates, $z = (z_c, z_x, z_y)$, z_i is a technology shock for $i = c, x, y$, where $\ell n\, z_i = \zeta_i$, and

$$\zeta_{i,t+1} = \lambda_i \zeta_{i,t} + \hat{\varepsilon}_{i,t+1}; \qquad 0 \le \lambda_i \le 1 \tag{33}$$

$i = c,\ x,\ y$, and $\hat{\varepsilon}_{i,t+1}$ is iid, normally distributed and has mean zero. z' is the value attained by z in the subsequent period. Note that we can write the consumption output as:

$$c = z_c q_c \cdot (L - L_x - L_y)^{\alpha_0 + a_0} (k_x - K_{xx} - K_{xy})^{\alpha_1 + a_1} (k_y - K_{yx} - K_{yy})^{\alpha_2 + a_2}$$

The first order conditions, after simple substitutions, are:

$$c_t^{-\sigma} p_{x,t} = \rho E \left(c_{t+1}^{-\sigma} p_{x,t+1} \left(\frac{w_{x,t+1}}{p_{x,t+1}} + (1-g_x) \right) \right) \tag{34}$$

$$c_t^{-\sigma} p_{y,t} = \rho E \left(c_{t+1}^{-\sigma} p_{y,t+1} \left(\frac{w_{y,t+1}}{p_{y,t+1}} + (1-g_y) \right) \right) \tag{35}$$

where

$$\frac{w_x}{p_x} = \beta_1 q_x L_x^{\beta_0 + b_0} K_{xx}^{\beta_1 + b_1 - 1} K_{yx}^{\beta_2 + b_2} \tag{36}$$

$$
\begin{aligned}
\frac{w_y}{p_y} &= \gamma_2 q_y L_y^{\gamma_0+c_0} K_{xy}^{\gamma_1+c_1} K_{yy}^{\gamma_2+c_2-1} && (37) \\
p_x &= \frac{\alpha_1 q_c L_c^{\alpha_0+a_0} K_{xc}^{\alpha_1+a_1-1} K_{yc}^{\alpha_2+a_2}}{\beta_1 q_x L_x^{\beta_0+b_0} K_{xx}^{\beta_1+b_1-1} K_{yx}^{\beta_2+b_2}} && (38) \\
p_y &= \frac{\alpha_1 q_c L_c^{\alpha_0+a_0} K_{xc}^{\alpha_1+a_1-1} K_{yc}^{\alpha_2+a_2}}{\gamma_1 q_y L_y^{\gamma_0+c_0} K_{xy}^{\gamma_1+c_1-1} K_{yy}^{\gamma_2+c_2}} && (39)
\end{aligned}
$$

The equations for accumulation are given by

$$k_{x,t+1} = (1 - g_x)\, k_{x,t} + x_t \tag{40}$$

$$k_{y,t+1} = (1 - g_y)\, k_{y,t} + y_t \tag{41}$$

The computations for the analysis and calibration of this model are presented in Appendix II. Here we proceed directly to study the local dynamics. The linearized dynamics of the model are :

$$
\begin{pmatrix} \hat{k}_{x,t+1} \\ \hat{k}_{y,t+1} \\ \hat{p}_{x,t+1} \\ \hat{p}_{y,t+1} \\ \hat{z}_{c,t+1} \\ \hat{z}_{x,t+1} \\ \hat{z}_{y,t+1} \end{pmatrix} = [Q]^{-1}[R] \begin{pmatrix} \hat{k}_{x,t} \\ \hat{k}_{y,t} \\ \hat{p}_{x,t} \\ \hat{p}_{y,t} \\ \hat{z}_{c,t} \\ \hat{z}_{x,t} \\ \hat{z}_{y,t} \end{pmatrix} + [Q]^{-1} \begin{pmatrix} 0 \\ 0 \\ -\hat{s}_{x,t+1} \\ -\hat{s}_{y,t+1} \\ \hat{\varepsilon}_{c,t+1} \\ \hat{\varepsilon}_{x,t+1} \\ \hat{\varepsilon}_{y,t+1} \end{pmatrix} \tag{42}
$$

where $\hat{s}_{i,t}, i = x, y$, are an *iid* sunspot shocks with zero mean, acting on the "Euler" equations for the two capital stocks. The matrices $[R]$ and $[Q]$ are defined in Appendix II. Their elements are functions of parameters of the system, and of steady state quantities which are also functions of the parameters. We can therefore evaluate the roots of $[Q]^{-1}[R]$ to check for the possibility of indeterminacy. When externality parameters are set to zero, four of the roots of the Jacobian matrix come in pairs of $\left(\mu, \frac{1}{\rho\mu}\right)$, and the other three roots are the autoregressive coefficients of the technology shocks[10]. For very modest externalities however indeterminacy arises, as it does in the continuous time case. The four roots no longer split with half inside and half outside the unit circle. We find that indeterminacy can easily occur for a large set of parameter values. The example below illustrates this point.

We calibrate the model along the lines of a standard RBC model. We set the quarterly discount factor to $r = 0.036$ and the depreciation rate to $g = 0.025$,so that quarterly net discount is $(r - g) = 0.011$. The instantaneous utility of consumption is logarithmic, so that $\sigma = 1$. Labor supply is taken to be quite elastic, although not infinitely elastic as is often the case in the real business cycle literature: we set $v = 0.2$, implying a labor supply elasticity of 5. The persistence parameters for the technology shocks, $\lambda_c,\ \lambda_x$,and λ_y are each set to 0.95.The production parameters and the resulting roots of the Jacobian $\left[[Q]^{-1}[R]\right]$ are as follows:

Parameters for the consumption good c and investment goods x and y:

$$
\begin{aligned}
q_c &= 1,\ \alpha_0 = 0.58;\ \alpha_1 = 0.15,\ \alpha_2 = 0.20;\ a_0 = 0.00;\ a_1 = 0.07,\ a_2 = 0.00 \\
q_x &= 1,\ \beta_0 = 0.50;\ \beta_1 = 0.22,\ \beta_2 = 0.21;\ b_0 = 0.00;\ b_1 = 0.07,\ b_2 = 0.00 \\
q_y &= 1,\ \gamma_0 = 0.51;\ \gamma_1 = 0.26,\ \gamma_2 = 0.15;\ c_0 = 0.00;\ c_1 = 0.00,\ c_2 = 0.08
\end{aligned}
$$

Roots of the Jacobian $\left[[Q]^{-1}[R]\right]$:

(0.251, 1.057, 0.967, 0.425, 0.950, 0.950, 0.950)

The last three roots are simply the persistence parameters $\lambda_c,\ \lambda_x$,and λ_y. Of the remaining four roots, three are within the unit circle, which implies indeterminacy since there are two capital stocks and two prices[11]. Many other parametrizations giving indeterminacy are also possible, but the one above is the parametrization that we use in the calibrations below.

To calibrate the model we set the standard deviations of sunspot shock $\hat{s}_{x,t}$, and the innovations to technology shocks $\hat{\varepsilon}_{i,t+1}$, $i = c, x, y$,all of which we take to be normally distributed, to 0.0039 [12]. They are so set to imply a standard deviation for GNP of 1.76, to match the US data. In the simulations we take the technology shocks to be perfectly correlated, and the sunspot shock to be independent. Experimenting with independent technology shocks or with technology shocks correlated with the sunspot does not change the simulation results by much. The results of our calibration exercise are given in the table below.

	GNP	CONSUMPTION	INVESTMENT	LABOR
ST. DEV.	1.00	0.74 (0.73)	3.32 (3.20)	0.70 (1.16)
CORR. WITH GNP	1.00	0.53 (0.76)	0.83 (0.90)	0.71 (0.86)
AR1 COEFF.	0.93 (0.90)	0.97 (0.84)	0.92 (0.76)	0.80 (0.90)

Standard deviations of the variables in the table are relative to those of GNP, and the numbers in parentheses are the same ratios for Hodrick-Prescott filtered US data. Investment corresponds to its aggregated value, evaluated at the current relative prices of x and y. GNP contains consumption, c, plus investment, with the price of the consumption good normalized to unity each period. Individual components of GNP, or of investment tend to be much more highly volatile than the aggregated series. We find however that this is the case for standard RBC calibrations, irrespective of whether the chosen parameter values generate determinate or indeterminate equilibria.[13]

The data generated by the model matches US data reasonably well. Consumption is more weakly correlated with output for the data generated by the model than it is for actual US data: this in part may be because positive technology shocks initially lead to strong expansions in investment at the expense of consumption. (See Figure 3 below and the preceding discussion.). In addition, labor data from the model is less volatile, and less correlated with output than it is for actual US data. One possible reason for this, as we pointed out earlier, is that we used a labor supply elasticity of 5, compared to the infinite labor supply elasticities used in much of the RBC literature.

Figure 1 below gives a typical simulation with indeterminate equilibria and sunspots, calibrated to the parameters given above, for GNP, investment and consumption. Clearly investment displays oscillations of largest amplitude, while consumption is fairly smooth, and GNP is in the intermediate range.

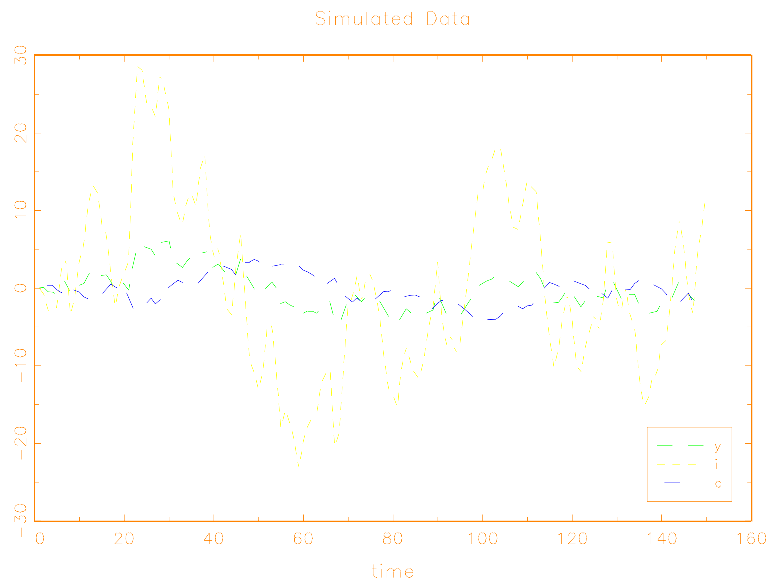

Figure 1

Figure 2 shows the impulse responses for consumption, investment and GNP, generated by an aggregate productivity shock impacting the three sectors simultaneously. The aggregative shock leads to a surge of investment, initially at the expense of consumption. Again we find that this feature, that is the initial negative response of consumption to the aggregative technology shock, typically arises for standard RBC calibrations of multi-sector models that do not have any external effects and therefore exhibit determinate equilibria.[14] GNP also drops by a small amount when the shock hits, but rises immediately afterward as investment surges, and then subsides to generate the hump-shaped response found in the data.

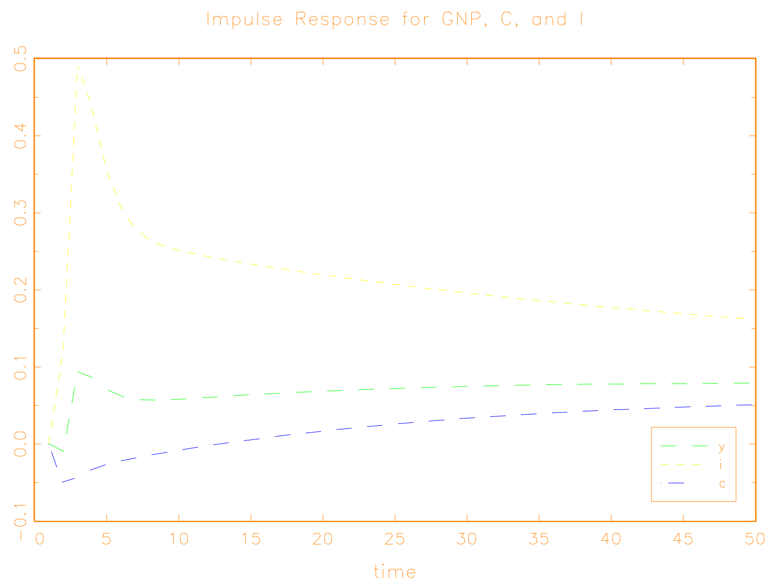

Figure 2

While we have by no means performed an extensive search, the model with above parameters that generates sunspot equilibria can provide a reasonable match to the various moments of actual data. There also exists many other reasonable parameter combinations that give a good match, and yet others that give a very poor match to the data. Furthermore, some of the moments generated by the model can be sensitive to parameter changes in certain regions of the parameter set. This is true whether we have external effects and indeterminacy, or whether we restrict ourselves to standard parametrizations of the model without externalities and indeterminacy. Another feature, shared with calibrated multisector models without external effects or market distortions that have determinate equilibria, is that prices and outputs of the individual investment goods tend to be more volatile than the aggregated value of investment, with some sectors even exhibiting countercyclical behavior (see for example Benhabib, Perli and Plutarchos [12]. This counterfactual observation about calibrated multi-sector models in the context of a determinate economy has led Huffman and Wynne [25] to introduce adjustment costs for the sectoral reallocations of factors of production. It seems therefore that multi-sector RBC models, with or without indeterminacy and sunspots, raise some new issues for the RBC literature[15]. More information concerning the moments of individual output series must be considered to identify the best parametrizations, and to assess how good the match is between the data and the simulations. Further disaggregation may be necessary to identify the sectors of the model with the actual sectors of the economy for which data is available. On the other hand it also seems likely that increasing the number of sectors will expand the range of parameters yielding indeterminacy, much as going from one to two to three sectors does. We view the above calibration exercise only as suggestive of interesting possibilities that can expand the scope of the RBC literature.

5 Appendix I: The continuous time case

In this appendix we will derive the expressions necessary to evaluate the steady state Jacobian of the linearized dynamics of the three sector model in continuous time.

5.1 The Static Structure

From equations (18), (19) and (20) we can write:

$$\omega_{12} = \frac{w_x}{w_y} = \frac{\alpha_1 K_{yc}}{\alpha_2 K_{xc}} = \frac{\beta_1 K_{yx}}{\beta_2 K_{xx}} = \frac{\gamma_1 K_{yy}}{\gamma_2 K_{xy}} \tag{43}$$

$$\omega_{10} = \frac{w_x}{w_0} = \frac{\alpha_1 L_c}{\alpha_0 K_{xc}} = \frac{\beta_1 L_x}{\beta_0 K_{xx}} = \frac{\gamma_1 L_y}{\gamma_0 K_{xy}} \tag{44}$$

$$\omega_{20} = \frac{w_y}{w_0} = \frac{\alpha_2 L_c}{\alpha_0 K_{yc}} = \frac{\beta_2 L_x}{\beta_0 K_{yx}} = \frac{\gamma_2 L_y}{\gamma_0 K_{yy}} \tag{45}$$

If we denote $\hat{x} = \frac{dx}{x}$, then logarithmic differentiation yields:

$$\hat{\omega}_{12} = \hat{K}_{yc} - \hat{K}_{xc} = \hat{K}_{yx} - \hat{K}_{xx} = \hat{K}_{yy} - \hat{K}_{xy} \tag{46}$$

$$\hat{\omega}_{10} = \hat{L}_c - \hat{K}_{xc} = \hat{L}_x - \hat{K}_{xx} = \hat{L}_y - \hat{K}_{xy} \tag{47}$$

$$\hat{\omega}_{20} = \hat{L}_c - \hat{K}_{yc} = \hat{L}_x - \hat{K}_{yx} = \hat{L}_y - \hat{K}_{yy} \tag{48}$$

Note that $(\omega_{ij}) = (\omega_{ji})^{-1}$ and that $(\omega_{ij})(\omega_{jh}) = (\omega_{ih})$. Now combining equations (19), (20), (43), (44) and (45) noting that at a steady state

$$w_x = rp_x; \quad w_y = rp_y, \tag{49}$$

we have:

$$w_x = p_x \left(q_x \beta_1 \left(\frac{\beta_0}{\beta_1} \right)^{\beta_0 + b_0} \left(\frac{\beta_2}{\beta_1} \right)^{\beta_2 + b_2} \right) (\omega_{10})^{\beta_0 + b_0} (\omega_{12})^{\beta_2 + b_2} = rp_x \tag{50}$$

$$w_y = p_y \left(q_y \gamma_2 \left(\frac{\gamma_0}{\gamma_2} \right)^{\gamma_0 + c_0} \left(\frac{\gamma_1}{\gamma_2} \right)^{\gamma_2 + c_2} \right) (\omega_{10})^{\gamma_0 + c_0} (\omega_{12})^{-\gamma_1 - c_1 - \gamma_0 - c_0} = rp_y \tag{51}$$

The exponential term $(-\gamma_1 - c_1 - \gamma_0 - c_0)$ appears in equation (51) because the factor price ratios $\omega_{21}$and ω_{20} in the equation are replaced by $(\omega_{12})^{-1}$ and $(\omega_{10})(\omega_{12})^{-1}$.Taking logs in (50) and (51) we can write them as:

$$\left(\begin{array}{c} \left(\ell n\, r - \ell n \left(q_x \beta_1 \left(\frac{\beta_0}{\beta_1} \right)^{\beta_0 + b_0} \left(\frac{\beta_2}{\beta_1} \right)^{\beta_2 + b_2} \right) \right) \\ \left(\ell n\, r - \ell n \left(q_y \gamma_2 \left(\frac{\gamma_0}{\gamma_1} \right)^{\gamma_0 + c_0} \left(\frac{\gamma_2}{\gamma_1} \right)^{\gamma_2 + c_2} \right) \right) \end{array} \right) = M \left(\begin{array}{c} \ell n\, \omega_{10} \\ \ell n\, \omega_{12} \end{array} \right) \tag{52}$$

where

$$[M] = \left[\begin{array}{cc} \beta_0 + b_0 & \beta_2 + b_2 \\ \gamma_0 + c_0 & -\gamma_1 - c_1 - \gamma_0 - c_0 \end{array} \right] \tag{53}$$

The equation (52) determines the steady state values of ω_{10} and ω_{12}. We now solve, using (19), for the prices p_x and p_y. We have:

$$\begin{aligned} & \alpha_1 q_c \left(\frac{K_{yc}}{K_{xc}} \right)^{(\alpha_2 + a_2)} \left(\frac{L_c}{K_{xc}} \right)^{(\alpha_0 + a_0)} \\ = \; & p_x \beta_1 q_x \left(\frac{K_{yx}}{K_{xx}} \right)^{(\beta_2 + b_2)} \left(\frac{L_x}{K_{xx}} \right)^{(\beta_0 + b_0)} \\ = \; & p_y \gamma_1 q_y \left(\frac{K_{yy}}{K_{xy}} \right)^{(\gamma_2 + c_2)} \left(\frac{L_y}{K_{xy}} \right)^{(\gamma_0 + c_0)} \end{aligned}$$

Substituting from equations (43), (44) and (45) we obtain:

$$\begin{aligned} & \alpha_1 q_c \left(\left(\frac{\alpha_2}{\alpha_1}\right)\omega_{12}\right)^{(\alpha_2+a_2)} \left(\left(\frac{\alpha_0}{\alpha_1}\right)\omega_{10}\right)^{(\alpha_0+a_0)} \\ = \;& p_x\beta_1 q_x \left(\left(\frac{\beta_2}{\beta_1}\right)\omega_{12}\right)^{(\beta_2+b_2)} \left(\left(\frac{\beta_0}{\beta_1}\right)\omega_{10}\right)^{(\beta_0+b_0)} \\ = \;& p_y\gamma_1 q_y \left(\left(\frac{\gamma_2}{\gamma_1}\right)\omega_{12}\right)^{(\gamma_2+c_2)} \left(\left(\frac{\gamma_0}{\gamma_1}\right)\omega_{10}\right)^{(\gamma_0+c_0)} \end{aligned} \tag{54}$$

Taking logs we then have:

$$\begin{aligned} \begin{pmatrix} \ell n\, p_x \\ \ell n\, p_y \end{pmatrix} = & \begin{pmatrix} \ell n \left(\left(\frac{q_c\alpha_1}{q_x\beta_1}\right) \left(\frac{\alpha_2}{\alpha_1}\right)^{(\alpha_2+a_2)} \left(\frac{\alpha_0}{\alpha_1}\right)^{(\alpha_0+a_0)} \left(\frac{\beta_2}{\beta_1}\right)^{-(\beta_2+b_2)} \left(\frac{\beta_0}{\beta_1}\right)^{-(\beta_0+b_0)} \right) \\ \ell n \left(\left(\frac{q_c\alpha_1}{q_y\gamma_1}\right) \left(\frac{\alpha_2}{\alpha_1}\right)^{(\alpha_2+a_2)} \left(\frac{\alpha_0}{\alpha_1}\right)^{(\alpha_0+a_0)} \left(\frac{\gamma_2}{\gamma_1}\right)^{-(\gamma_2+c_2)} \left(\frac{\gamma_0}{\gamma_1}\right)^{-(\gamma_0+c_0)} \right) \end{pmatrix} \\ & + [N] \begin{pmatrix} \ell n\; \omega_{10} \\ \ell n\; \omega_{12} \end{pmatrix} \end{aligned} \tag{55}$$

where

$$[N] = \begin{bmatrix} \alpha_0 + a_0 - \beta_0 - b_0 & \alpha_2 + a_2 - \beta_2 - b_2 \\ \alpha_0 + a_0 - \gamma_0 - c_0 & \alpha_2 + a_2 - \gamma_2 - c_2 \end{bmatrix} \tag{56}$$

The equation (55) allows us to solve for p_x and p_y in terms of ω_{10} and ω_{12}, and then using (52) and (49), for the steady state values of the prices p_x and p_y and capital rentals w_x and w_y: we will need them to evaluate the Jacobian matrix describing the local dynamics around the steady state. Furthermore, using equation (55) we obtain:

$$\begin{pmatrix} \hat{\omega}_{10} \\ \hat{\omega}_{12} \end{pmatrix} = [N]^{-1} \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \tag{57}$$

Now taking logarithmic derivatives of equations (50) and (51) we get:

$$\begin{aligned} \begin{pmatrix} \hat{w}_x \\ \hat{w}_y \end{pmatrix} &= \begin{bmatrix} \beta_0 + b_0 & \beta_2 + b_2 \\ \gamma_0 + c_0 & \gamma_2 + c_2 \end{bmatrix} \begin{pmatrix} \hat{\omega}_{10} \\ \hat{\omega}_{12} \end{pmatrix} + \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \\ &= \left(\begin{bmatrix} \beta_0 + b_0 & \beta_2 + b_2 \\ \gamma_0 + c_0 & \gamma_2 + c_2 - 1 \end{bmatrix} [N]^{-1} + I \right) \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \\ &= [G] \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \end{aligned} \tag{58}$$

Note that equation (58) defines the matrix $[G]$.

5.1.1 The static structure for the two-sector case

In a two-sector economy without the capital good y,either as an input or an output, the above expression for the elasticity of $\hat{w}_x$ with respect to $\hat{p}_x$ can be simplified. The matrix $[N]$ becomes a scalar and since at a steady state $w_x = rp_x$, we obtain:

$$\frac{\partial w_x}{\partial p_x} = r\frac{\hat{w}_x}{\hat{p}_x} = r\frac{\alpha_0 + a_0}{\alpha_0 + a_0 - \beta_0 - b_0} \tag{59}$$

We use this expression in Section 2 above.

5.2 Unit Input Coefficients

Computing unit input coefficients is straightforward. Taking logs of the production function for capital good x and using (43), (44) and (45) , we have

$$\begin{bmatrix} \beta_1 + b_1 & \beta_2 + b_2 & \beta_0 + b_0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ell n\, K_{xx} \\ \ell n\, K_{yx} \\ \ell n\, L_x \end{bmatrix} = \begin{bmatrix} \ell n\, x - \ell n\, q_x \\ \ell n\, w_x - \ell n\, w_y + \ell n\, \beta_2 - \ell n\, \beta_1 \\ \ell n\, w_x - \ell n\, w_0 + \ell n\, \beta_0 - \ell n\, \beta_1 \end{bmatrix}$$

Solving we get:

$$a_{01} = \frac{L_x}{x} = \left(\frac{1}{q_x}\right)\left(\frac{\beta_0 w_x}{\beta_1 w_0}\right)^{\beta_1 + b_1}\left(\frac{\beta_0 w_y}{\beta_2 w_0}\right)^{\beta_2 + b_2} \tag{60}$$

$$a_{11} = \frac{K_{xx}}{x} = \left(\frac{1}{q_x}\right)\left(\frac{\beta_1 w_0}{\beta_0 w_x}\right)^{\beta_0 + b_0}\left(\frac{\beta_1 w_y}{\beta_2 w_x}\right)^{\beta_2 + b_2} \tag{61}$$

$$a_{21} = \frac{K_{yx}}{x} = \left(\frac{1}{q_x}\right)\left(\frac{\beta_2 w_0}{\beta_0 w_y}\right)^{\beta_0 + b_0}\left(\frac{\beta_2 w_x}{\beta_1 w_y}\right)^{\beta_1 + b_1} \tag{62}$$

Similarly, for the consumption and second capital good we obtain:

$$a_{00} = \frac{L_c}{c} = \left(\frac{1}{q_c}\right)\left(\frac{\alpha_0 w_x}{\alpha_1 w_0}\right)^{\alpha_1 + a_1}\left(\frac{\alpha_0 w_y}{\alpha_2 w_0}\right)^{\alpha_2 + a_2} \tag{63}$$

$$a_{10} = \frac{K_{xc}}{x} = \left(\frac{1}{q_c}\right)\left(\frac{\alpha_1 w_0}{\alpha_0 w_x}\right)^{\alpha_0 + a_0}\left(\frac{\alpha_1 w_y}{\alpha_2 w_x}\right)^{\alpha_2 + a_2} \tag{64}$$

$$a_{20} = \frac{K_{yc}}{c} = \left(\frac{1}{q_c}\right)\left(\frac{\alpha_2 w_0}{\alpha_0 w_y}\right)^{\alpha_0 + a_0}\left(\frac{\alpha_2 w_x}{\alpha_1 w_y}\right)^{\alpha_1 + a_1} \tag{65}$$

$$a_{02} = \frac{L_y}{y} = \left(\frac{1}{q_y}\right)\left(\frac{\gamma_0 w_x}{\gamma_1 w_0}\right)^{\gamma_1 + c_1}\left(\frac{\gamma_0 w_y}{\gamma_2 w_0}\right)^{\gamma_2 + c_2} \tag{66}$$

$$a_{12} = \frac{K_{xy}}{y} = \left(\frac{1}{q_y}\right)\left(\frac{\gamma_1 w_0}{\gamma_0 w_x}\right)^{\gamma_0 + c_0}\left(\frac{\gamma_1 w_y}{\gamma_2 w_x}\right)^{\gamma_2 + c_2} \tag{67}$$

$$a_{22} = \frac{K_{yy}}{y} = \left(\frac{1}{q_y}\right)\left(\frac{\gamma_0 w_x}{\gamma_1 w_0}\right)^{\gamma_1 + c_1}\left(\frac{\gamma_0 w_y}{\gamma_2 w_0}\right)^{\gamma_2 + c_2} \tag{68}$$

Note that the input coefficients are functions of the ratios of factor rentals and can be written in terms of ω_{10} and ω_{12}, remembering that $(\omega_{ij}) = (\omega_{ji})^{-1}$ and that $(\omega_{ih}) = (\omega_{ij})(\omega_{jh})$.

5.3 Steady State Quantities

At a steady state we have

$$x = gk_x; \qquad y = gk_y \tag{69}$$

Full employment then requires:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}\begin{pmatrix} c \\ gk_x \\ gk_y \end{pmatrix} = \begin{pmatrix} L \\ k_x \\ k_y \end{pmatrix} \tag{70}$$

We can solve for k_x and k_y as:

$$\begin{pmatrix} k_x \\ k_y \end{pmatrix} = \left[I - g\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right]^{-1}\begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} c \tag{71}$$

Let $\nu = \left(\frac{1}{v}\right)$. Then using (21), (69), and (70), we can solve for steady state c :

$$\left(a_{00} + g\,(a_{01}\ a_{02})\left[I - g\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right]^{-1}\begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}\right) c = \alpha_0^{\nu} c^{(1-\sigma)\nu} L_c^{-\nu}$$

or, since $L_c = (a_{00})\, c$,

$$c = \left(\left(\frac{\alpha_0}{a_{00}}\right)^{(-\nu)} \left(a_{00} + g\,(a_{01}\ a_{02}) \left[I - g \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right]^{-1} \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}\right)\right)^{\mu} \tag{72}$$

where $\mu = \left(-\frac{1}{\sigma\nu+1}\right)$. Using equations (72) and (71) we can solve for steady state k_x and k_y. Since unit input coefficients are functions of ω_{10} and ω_{12}, whose steady state values are given by equation (52), the steady state stocks k_x and k_y can be computed in terms of the parameters of the model. The steady state outputs then are given by equations (69) and (72). We can express the steady state factor inputs in the production function as:

$$\begin{aligned} L_c &= a_{00}c;\;\; K_{xc} = a_{10}c;\;\; K_{yc} = a_{20}c \\ Lx &= a_{01}x;\;\; K_{xx} = a_{11}x;\;\; K_{xy} = a_{21}x \\ L_y &= a_{02}y;\;\; K_{xy} = a_{12}y;\;\;\; K_{yy} = a_{22}y \end{aligned}$$

where by construction:

$$\begin{aligned} L &= L_c + L_x + L_y \\ k_x &= K_{xc} + K_{xx} + K_{xy} \\ k_y &= K_{yc} + K_{yx} + K_{yy} \end{aligned}$$

5.4 Output Elasticities

First we compute the elasticities of inputs with respect to ω_{10} and ω_{12}. From the first order condition for labor given by equation (21) we have:

$$\hat{L} = x_0\hat{L}_c + x_1\hat{K}_{xc} + x_2\hat{K}_{yc} \tag{73}$$

where

$$x_0 = \frac{(\alpha_0 + a_0)(1-\sigma) - 1}{v}; \qquad x_1 = \frac{(\alpha_1 + a_1)(1-\sigma)}{v}; \qquad x_2 = \frac{(\alpha_2 + a_2)(1-\sigma)}{v}$$

Using equations (47), (46) and (48) it follows that

$$\hat{L} = (x_0 + x_1 + x_2)\,\hat{K}_{yc} + x_0\hat{\omega}_{20} - x_1\hat{\omega}_{12} \tag{74}$$

The following identity,

$$\left[\begin{array}{ccc} \left(\frac{L_c}{L}\right)\hat{L}_c & \left(\frac{L_x}{L}\right)\hat{L}_x & \left(\frac{L_y}{L}\right)\hat{L}_y \\ \left(\frac{K_{xc}}{k_x}\right)\hat{K}_{xc} & \left(\frac{K_{xx}}{k_x}\right)\hat{K}_{xx} & \left(\frac{K_{xy}}{k_x}\right)\hat{K}_{xy} \\ \left(\frac{K_{yc}}{k_y}\right)\hat{K}_{yc} & \left(\frac{K_{yx}}{k_y}\right)\hat{K}_{yx} & \left(\frac{K_{yy}}{k_y}\right)\hat{K}_{yy} \end{array}\right]\left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) = \left(\begin{array}{c} \hat{L} \\ \hat{k}_x \\ \hat{k}_y \end{array}\right) \tag{75}$$

may now be rewritten, after some substitutions using equations (74), (47), (46) and (48), as:

$$\left[\begin{array}{ccc} \left(\frac{L_c}{L}+\frac{\sigma}{v}\right) & \left(\frac{L_x}{L}\right) & \left(\frac{L_y}{L}\right) \\ \left(\frac{K_{xc}}{k_x}\right) & \left(\frac{K_{xx}}{k_x}\right) & \left(\frac{K_{xy}}{k_x}\right) \\ \left(\frac{K_{yc}}{k_y}\right) & \left(\frac{K_{yx}}{k_y}\right) & \left(\frac{K_{yy}}{k_y}\right) \end{array}\right]\left(\begin{array}{c} \hat{K}_{yc} \\ \hat{K}_{yx} \\ \hat{K}_{yy} \end{array}\right) = \left(\begin{array}{c} (\hat{\omega}_{12}-\hat{\omega}_{10})(1-x_0)-(\hat{\omega}_{12})x_1 \\ \hat{\omega}_{12}+\hat{k}_x \\ \hat{k}_y \end{array}\right) \tag{76}$$

In particular, to derive the first equation of (76) we use the fact that

$$(\hat{\omega}_{20}) = (\hat{\omega}_{10}) - (\hat{\omega}_{12})$$

and that:

$$x_0 + x_1 + x_2 = -\frac{\sigma}{v}$$

The equations given by (76) allow us to express the steady state elasticities of K_{yc}, K_{yx} and K_{yy} with respect to ω_{10}, ω_{12}, k_x and k_y.From the production functions on the other hand, we have:

$$\begin{aligned} \hat{c} &= (\alpha_0+a_0)\left(\hat{K}_{yc}-\hat{\omega}_{12}+\hat{\omega}_{10}\right)+(\alpha_1+a_1)\left(\hat{K}_{yc}-\hat{\omega}_{12}\right)+(\alpha_2+a_2)\left(\hat{K}_{yc}\right) \\ &= \hat{K}_{yc}+(\alpha_0+a_0)(-\hat{\omega}_{12}+\hat{\omega}_{10})-(\alpha_1+a_1)(\hat{\omega}_{12}) \end{aligned} \tag{77}$$

$$\hat{x} = \hat{K}_{yx}+(\beta_0+b_0)(-\hat{\omega}_{12}+\hat{\omega}_{10})-(\beta_1+b_1)(\hat{\omega}_{12}) \tag{78}$$

$$\hat{y} = \hat{K}_{yy}+(\gamma_0+c_0)(-\hat{\omega}_{12}+\hat{\omega}_{10})-(\gamma_1+c_1)(\hat{\omega}_{12}) \tag{79}$$

Now notice that since $\hat{K}_{yc}, \hat{K}_{yx}$ and $\hat{K}_{yy}$ depend on $\hat{\omega}_{10}$, $\hat{\omega}_{12}$, $\hat{k}_x$ and $\hat{k}_y$, and $\hat{\omega}_{10}$, $\hat{\omega}_{12}$ in turn depend on $\hat{p}_x$ and $\hat{p}_y$, we can now express the output elasticities of c, x, and y with respect to p_x, p_y, k_x and k_y. We define the elasticity function of a m-vector q with respect to an n-vector p as an $m \times n$

matrix as $E(q:p)$. Let the matrix in equation (76) be defined as:

$$[F] = \begin{bmatrix} \left(\frac{L_c}{L} + \frac{\sigma}{v}\right) & \left(\frac{L_x}{L}\right) & \left(\frac{L_y}{L}\right) \\ \left(\frac{K_{xc}}{k_x}\right) & \left(\frac{K_{xx}}{k_x}\right) & \left(\frac{K_{xy}}{k_x}\right) \\ \left(\frac{K_{yc}}{k_y}\right) & \left(\frac{K_{yx}}{k_y}\right) & \left(\frac{K_{yy}}{k_y}\right) \end{bmatrix}$$

Then we have:

$$[E_{KK}] \equiv E\left((K_{yc}, K_{yx}, K_{yy}) : (k_x, k_y)\right) = [F]^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now define $[S] = [N]^{-1}$. The price elasticities are:

$$\begin{aligned} [E_{KP}] &\equiv E\left((K_{yc}, K_{yx}, K_{yy}) : (p_x, p_y)\right) \\ &= \left[F^{-1}\right] \begin{bmatrix} (-S_{11} + S_{21})(1 - x_0) - x_1 S_{21} & (-S_{12} + S_{22})(1 - x_0) - x_1 S_{22} \\ S_{21} & S_{22} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where the elements of the matrix $[S]$ are denoted by S_{ij}. We will denote the ij'th element of $[E_{KK}]$ and $[E_{KP}]$ by $E_{KK}(i,j)$ and $E_{KP}(i,j)$. Substituting the elements of the matrices $[S]$, $[E_{KK}]$ and $[E_{KP}]$ into the production equations (77), (78), and (79), we obtain the output elasticities:

$$E(c:p_x) = E_{KP}(1,1) + (\alpha_0 + a_0)\left((S_{11} - S_{21})\right) - (\alpha_1 + a_1) S_{21}$$

$$E(c:p_y) = E_{KP}(2,1) + (\alpha_0 + a_0)\left((S_{12} - S_{22})\right) - (\alpha_1 + a_1) S_{22}$$

$$E(c:k_x) = E_{KK}(1,1)$$

$$E(c:k_y) = E_{KK}(2,1)$$

$$E(x:p_x) = E_{KP}(2,1) + (\beta_0 + b_0)\left((S_{11} - S_{21})\right) - (\beta_1 + b_1) S_{21}$$

$$E(x:p_y) = E_{KP}(2,2) + (\beta_0 + b_0)\left((S_{12} - S_{22})\right) - (\beta_1 + b_1) S_{22}$$

$$E(x:k_x) = E_{KK}(2,1)$$

$$E(x:k_y) = E_{KK}(2,2)$$

$$E(y:p_x) = E_{KP}(3,1) + (\gamma_0 + c_0)\left((S_{11} - S_{21})\right) - (\gamma_1 + c_1) S_{21}$$

$$E(y:p_y) = E_{KP}(3,2) + (\gamma_0 + c_0)\left((S_{12} - S_{22})\right) - (\gamma_1 + c_1) S_{22}$$

$$E\left(y:k_x\right)=E_{KK}\left(3,1\right)$$
$$E\left(y:k_y\right)=E_{KK}\left(3,2\right)$$

5.4.1 Output Elasticities for the Two-Sector Case

Simpler expressions can be obtained for the case of a two-sector model. We will derive an expression only for $\left(\frac{\partial x}{\partial k_x}\right)$ in the two-sector case, to be used in section (2) above[16]. Setting $\hat{\omega}_{10}$ and $\hat{\omega}_{12}$ to zero and using (44) and (47), the matrix equation (76) can be modified and written as:

$$\begin{bmatrix} \left(\frac{L_c}{L}+\frac{\sigma}{v}\right) & \left(\frac{L_x}{L}\right) \\ \left(\frac{K_{xc}}{k_x}\right) & \left(\frac{K_{xx}}{k_x}\right) \end{bmatrix}\begin{pmatrix} \hat{K}_{xc} \\ \hat{K}_{xx} \end{pmatrix}=\begin{pmatrix} 0 \\ \hat{k}_x \end{pmatrix} \tag{80}$$

Solving for $\hat{K}_{xx}, L_x$ we have,

$$\hat{K}_{xx}=\left(\frac{kL}{Q}\right)\left(\frac{L_c}{L}-\left(\frac{\sigma}{v}\right)\right)\hat{k}_x$$

where

$$Q=L_cK_x-L_xK_c+\left(\frac{\sigma}{v}\right)K_xL \tag{81}$$

Since $\hat{\omega}_{10}$ has been set to zero, from (47), we get:

$$\hat{L}_x=\hat{K}_{xx}-\omega_{10}=\left(\frac{kL}{Q}\right)\left(\frac{L_c}{L}-\left(\frac{\sigma}{v}\right)\right)\hat{k}_x$$

Using the input coefficients given by (63), (64), (67), and (68), but modified for the two-sector model, we obtain:

$$\begin{aligned}\left(\frac{Q}{L_cL_x}\right) &= \left(\frac{K_x}{L_x}\left(1+\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)-\frac{K_c}{L_c}\right)\\ &= \frac{\omega}{\alpha_0\beta_0}\left(\beta_1\left(1-a_0-a_1\right)-\alpha_1\left(1-b_0-b_1\right)+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)\end{aligned}$$

Now the growth of the output x will be :

$$\hat{x} = \left(\beta_1+b_1\right)\hat{K}_{xx}+\left(1-\beta_1-b_1\right)\hat{L}_x$$

$$
\begin{aligned}
&= \left(\frac{k_x L}{Q}\right)\left(\frac{L_c}{L}+\left(\frac{\sigma}{v}\right)\right)\hat{k}_x \\
&= \left(\frac{k_x L}{L_c L_x}\right)\left(\frac{\alpha_0\beta_0}{\omega}\right)\left(\frac{\left(\frac{L_c}{L}+\left(\frac{\sigma}{v}\right)\right)}{\left(\beta_1\left(1-a_0-a_1\right)-\alpha_1\left(1-b_0-b_1\right)+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)}\right)\hat{k}_x
\end{aligned}
$$

and we have

$$
\left(\frac{\partial x}{\partial k}\right)=\left(\frac{xL}{L_c L_x}\right)\left(\frac{\alpha_0\beta_0}{\omega}\right)\left(\frac{\left(\frac{L_c}{L}+\left(\frac{\sigma}{v}\right)\right)}{\left(\beta_1\left(1-a_0-a_1\right)-\alpha_1\left(1-b_0-b_1\right)+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)}\right)
$$

However, noting that at the steady state:

$$
\begin{aligned}
w &= rp = p\beta q_x\left(\left(\frac{\beta_0}{\beta_1\omega}\right)\right)^{(\beta_0+b_0)} \\
&= p\beta q_x\left(\left(\frac{\beta_1\omega}{\beta_0}\right)\right)^{(\beta_1+b_1)}\left(\frac{\beta_0}{\beta_1\omega}\right)
\end{aligned}
$$

and

$$
\left(\frac{x}{L_x}\right)=q_x\left(\frac{\beta_1\omega}{\beta_0}\right)^{(\beta_1+b_1)}=\left(\frac{r\omega}{\beta_0}\right)
$$

Therefore, we obtain:

$$
\begin{aligned}
\left(\frac{\partial x}{\partial k}\right) &= r\left(\frac{L}{L_c}\right)\left(\frac{\alpha_0\left(\frac{L}{L_c}+\left(\frac{\sigma}{v}\right)\right)}{\left(\beta_1\left(1-a_0-a_1\right)-\alpha_1\left(1-b_0-b_1\right)+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)}\right) \\
&= \left(\frac{r\alpha_0\left(1+\left(\frac{L}{L_c}\right)\left(\frac{\sigma}{v}\right)\right)}{\left(\beta_1\left(1-a_0-a_1\right)-\alpha_1\left(1-b_0-b_1\right)+\beta_1\alpha_0\left(\frac{\sigma}{v}\right)\frac{L}{L_c}\right)}\right) \qquad (82)
\end{aligned}
$$

We will use the expression $\left(\frac{\partial x}{\partial k}\right)$ in section 2 above.

6 Appendix II: The discrete time case

The discrete time problem can be defined as:

$$\begin{aligned} V(k_x, k_y, z) &= Max \left(\frac{1}{1-\sigma}\right)\left(z_c q_c L_c^{\alpha_0} K_{xc}^{\alpha_1} K_{yc}^{\alpha_2} \overline{L_c^{a_0} K_{xc}^{a_1} K_{yc}^{a_2}}\right)^{(1-\sigma)} - (1+v)^{-1} L^{(1+v)} \\ &\quad +\rho EV((1-g_x)k_x + x, (1-g_y)\, k_y + y, z') \end{aligned}$$

$$x = z_x q_x L_x^{\beta_0} K_{xx}^{\beta_1} K_{yx}^{\beta_2} \overline{L_c^{b_0} K_{xx}^{b_1} K_{yx}^{b_2}} \tag{83}$$

$$y = z_y q_y L_y^{\gamma_0} K_{xy}^{\gamma_1} K_{yy}^{\gamma_2} \overline{L_y^{c_0} K_{xy}^{c_1} K_{yy}^{c_2}} \tag{84}$$

where the discount factor is $\rho = (1 + (r - g))^{-1}$, g_x , g_y are depreciation rates, $z = (z_c, z_x, z_y)$, z_i is a technology shock where $\ell n\, z_i = \zeta_i$,

$$\zeta_{i,t+1} = \lambda_i \zeta_{i,t} + \varepsilon_{i,t+1}; \qquad 0 \leq \lambda_i \leq 1 \tag{85}$$

$i = c,\ x,\ y$, and $\varepsilon_{i,t+1}$ is iid, normally distributed and has mean zero. z' is the value attained by z in the subsequent period. Note that we can write the consumption output as:

$$c = z_c q_c (L - L_x - L_y)^{\alpha_0 + a_0} (k_x - K_{xx} - K_{xy})^{\alpha_1 + a_1} (k_y - K_{yx} - K_{yy})^{\alpha_2 + a_2}$$

The first order conditions, after simple substitutions, are:

$$c_t^{-\sigma} p_{x,t} = \rho E \left(c_{t+1}^{-\sigma} p_{x,t+1} \left(\frac{w_{x,t+1}}{p_{x,t+1}} + (1 - g_x) \right) \right) \tag{86}$$

$$c_t^{-\sigma} p_{y,t} = \rho E \left(c_{t+1}^{-\sigma} p_{y,t+1} \left(\frac{w_{y,t+1}}{p_{y,t+1}} + (1 - g_y) \right) \right) \tag{87}$$

and the equations for accumulation are given by

$$k_{x,t+1} = (1 - g_x)\, k_{x,t} + x_t \tag{88}$$

$$k_{y,t+1} = (1 - g_y)\, k_{y,t} + y_t \tag{89}$$

Before analyzing the dynamics it is easy to show that the steady state of the dynamic system, (86), (87), (88) and (89), (85), with the random variables z_c, z_x, z_y set to their long-run means, is identical to the steady state of the deterministic continuous time system if $g_x = g_y = g$. To see this set $\rho =$

$(1+(r-g))^{-1}$ and note that at a steady state this implies:

$$1+r-g=\frac{w_x}{p_x}+(1-g)=\frac{w_y}{p_y}+(1-g)$$

We can define $q_i=\bar{q}_i\bar{z}_i$ where $\bar{z}_i$ is the long run mean of z_i for $i=c,x,y$. The steady state values of variables computed in sections above therefore remain as before. Quantity and price variables will now depend on the realizations of shocks as well, so to study the dynamics around the steady state we now have to compute the elasticities of input coefficients and of outputs that will now incorporate the effects of the stochastic shocks.

First we note that the equations (43), (44), (45) as well as (76) remain unchanged. The equation (57) has to be slightly modified :

$$\begin{pmatrix}\hat{\omega}_{10}\\ \hat{\omega}_{12}\end{pmatrix}=[N]^{-1}\begin{pmatrix}\hat{p}_x+\hat{z}_x-\hat{z}_c\\ \hat{p}_y+\hat{z}_y-\hat{z}_c\end{pmatrix} \tag{90}$$

In terms of elasticities, this implies that

$$\begin{aligned}E\left(\omega_{10},\omega_{12}:z_x,z_y\right) &= E\left(\omega_{10},\omega_{12}:p_x,p_y\right)\\ E\left(\omega_{10}:\hat{z}_c\right) &= -E\left(\omega_{10}:p_x\right)\\ E\left(\omega_{12}:\hat{z}_c\right) &= -E\left(\omega_{12}:p_y\right)\end{aligned}$$

It follows therefore that:

$$E(c,x,y\ :\ z_x,z_y)=\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}+E(c,x,y:p_x,p_y) \tag{91}$$

$$E(c,x,y\ :\ z_c)=\begin{pmatrix}1&0&0\end{pmatrix}-E(c,x,y:p_x)-E(c,x,y:p_y) \tag{92}$$

Note that the elasticities of $\left(\frac{w_x}{p_x}\right)$and $\left(\frac{w_y}{p_y}\right)0$ can be obtained from (58) after a slight rearrangement to incorporate the technology shock . Let $r_x=\left(\frac{w_x}{p_x}\right)$ and $r_y=\left(\frac{w_y}{p_y}\right)$. Then

$$\begin{pmatrix}\hat{r}_x\\ \hat{r}_y\end{pmatrix}=\left(\begin{bmatrix}\beta_0+b_0 & \beta_2+b_2\\ \gamma_0+c_0 & \gamma_2+c_2-1\end{bmatrix}[N]^{-1}\right)\begin{pmatrix}\hat{p}_x+\hat{z}_x-\hat{z}_c\\ \hat{p}_y+\hat{z}_y-\hat{z}_c\end{pmatrix}+\begin{pmatrix}\hat{z}_x\\ \hat{z}_y\end{pmatrix}$$

In terms of elasticities, this implies that

$$E\left(r_x, r_y : z_x, z_y\right) = I + E\left(r_x, r_y : p_x, p_y\right) \tag{93}$$

$$E\left(r_x, r_y : z_c\right) = -E\left(r_x, r_y : p_x\right) - E\left(r_x, r_y : p_y\right) \tag{94}$$

Note that $E\left(r_x, r_y : p_x, p_y\right)$ is simply given by the elements of the matrix

$$\left(\begin{bmatrix} \beta_0 + b_0 & \beta_2 + b_2 \\ \gamma_0 + c_0 & \gamma_2 + c_2 - 1 \end{bmatrix} [N]^{-1}\right)$$

Now we have all the elements to evaluate the Jacobian corresponding to the linear system evaluated at the steady state. The relevant partial derivatives can be computed from the associated elasticities using steady state values. To compute the appropriate Jacobian we note first that at the steady state, $\rho\left(\frac{w_{y,t+1}}{p_{y,t+1}} + (1 - g_y)\right) = 1$. A similar equation holds for the linearization of (87). We can express the linearized dynamics as percentage deviations from the steady state with the help of the following matrices:

$$[R_{11}] = \begin{bmatrix} \left(\frac{k_x\left(\left(\frac{\partial x}{\partial k_x}\right)+1-g\right)}{x+(1-g)k_x}\right) & \left(\frac{k_y\left(\frac{\partial x}{\partial k_y}\right)}{x+(1-g)k_x}\right) \\ \left(\frac{k_x\left(\frac{\partial y}{\partial k_x}\right)}{y+(1-g)k_y}\right) & \left(\frac{k_y\left(\left(\frac{\partial y}{\partial k_y}\right)+1-g\right)}{y+(1-g)k_y}\right) \end{bmatrix}$$

$$[R_{12}] = \begin{bmatrix} \left(\frac{p_x\left(\frac{\partial x}{\partial p_x}\right)}{zx+(1-g)k_x}\right) & \left(\frac{p_y\left(\left(\frac{\partial x}{\partial p_y}\right)\right)}{zx+(1-g)k_x}\right) \\ \left(\frac{p_x\left(\frac{\partial y}{\partial p_x}\right)}{y+(1-g)k_y}\right) & \left(\frac{p_y\left(\frac{\partial y}{\partial p_y}\right)}{y+(1-g)k_y}\right) \end{bmatrix}$$

$$[R_{21}] = \begin{bmatrix} -\sigma\left(\frac{k_x}{c}\right)\left(\frac{\partial c}{\partial k_x}\right) & -\sigma\left(\frac{k_y}{c}\right)\left(\frac{\partial c}{\partial k_y}\right) \\ -\sigma\left(\frac{k_x}{c}\right)\left(\frac{\partial c}{\partial k_x}\right) & -\sigma\left(\frac{k_y}{c}\right)\left(\frac{\partial c}{\partial k_y}\right) \end{bmatrix}$$

$$[R_{22}] = \begin{bmatrix} -\sigma\left(\frac{p_x}{c}\right)\left(\frac{\partial c}{\partial p_x}\right) + 1 & -\sigma\left(\frac{p_y}{c}\right)\left(\frac{\partial c}{\partial p_y}\right) \\ -\sigma\left(\frac{p_x}{c}\right)\left(\frac{\partial c}{\partial p_x}\right) & -\sigma\left(\frac{p_y}{c}\right)\left(\frac{\partial c}{\partial p_y}\right) + 1 \end{bmatrix}$$

$$R_{13} = \begin{bmatrix} \left(\frac{z_c\left(\frac{\partial x}{\partial z_c}\right)}{x+(1-g)k_x}\right) & \left(\frac{z_x\left(\frac{\partial x}{\partial z_x}\right)}{x+(1-g)k_x}\right) & \left(\frac{z_y\left(\left(\frac{\partial x}{\partial z_y}\right)\right)}{x+(1-g)k_x}\right) \\ \left(\frac{z_c\left(\frac{\partial y}{\partial z_c}\right)}{y+(1-g)k_y}\right) & \left(\frac{z_x\left(\frac{\partial y}{\partial z_x}\right)}{y+(1-g)k_y}\right) & \left(\frac{z_y\left(\frac{\partial y}{\partial z_y}\right)}{y+(1-g)k_y}\right) \end{bmatrix}$$

$$R_{23} = \begin{bmatrix} -\sigma\left(\frac{z_c}{c}\right)\left(\frac{\partial c}{\partial z_c}\right) & -\sigma\left(\frac{z_x}{c}\right)\left(\frac{\partial c}{\partial z_x}\right) & -\sigma\left(\frac{z_y}{c}\right)\left(\frac{\partial c}{\partial z_y}\right) \\ -\sigma\left(\frac{z_c}{c}\right)\left(\frac{\partial c}{\partial z_c}\right) & -\sigma\left(\frac{z_x}{c}\right)\left(\frac{\partial c}{\partial z_x}\right) & -\sigma\left(\frac{z_y}{c}\right)\left(\frac{\partial c}{\partial z_y}\right) \end{bmatrix}$$

$$[R_{31}] = [R_{32}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[R_{33}] = \begin{bmatrix} \lambda_c & 0 & 0 \\ 0 & \lambda_x & 0 \\ 0 & 0 & \lambda_y \end{bmatrix}$$

$$[R] = \begin{bmatrix} [R_{11}] & [R_{12}] & [R_{13}] \\ [R_{21}] & [R_{22}] & [R_{23}] \\ [R_{31}] & [R_{32}] & [R_{33}] \end{bmatrix}$$

$$[Q_{11}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[Q_{12}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[Q_{13}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[Q_{21}] = \begin{bmatrix} \left(\frac{-\sigma k_x}{c}\right)\left(\frac{\partial c}{\partial k_x}\right) & \left(\frac{-\sigma k_y}{c}\right)\left(\frac{\partial c}{\partial k_y}\right) \\ \left(\frac{-\sigma k_x}{c}\right)\left(\frac{\partial c}{\partial k_x}\right) & \left(\frac{-\sigma k_y}{c}\right)\left(\frac{\partial c}{\partial k_y}\right) \end{bmatrix}$$

$$[Q_{22}] = \begin{bmatrix} \rho p_x\left(\frac{\partial r_x}{\partial p_x}\right) - \left(\frac{\sigma p_x}{c}\right)\left(\frac{\partial c}{\partial p_x}\right) + 1 & \rho p_y\left(\frac{\partial r_x}{\partial p_y}\right) - \left(\frac{\sigma p_y}{c}\right)\left(\frac{\partial c}{\partial p_y}\right) \\ \rho p_x\left(\frac{\partial r_y}{\partial p_x}\right) - \left(\frac{\sigma p_x}{c}\right)\left(\frac{\partial c}{\partial p_x}\right) & \rho p_y\left(\frac{\partial r_y}{\partial p_y}\right) - \left(\frac{\sigma p_y}{c}\right)\left(\frac{\partial c}{\partial p_y}\right) + 1 \end{bmatrix}$$

$$\begin{aligned} &[Q_{23}] \\ = &\begin{bmatrix} \rho z_c\left(\frac{\partial r_x}{\partial z_c}\right) - \left(\frac{\sigma z_c}{c}\right)\left(\frac{\partial c}{\partial z_c}\right) & \rho z_x\left(\frac{\partial r_x}{\partial z_x}\right) - \left(\frac{\sigma z_x}{c}\right)\left(\frac{\partial c}{\partial z_x}\right) & \rho z_y\left(\frac{\partial r_x}{\partial z_y}\right) - \left(\frac{\sigma z_y}{c}\right)\left(\frac{\partial c}{\partial z_y}\right) \\ \rho z_c\left(\frac{\partial r_y}{\partial z_c}\right) - \left(\frac{\sigma z_c}{c}\right)\left(\frac{\partial c}{\partial z_c}\right) & \rho z_x\left(\frac{\partial r_y}{\partial z_x}\right) - \left(\frac{\sigma z_x}{c}\right)\left(\frac{\partial c}{\partial z_x}\right) & \rho z_y\left(\frac{\partial r_y}{\partial z_y}\right) - \left(\frac{\sigma z_y}{c}\right)\left(\frac{\partial c}{\partial z_y}\right) \end{bmatrix} \end{aligned}$$

$$[Q_{31}] = [Q_{32}] = [R_{31}]$$

$$[Q_{33}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[Q] = \begin{bmatrix} [Q_{11}] & [Q_{12}] & [Q_{13}] \\ [Q_{21}] & [Q_{22}] & [Q_{23}] \\ [Q_{31}] & [Q_{32}] & [Q_{33}] \end{bmatrix}$$

The linearized dynamics are then:

$$\begin{pmatrix} \hat{k}_{x,t+1} \\ \hat{k}_{y,t+1} \\ \hat{p}_{x,t+1} \\ \hat{p}_{y,t+1} \\ \hat{z}_{c,t+1} \\ \hat{z}_{x,t+1} \\ \hat{z}_{y,t+1} \end{pmatrix} = [Q]^{-1}[R] \begin{pmatrix} \hat{k}_{x,t} \\ \hat{k}_{y,t} \\ \hat{p}_{x,t} \\ \hat{p}_{y,t} \\ \hat{z}_{c,t} \\ \hat{z}_{x,t} \\ \hat{z}_{y,t} \end{pmatrix} + [Q]^{-1} \begin{pmatrix} 0 \\ 0 \\ -\hat{s}_{x,t+1} \\ -\hat{s}_{y,t+1} \\ \hat{\varepsilon}_{c,t+1} \\ \hat{\varepsilon}_{x,t+1} \\ \hat{\varepsilon}_{y,t+1} \end{pmatrix} \tag{95}$$

where $\hat{s}_{i,t}, i = x, y$, is an *iid* sunspot shock with zero mean, acting on the

"Euler" equations for the two capital stocks. Note that the elements of the matrices $[R]$ and $[Q]$ are functions of the parameters of the system, and also of the steady state quantities which are functions of the parameters as well. We can therefore evaluate the roots of $[Q]^{-1}[R]$ to check for the possibility of indeterminacy. When externality parameters are set to zero, as is well-known the four of the roots of the Jacobian matrix come in pairs of $\left(\mu, \frac{1}{\rho\mu}\right)$, and the other three are the autoregressive coefficients of the technology shocks. For modest externalities however, it is easy to find large parameter regions for which there exists indeterminate equilibria, as the calibrated example in section (4) illustrates.

6.1 The Calibration

Equation (95) can easily be used to simulate or assess the stochastic properties of our dynamic system. In order to then obtain series for outputs c, x,and y, as well as their inputs, we must first express them as functions of (p_x, p_y, k_x, k_y). Using (43), (44), (45), and (73), we can set up the matrix equation:

$$\begin{pmatrix} \hat{\omega}_{10} \\ \hat{\omega}_{12} \\ \hat{\omega}_{10} \\ \hat{\omega}_{12} \\ \hat{\omega}_{10} \\ \hat{\omega}_{12} \\ \hat{k}_x \\ \hat{k}_y \\ 0 \end{pmatrix} = \begin{pmatrix} [N]^{-1}\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \\ [N]^{-1}\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \\ [N]^{-1}\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \\ \hat{k}_x \\ \hat{k}_y \\ 0 \end{pmatrix} \tag{96}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{K_{xc}}{k_x} & \frac{K_{xx}}{k_x} & \frac{K_{xy}}{k_x} & 0 & 0 & 0 \\ \frac{K_{yc}}{k_y} & \frac{K_{yx}}{k_y} & \frac{K_{yy}}{k_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & -x_1 & 0 & 0 & \frac{L_c}{L} - x_0 & \frac{L_x}{L} & \frac{L_y}{L} \end{pmatrix} \begin{pmatrix} \hat{K}_{yc} \\ \hat{K}_{yx} \\ \hat{K}_{yy} \\ \hat{K}_{xc} \\ \hat{K}_{xx} \\ \hat{K}_{xy} \\ \hat{L}_c \\ \hat{L}_x \\ \hat{L}_y \end{pmatrix}$$

where x_0, x_1, x_2 are as in equation (73). Equation (96), after inverting, will solve for $\hat{K}_{xc}$ and $\hat{K}_{yc}$ in terms of $(\hat{p}_x, \hat{p}_y, \hat{k}_x, \hat{k}_y)$, so that we can obtain the associated elasticities. We now have all the elements of the Jacobian and we can analytically compute the variance-covariance matrix of the variables in equation(95), for contemporaneous as well as lagged values. Furthermore, using these we can easily compute a larger variance-covariance matrix that includes linear functions of the original variables, like the outputs of the three goods, and the aggregate value of investment or GNP.

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Notes

[1]A long but incomplete list of the recent literature includes Beaudry and Deveruex [4], Benhabib and Farmer [6], [10], [7], [9], Boldrin and Rustichini [14], Chatterjee and Cooper [17], Christiano and Harrison [18], Farmer and Guo [19]), [20], Gali [21], Perli [27], Rotemberg and Woodford ([28]), Schmitt-Grohé [29], Weder [31] and Xie [32].

[2]Since Benhabib and Farmer [10] postulate constant returns at the private level, we can mesure increasing returns as the sum of all Cobb-Douglas coefficients minus one. Indeterminacy then, for standard parametrizations, requires increasing returns of about 0.07.

[3]See for example Basu and Fernald [1], [2], Burnside, Eichenbaum and Rebello [15] or Burnside [16], among others.

[4]In the same way the Stolper-Samuelson theorem implies that $-\left(\frac{\partial w_x}{\partial p}\right)$ will overwhelm r, since at a steady state $w_x = rp$, but the expression above for $\left(-\frac{\partial w_x}{\partial p} + r\right)$ already incorporates the steady state relationship.

[5]We should note that other distortions which interfere with true cost minimization are also likely to give rise to similar results.

[6]The argument for the case in which the capital good is labor intensive is similarly based on the Rybczynski and Stolper-Samuelson theorems,but in this case departing from

the initial equilibrium trajectory leads to explosive prices instead of outputs.

[7]A generalization of the results to a multisector framework will follow from the roots of $\left(\frac{\partial x}{\partial k}\right)$ and $\left(\frac{\partial w}{\partial p}\right)$, which now are matrices, and which are equal to each other if there are no external effects. It can be shown that only $\left(\frac{\partial w}{\partial p}\right)$ depends on external effects. Given any $r > g$, it is then possible to construct robust families of Cobb-Douglas technologies giving rise to indeterminacy for arbitrarily small external effects.

[8]With non-linear utility we will have indeterminacy in the two-sector model if the trace of $[J]$ is negative and its determinant is positive. It can in fact be shown that the trace will be negative under the assumptions of the proposition above because $\left(\frac{\partial c}{\partial k}\right)\left(\frac{\partial x}{\partial p}\right)$will be positive. For a positive determinant we must also assume that the term E in (12) is positive. However ε_{cp} appearing in E endogenous. It can be computed as a function of parameters, and it is possible to produce examples of indeterminate steady states for positive but small values of σ.

[9]It is also easy to construct examples of systems without externalities generating closed orbits or cycles as optimal paths, as in Benhabib and Nishimura [5]. For example if we set $r = 0.05, g = 0.01, v = 1, \sigma = 0.001,\ \alpha_1 = .0017,\ \alpha_2 = .459,\ \alpha_0 = 1 - \alpha_1 - \alpha_2,\ \beta_1 = .0265,\ \beta_2 = .0012, \beta_0 = 1 - \beta_1 - \beta_2,\ \gamma_1 = .5635, \gamma_2 = .423, \gamma_0 = 1 - \gamma_1 - \gamma_2$, then at $\sigma \approx .020386$, the Jacobian $[JN]$ has two complex roots with zero real parts which become negative for higher σ, and positive for lower σ, satisfying the conditions of the Hopf Bifurcation Theorem for existence of closed orbits. We note that the family of cycles as a function of σ in the example occur for low discount rates, but a for utility function of consumption that is close to linear.

[10]Here again since $\rho < 1$, there is the possibility of too many roots crossing and falling outside the unit circle even without external effects, making the steady state unstable and creating cycles on invariant circles via a Hopf bifucation, or cycles via flip bifurcation. Our concern here however is with too many roots inside the unit circle, a situation that implies indeterminacy.

[11]The same parameters in the deterministic version of the model in continous time would yield the four roots: -0.72, 0.06, -0.03, - 1.47. The three negative roots imply indeterminacy.

[12]Of course both sunspot shocks cannot be independently chosen: there is a joint restriction on the properties of the sunspot shocks and the innovations to technology shocks that is needed to guarantee that the solution remains stationary, and that the effect of the root outside the unit modulus is nullified. We choose the sunspot shock $\hat{s}_{y,t}$, as a linear combination of the innovations to technology shocks and the sunspot shock $\hat{s}_{y,t}$ in order to satisfy this restriction. We note that alternatively, we could have picked $\hat{s}_{y,t}$ independently and $\hat{s}_{x,t}$ to satisfy the restriction, or simply chosen them jointly.

[13]In multisector RBC models, in order to reduce volatility at the sectoral level and to insure that all sectors are procylical , it may be necessary to introduce adjustment costs for the movement of factors across sectors. See Huffman and Wynne [25].

[14]The impulse response function of consumption to a technology shock also exhibits the same behaviour in the multisector model of Weder in this issue.

[15]For an exploration of these issues in a three-sector model without external effects,

market distortions or indeterminacies, see Benhabib, Perli and Sakellaris [12].

[16] A full characterization of the two-sector case is available from the authors.