1 The Lucas-Baseline Model Described: Poisson arrivals

What happens in a meeting?

— Agent (a draw z from $F(z,t)$) meets another, z'. Note:cdf $F(z,t)$ defined as right cdf as opposed to regular left cdf

— Not symmetric: z is active, learns from meeting, but z' is passive.

— If $z' \geq z$, nothing happens; if $z' \leq z$ agent z adopts z' (draws are costs; to convert to productivities reverse inequalities for z,and use left cdf $F(z,t)$)

1.1 How do people meet others?

— Poisson Idea Arrivals: over time interval $(t, t+\Delta)$ meet another with probability $\alpha\Delta$

— More than one meeting with with probability $o(\Delta)$ (where the function $o(\Delta)$ satisfies $lim_{\to\infty} \frac{o(\Delta)}{\Delta} = 0$)

Now motivate differential equation for right cdf $F(z,t) = \Pr\{\tilde{z} \geq z \; at \; date \; t\}$

$$F(z, t+\Delta) = F(z,t) \times Pr\{no \; lower \; cost \; idea \; darrives \; in \; (t, t+\Delta)\}$$

$$\begin{aligned} Pr\{no \; lower \; cost \; arrives \; in (t, t+\Delta) &= Pr\{no \; ideas \; arrive \; in \; (t, t+\Delta)\} \\ +Pr\{one \; idea &> z \; arrives \; from \; F(z,t) \; in \; (t, t+\Delta)\} \\ +Pr\{more \; than \; one \; idea &> z \; arrives \; in \; (t, t+\Delta)\} \\ &= 1 - \alpha + \alpha\Delta F(t,z) + o(\Delta) \end{aligned}$$

Combine to get

$$F(z, t+\Delta) = F(z,t)\left[1 - \alpha\Delta + \alpha\Delta F(t,z) + o(\Delta)\right]$$

$$\frac{F(z,t+\Delta) - F(z,t)}{\Delta} = -F(z,t)\left[\alpha - \alpha F(z,t) - \frac{o(\Delta)}{\Delta}\right]$$

Let $h \to 0$ to get

$$\frac{\partial F(z,t)}{\partial t} = -\alpha F(z,t)\left[1 - F(z,t)\right]$$

Now fix z, treat as ODE, solve given initial $F(z,0) = G(z,)$ and characterize.

For fixed z solution is:

$$F(z,t) = \frac{G(z)}{G(z) + e^{\alpha t}(1 - G(z))}$$

and if the initial distribution is bounded below at $\overline{z}$ so $G(\overline{z}) = 1$, then all mass converges down to $\overline{z}$.

1.2 When there is growth in z

Assume that $z's$ grow at an exogenous rate g, say due to R&D. S. Thus without any search and technology diffusion, the distribution of z moves to the right at the rate g.

$$F(z, t+\Delta) = F\left(ze^{-g\Delta}, t\right)[1 - \alpha\Delta + \alpha\Delta F(t, z) + o(\Delta)]$$

We can re-write as

$$F(z, t+\Delta) - F\left(ze^{-g\Delta}, t\right) = F\left(ze^{-g\Delta}, t\right)[-\alpha\Delta + \alpha\Delta F(t, z) + o(\Delta)]$$

$$\frac{(F(z, t+\Delta) - F(z, t)) - (F((1 - g\Delta)z, t) - F(z, t))}{\Delta} = F\left(ze^{-g\Delta}, t\right)\left[-\alpha + \alpha F(t, z) + \frac{o(\Delta)}{\Delta}\right]$$

where for small Δ, expanding in power series and ignoring squared and higher order terms $e^{-g\Delta} = 1 - g\Delta$. Letting $\Delta \to 0$ we have

$$\begin{aligned}
\frac{\partial F(z,t)}{\partial t} + gz\frac{\partial F(z,t)}{\partial z} &= -\alpha F(z,t)[1 - \alpha F(t,z)] \\
\frac{\partial F(z,t)}{\partial t} &= -gz\frac{\partial F(z,t)}{\partial z} - \alpha F(z,t)[1 - \alpha F(t,z)]
\end{aligned}$$

which is a standard PDE with initial condition $F(z, 0) = G(z,)$ that can be solved by the method of characteristics. The question is whether there can be a non-degenerate stationary distribution.

1.2.1 Method of Characteristics

The idea is to first transform the pDE to an ODE along characteritic curves. Let

$$\begin{aligned} \frac{dz}{dr} &= -gz, \quad \frac{dt}{dr} = 1 \\ z(r) &= C_1 e^{-gr} \quad t(r) = r + C_2, \end{aligned}$$

with

$$\begin{aligned} C_2 &= 0, \; z(0) = C_1. \\ z(r) &= z(0) e^{-gr} \end{aligned}$$

Note that

$$\begin{aligned} \frac{dF(z(r), t(r))}{dr} &= \frac{\partial F(z,t)}{\partial z}\frac{dz}{dr} + \frac{\partial F(z,t)}{\partial t}\frac{dt}{dr} = -\alpha F(z(r), t(r))[1 - F(t(r), z(r))] \\ \frac{dF}{dr} &= -\alpha F[1 - \alpha F] \end{aligned}$$

which is an ODE in r, and has solution given the initial condition $G(z(0))$:

$$\begin{aligned} F(z(0), t) &= \frac{G(z(0))}{G(z(0)) + e^{\alpha r}(1 - G(z(0)))} \\ &= \frac{G(z(0))}{G(z(0)) + e^{\alpha t}(1 - G(z(0)))} \end{aligned}$$

Thus the initial distribution determines F as it slides with $z(t) e^{gt}$, so if the initial distribution was over (z_l, z_h), Then

$$F\left(e^{-gt} z(t), t\right) = \frac{G\left(z(t) e^{-gt}\right)}{G\left(z(t) e^{-gt}\right) + e^{\alpha t}\left(1 - G\left(z(t) e^{-gt}\right)\right)}, \quad z(t) \in \left(e^{gt} z_l, e^{gt} z_h\right)$$

Boundary Conditions If there are the boundary conditions with births, say $f(z_l, t) = h(t)$ the characteristic space is divided so that at t, the initial condition $G(z(0))$ determines the solution for $z \geq e^{gt} z_l$, and the boundary condition determines the solution for $z \in (z_l, e^{gt} z_l)$. We have:

$$F\left(e^{-gt} z(t), t\right) = \frac{G\left(z(t) e^{-gt}\right)}{G\left(z(t) e^{-gt}\right) + e^{\alpha t}\left(1 - G\left(z(t) e^{-gt}\right)\right)}, \quad z(t) \in \left(e^{gt} z_l, e^{gt} z_h\right)$$

The boundary condition determines the solution for $z \in (z_l, e^{gt} z_l)$. The density of those with productivity z at t but born with z_l are of age τ, so that $z = z_l e^{g\tau}$, so $\tau = -\frac{1}{g} \ln\left(\frac{z_l}{z}\right)$. Therefore we need the density of z at birth, $h(t - \tau) = h\left(t - \left(\frac{-1}{g} \ln\left(\frac{z_l}{z}\right)\right)\right)$ (no deaths.)

$$F\left(e^{-gt} z(t), t\right) = \frac{h\left(t + \frac{1}{g} \ln\left(\frac{z_l}{z}\right)\right)}{h\left(t + \frac{1}{g} \ln\left(\frac{z_l}{z}\right)\right) + e^{\alpha t}\left(1 - h\left(t + \frac{1}{g} \ln\left(\frac{z_l}{z}\right)\right)\right)} \text{ for } z \in \left(z_l, z_l e^{gt}\right)$$

Note: if you have wealth $z = z_l$ you are just born, so $h\left(t - \frac{1}{g} \ln\left(\frac{z_l}{z}\right)\right) = h(t)$ .Note that if your wealth is $z_l e^{gt}$, $h\left(t + \frac{1}{g} \ln\left(\frac{z_l}{z}\right)\right) = h\left(t + \frac{1}{g} \ln\left(\frac{z_l}{z_l e^{gt}}\right)\right) = h(0)$.