The distribution of wealth and redistributive policies

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The *modern* wealth distribution question:
- In the U.S. wealth is highly concentrated and unequally distributed:
  
  richest 1% owns 1/3 of total wealth
- Earnings are much less concentrated than wealth

- ”What mechanisms are necessary to generate savings behavior that leads to a distribution asset holdings consistent with the actual data;” Cagetti-De Nardi (2005).
The *old* wealth distribution question:
- "In all places and all times, the distribution of income remains [Pareto]. Neither institutional change nor egalitarian taxation can alter this fundamental constant of social sciences."


- What are the economic and sociological factors generating Pareto distributions of wealth?
Literature:

Exogenous stochastic process for wealth, usually different for low and high wealth levels:


2. Econophysics.
We are not after matching empirical distributions as:
Krusell-Smith(1998), Quadrini(2000),
Hughett(1996), De Nardi(2004),
Laitner(2001),

So much so that agents have no earnings in our economy!
The Economy:

OLG economy as in Yaari (1965) and Blanchard (1985):

continuous time; ex-ante identical agents with probability of death, at \( t, pe^{-pt} \); exogenous interest rate \( r \); fair annuity markets and assets; preferences for bequests; capital income and estate taxes
The individual’s maximization problem of an agent born at time $s$ is:

$$\max \int_t^\infty e^{(\theta + p)(t-v)}(u(c(s, v)) + p\phi((1 - b)\omega(s, v)))dv$$  

subject to:

$$w(s, t) = w(s, s)$$

$$+ \int_s^t ((r + p - \tau)w(s, v) - p\omega(s, v) - c(s, v))dv$$

In the interest of closed form solutions:

$$u(c) = \ln(c), \quad \phi(\omega) = \chi \ln \omega$$
The consumption-savings path which solves the agent’s maximization problem is characterized by:

\[ c = \eta w, \quad \omega = \chi \eta w, \quad \eta = \frac{(p + \theta)}{p\chi + 1} \]

\[ s(s,t) = (r - \theta - \tau)w(s,t) \]

Note:
- growth rate of an agent’s wealth, 
  \( g = r - \theta - \tau \), is independent of the preference parameter for bequests \( \chi \),
- \( g \) decreases with the capital income tax \( \tau \) but is independent of estate taxes \( b \),
- the fraction of wealth in the asset yielding an annuity is: 
  \[ 1 - \mu = \frac{\omega}{w} = \frac{(p + \theta)\chi}{p\chi + 1} \]
For given initial wealth, wealth grows exponentially with age.
Demographics:

- population is constant, normalized to 1,
- at any time $s$, $p$ agents die,
- of the $p$ agents dying at any $s$, only $q < p$ leave an inheritance,
- correspondingly, $p - q$ agents are born with no wealth and $q$ with the inherited wealth.
Aggregate wealth is:

\[ W(t) = \int_{\infty}^{t} w(s, t)pe^{p(s-t)} ds \]

The growth rate of wealth is constant across all agents in our economy:

\[ \dot{W} = (r - \tau - \theta - p)W + q(1 - \mu)(1 - b)W(t) + \gamma W \]

where is gov’t redistribution of taxes:

\[ \gamma = \tau + qb(1 - \mu) \]

The growth rate of aggregate wealth is then:

\[ g^r = r - \theta - p + q(1 - \mu) \]

and

\[ g - g^r = p - q(1 - \mu) - \tau > 0 \]
Welfare policies:

**Lump-sum subsidies.** All agents born at any $t$ with an inheritance receive a lump-sum subsidy $x(t) = x e^{g t}$.

Agents who do not inherit receive minimum wealth $w(t) = w_0 e^{g t}$. No agent who inherits has wealth less than $w(t)$.

**Means-tested subsidies.** All agents born at any $t$ with inheritance less than $w(t)$ get a transfer of wealth to bring them to $w(t)$.
In the case of lump-sum subsidies, a fiscal policy \((\tau, b)\) determines the set of feasible welfare policies \((w, x)\), which satisfies

\[(p - q)w(t) + qx(t) = \tau W(t) + qb(1 - \mu)W(t)\]

In the case of lump-sum subsidies, a fiscal the total amount paid by the dovernement depends on the distribution of wealth at \(t\):

\[(p - q)w(t) +
q \int_{w(t)}^{((1-b)(1-\mu))^{-1}w(t)} (w(t) - (1 - b)(1 - \mu)w)f(w, t)dw\]
The distribution of wealth:

Construction of the Chapman-Kolmogorov equations:
Let $f(w, t)$ be the distribution of wealth at $t$, $\sigma(w)$ be the wealth a parent needs at death for his heir born at $t + \Delta$ to inherit wealth $w$.

$$\int_{w_1}^{w} f(w, t + \Delta) \, dw$$

$$= (1 - p\Delta) \int_{(1 - g\Delta)w_1}^{(1 - g\Delta)w} f(w, t) \, dw + q\Delta \int_{\sigma(w_1)}^{\sigma(w)} f(w, t) \, dw$$

$$+ o(\Delta)$$

Differentiating with respect to $w$ and ignoring second-order terms (terms in $\Delta^2$),

$$f(w, t + \Delta) = (1 - p\Delta)(1 - g\Delta)f((1 - g\Delta w), t)$$

$$+ q\Delta \frac{\partial \sigma(w)}{\partial w} f(\sigma(w), t))$$

Rearranging, dividing by $\Delta$, letting $\Delta \to 0$
\[
\frac{\partial f(w, t)}{\partial t} = -(p + g)f(w, t) + q \frac{\partial \sigma(w)}{\partial w} f(\sigma(w), t) - g \frac{\partial f(w, t)}{\partial w}
\]
The dynamics are described by the linear partial differential equation (PDE), an initial condition for the initial wealth distribution, and a boundary condition that reflects the injection of wealth to newborns under our welfare policies. We assume for simplicity that at time $t = 0$ all agents have wealth greater than minimal wealth.

**The initial condition is:**

$$f(w, 0) = h(w) = 0, \quad w \geq w$$

**The boundary condition:**

$$f(w(t), t) = \frac{p - q}{g} \frac{1}{w(t)} + q \int_{w(t)}^{\sigma(w(t))} f(w, t)dw$$

Finally, with lump-sum subsidies we have:

$$\sigma(w) = \frac{w - x}{(1 - \mu)(1 - b)}$$

and with means-tested subsidies we have:

$$\sigma(w) = \frac{w}{(1 - \mu)(1 - b)}$$
The distribution of wealth $f(z, t)$ satisfies:

$$f(z, t) =$$

$$\left( \frac{z}{w} \right)^{-\frac{p}{g-g^r}-1} f(w, t - \tau(z, w))$$

$$+ q \int_{w}^{z} \frac{\partial \sigma(y)}{\partial y} f(\sigma(y), t - \tau(z, y))$$

$$6(y) \left( \frac{p}{g-g^r} \right) (g - g^r)^{-1}(z) \left( \frac{p}{g-g^r+1} \right) dy$$

for $z \in (w, we^{(g-g^r)t})$

$$f(z, t) =$$

$$e^{-(p+g-g^r)t} h(z e^{(g-g^r)t})$$

$$+ q \int_{w}^{z} \frac{\partial \sigma(y)}{\partial y} f(\sigma(y), t - \tau(z, y))(y) \left( \frac{p}{g-g^r} \right)$$

$$6(g - g^r)^{-1}(z) \left( \frac{p}{g-g^r+1} \right) dy$$

for $z \geq we^{(g-g^r)t}$

where $\tau(z, y) = \frac{\ln z}{\ln y} \frac{1}{g-g^r}$
Economic interpretation:

- \( \tau(z, y) = \frac{\ln z}{\ln y} \frac{1}{g - g^r} \) is the age of an agent with wealth \( z \) at \( t \), born with wealth \( y \).

- The first element of \( f(z, t) \) over \( z \in (w, we^{(g - g^r)t}) \) is
  \[
  \left( \frac{z}{w} \right) \left( \frac{g - g^r - 1}{g - g^r} \right) f(w, t - \tau(z, w)).
  \]
  It is the density of agents who entered the economy with wealth \( w \), have never died, and reached wealth \( z \) at \( t \). It is determined by the boundary condition at \( t - \tau(z, w) \).

- The second element of \( f(z, t) \) on \( z \in (w, we^{(g - g^r)t}) \) is the density of agents who entered the economy with some wealth \( y \), have never died, and reached wealth \( z \) at \( t \).

- Consider now the density of discounted wealth \( z \) at \( t \) greater than \( we^{(g - g^r)t} \). The only agents who have this wealth are: \( i \) agents born at time 0 who have never died, \( ii \)
heirs of agents who died at some time \( t^r < t \) with bequests larger than \( w e^{(g - g^r)t^r} \).
Special cases:
The distribution of wealth \( f(z,t) \) must then satisfy the PDE, the initial condition, but also the boundary condition. To set ideas we look at two easy special cases first.
No inheritance
- No preferences for bequests: \( \chi = 0 \) for all agents.
- Agents only invest in annuities, \( \mu = 1 \).
- All \( p \) newborns at time \( t \) receive \( w \) (\( q = 0 \)).
- Furthermore, \( g - g^r = p - \tau \).

In this economy, the density of wealth at the boundary \( w \) is constant over time and the boundary condition is reduced to:

\[
f(w, t) = \frac{p}{g - g^r} \frac{1}{w}
\]

while the initial condition is as before.
Proposition The economy without bequests has the following distribution of discounted wealth at each time \( t \):

\[
f(z, t) = \begin{cases} 
\frac{p}{p-\tau} \ w^{\frac{p}{p-\tau}} \ z^{-\left(\frac{p}{p-\tau}+1\right)} \\
\text{for } z \in (w, \ w e^{(p-\tau)t}) \\
e^{- (p+p-\tau)t} h(z e^{-(p-\tau)}) \\
\text{for } z \geq \ w e^{(p-\tau)t}
\end{cases}
\]

\( f(z, t) \) is a truncated Pareto distribution in the range \((w, \ w e^{(p-\tau)t})\). The ergodic distribution of discounted wealth is

\[
f(z) = \frac{p}{p-\tau} \ w^{\frac{p}{p-\tau}} \ z^{-\left(\frac{p}{p-\tau}+1\right)}
\]

which is a Pareto distribution with finite mean.
Full inheritance, no estate taxes
- Agents leave all of their wealth as inheritance, $\mu = 0$ ($\chi$ be large enough).
- No estate taxes, $b = 0$.
- However, at each time $t$, $p - q$ agents die without heirs and $p - q$ agents are born with minimal wealth $w$.
- Furthermore, $g - g^r = p - q - \tau$.

In this economy the boundary condition requires:

$$f(w, t) = \frac{p - q}{g - g^r} \frac{1}{w}$$
Proposition  The economy with full inheritance and no estate taxes has the following distribution of discounted wealth at each time $t$:

$$f(z, t) = \begin{cases} 
\frac{p-q}{p-q-\tau} \, w \frac{p-q}{p-q-\tau} \, z^{-\left(\frac{p-q}{p-q-\tau}+1\right)} & \text{if } z \in (w, we^{(p-q-\tau)t}) \\
\frac{e^{-(p+p-q-\tau)t}}{h(ze^{-(p-q-\tau)t})} & \text{if } z \geq we^{(p-q-\tau)t}
\end{cases}$$

It is a truncated Pareto distribution in the range $(w, we^{(p-q-\tau)t})$. The ergodic distribution of discounted wealth is

$$f(z) = \frac{p-q}{p-q-\tau} \, w \frac{p-q}{p-q-\tau} \, z^{-\left(\frac{p-q}{p-q-\tau}+1\right)},$$

a Pareto distribution with finite mean.

Note: equivalent to an economy without bequest where all agents die without heirs with probability $p-q$ (as if the fraction $q$ who die at $t$ leaving full inheritance do not "die").
**General case:** 1. Means-tested subsidies, \(0 < \mu, b < 1\)

**Proposition** The economy with inheritance, estate taxes, and means-tested subsidies has a stationary distribution of wealth

\[
f(z) = \frac{p-a^* q(1-\mu)(1-b)}{p-q(1-\mu)-\tau} w^{\frac{p-a^* q(1-\mu)(1-b)}{p-q(1-\mu)-\tau}} z^{-\left(\frac{p-a^* q(1-\mu)(1-b)}{p-q(1-\mu)-\tau} + 1\right)},
\]

\(0 < a^* < 1\)

a Pareto distribution with finite mean and

\[
a = \left(\frac{(1-\mu)(1-b)}{p-q(1-\mu)-\tau}\right)^{p-a q(1-\mu)(1-b)} - 1\right).
\]

Furthermore:
- \(a\) has a unique fixed point, \(a^*\), and \(0 < a^* < 1\).
- \(a^*\) has the interpretation of \(1 - \) the value of the cumulative Pareto distribution at \(w((1 - \mu)(1 - b))^{-1}\), it represents the fraction of the \(q\) agents that inherit wealth
greater than $w$. 
Do we have simply an "age" theory of the wealth distribution? We better not! In fact:

**Corollary** The stationary wealth distribution of agents $n$ years old is

$$A(z; n) = pf(z e^{-(g-g^r)n}) e^{-pn}, \quad z \geq e^{(g-g^r)n} w$$

where the density $f(z)$ is given by the above Proposition.

Since $f(z)$ is a Pareto distribution on $z \geq w$, it follows that $A(z; n)$ is also a Pareto density on $z \geq we^{(g-g^r)n}$ scaled down by the factor $pe^{-pn}$.

Inheritance generates the skewness of the wealth distribution within each age cohort in our economy.
General case: 1. Lump-sum subsidies, $0 < \mu, b < 1$

Proposition The economy with inheritance, estate taxes, and welfare policies with minimal wealth support and lump-sum subsidies has a stationary distribution of discounted wealth with the following properties:

i) for any $z$, it is bounded below by a Pareto distribution with exponent

$$\frac{p}{p - q(1 - \mu) - \tau}$$

and it is bounded above by a Pareto distribution with exponent

$$\frac{p - a^* q(1 - \mu)(1 - b)}{p - q(1 - \mu) - \tau} > 1, \quad 0 < a^* < 1;$$

ii) for large $z$, it is approximated by a Pareto distribution with exponent

$$\frac{p - a^* q(1 - \mu)(1 - b)}{p - q(1 - \mu) - \tau} > 1, \quad 0 < a^* < 1$$
FISCAL POLICY-Inequality

The Pareto exponent $P$ and Gini coefficient $G$ of the stationary distribution are:

\[ P = \frac{p - a^* q(1 - \mu)(1 - b)}{p - q(1 - \mu) - \tau}, \quad G = \frac{1}{2P - 1} \]

\[ a^* = (1 - \mu) \left( \frac{p - a^* q(1 - \mu)(1 - b)}{p - q(1 - \mu) - \tau} - 1 \right) \]

We proceed by characterizing the effects of policy variables $b \in [0, 1]$ and $\tau \in [0, p - q(1 - \mu)]$ on $P$.

**Proposition**

The Pareto coefficient of the economy’s stationary distribution of discounted wealth is increasing in capital income taxes $\tau$, $\frac{\partial P(\tau, b)}{\partial \tau} > 0$, and non-decreasing in estate taxes $b$, $\frac{\partial P(\tau, b)}{\partial b} \geq 0$. Perfect equality ($G = 0, P = \infty$) is attained for $\tau = p - q(1 - \mu)$ for any $b$. 
Simulations

Calibrate a simple economy:

\[ p = 0.016, \quad q = 0.013, \quad \theta = 0.04, \quad \chi = 10, \quad r = 1.08 \]

- \( p \) implies that expected lifetime \( p^{-1} = 62 \) years;
- \( \chi = 10 \) implies that agents with a positive bequest motive hold 0.49\% of their wealth in inheritable, non-annuitized assets;
- the fraction of the population that leave bequests is \( \frac{q}{p} = 0.8125 \).
Figure 1, shows the relationship between $P$ and the taxes $(b, \tau)$.

Capital income taxes affect $P$ essentially through $g - g^r = p - q(1 - \mu) - \tau$. 
As $\tau$ increases:
- $a^*(1 - \mu)(1 - b)$, the fraction of the $q$ agents that inherit wealth above $w$, declines, and
- the effect of estate taxes $b$ on $P = \frac{p-a^*q(1-\mu)(1-b)}{p-q(1-\mu)-\tau}$ becomes negligible:
FISCAL POLICY-Welfare

Discounted utility:

\[
U(z) = \frac{1}{\theta + p} \left( \frac{g(1 + p\chi)}{\theta + p} + \ln \eta + p\chi \ln(\eta\chi(1 - b)) \right) + \frac{1 + p\chi}{\theta + p} \ln z
\]

Utilitarian social welfare

\[
\Omega(w, P) = \frac{q}{p} \int_{w}^{\infty} U(z)f(z)dz + \frac{P - q}{p} \int_{w}^{\infty} U_0(z)f(z)dz
\]

Under balanced government budget mean wealth \( M \) is constant so \( w = M \frac{P - 1}{P} \). Then

\[
\Omega(M, P) = \frac{1}{p + \theta} \left( \frac{g(1 + p\chi)}{p + \theta} + \ln \eta + p\chi \ln(\eta\chi(1 - b)) \right) + \frac{(1 + p\chi)}{p + \theta} \left( \ln \left( \frac{P - 1}{P} M \right) + P^{-1} \right)
\]

where \( g = r - \theta - \tau \). Therefore (Chipman, 1974)
\[
\frac{\partial \Omega(M, P)}{\partial P} = \frac{(1 + p\chi)}{(p + \theta)^2} \frac{P^{-2}}{P - 1} > 0
\]
\[ \frac{\partial \Omega(M,P)}{\partial P} = \frac{(1 + p \chi)}{(p + \theta)^2} \frac{P^{-2}}{P - 1} > 0 \]

For our social welfare function, when \( P \) is maximized at \( \tau = p - q(1 - \mu) \), we have \( \frac{\partial P}{\partial b} = 0 \). Social welfare would then decline in \( b \) due to the bequest motive: the optimal \( b \) would be zero. If however \( \tau \) has an interior solution so that \( \frac{\partial P}{\partial b} > 0 \), we cannot determine whether or not \( b \) will be interior. In fact we can show that the value of \( \tau \) that maximizes social welfare has to be less than the maximum \( p - q(1 - \mu) \) because for \( \tau \to p - q(1 - \mu) \) we have \( (P - 1) \to \infty \), \( ((1 - \mu)(1 - b))^p P \to 0 \) and \( \frac{\partial \Omega(M,b,\tau)}{\partial \tau} < 0 \).
Another interesting feature of social welfare function is that even for small values of the bequest parameter $\chi$ (and interior $\tau$) we have $\frac{\partial \Omega(M,b,\tau)}{\partial b} < 0$, so that the maximizing social welfare requires setting $b = 0$. The reason is that for small values of $\chi$, the agent sets a high $\mu$ and therefore leaves a small bequest. The negative effect of $b$ on social welfare through its reduction of bequests, given by $-p\chi(1 - b)^{-1}$, dominates the positive egalitarian effect of $b$ on social welfare through the Pareto exponent.
Welfare is maximized at \((b, \tau) = (0, 0.0095)\). Maximum \(\tau\) is \(p - q(1 - \mu) = 0.0097\), so \(\tau\) is indeed interior. The Pareto exponent is \(P = 71.38\), the lower bound on wealth is \(\underline{w} = 0.986\), the fraction of the \(q\) agents who inherit more than \(\underline{w}\) is \((1 - \mu)(1 - b)a^* = (1 - \mu)^{71.38} = (2.66)(10)^{-23}\) and the fraction of wealth that the \(q\) agents hold in non-inheritable form is \(\mu = 0.5172\).
Lower Bequest Motive: $\chi = 0.01$

Despite the small $\chi$, welfare still declines with $b$ and it is maximized at $(b, \tau) = (0, 0.0158)$ where the maximum allowed value of $\tau$ is $p - q(1 - \mu) = 0.01599$. Now the welfare maximizing capital tax is higher at $\tau = 0.0157$, the Pareto exponent is lower at $P = 53.4631$, the lower bound on wealth is $\bar{w} = 0.9813$, the fraction of the $q$ agents who inherit more than $\bar{w}$ is $(1 - \mu)(1 - b)a^* = (1 - \mu)^{53.46} = 4.83(10)^{-228}$ and the fraction of wealth that the $q$ agents hold in non-inheritable annuitized form is $\mu = 0.9999$. Almost all the population is concentrated just below the mean wealth of 1.

However, the egalitarianism implicit in the social welfare function is implemented through capital rather than estate taxes. This now comes at the expense of growth.
of almost 1% to 1.5%.

For both $\chi = 10$ and $\chi = 0.001$, at the social welfare optimum estate taxes $b = 0$, capital taxes are interior but close to their maximum, and in both cases almost all the population is concentrated just below the mean wealth of 1. However, the egalitarianism implicit in the social welfare function is implemented through capital rather than estate taxes. Depending on the
bequest motive \( \chi \), this comes at the expense of growth of almost 1\% to 1.5\%.
Altruistic preferences-Imperfect Insurance

1. Consider an agent who values his son’s utility $\alpha \leq 1$, whose recursive maximization problem is

$$V(w(s, t)) = \max \int_t^\infty e^{(\theta + p)(t-v)}(\ln c(s, v) + p\alpha V(\omega(s, v)))dv$$

subject to

$$\frac{dw(s, t)}{dt} = (r + p)w(s, t) - p\omega - c(s, t)$$

Optimal decisions of an altruistic agent corresponds to those of an agent with ”joy of giving” preferences with endogeneous preference for bequest

$$\chi = \frac{1}{\theta + p(1-\alpha)}.$$ Therefore,

$$c = (\theta + p(1-\alpha))w, \quad \omega = \alpha w$$

$$\omega(s, t) = (r - \theta - \tau)w(s, t)$$

When $\alpha = 1$ and the parent cares about his son as for himself all of the wealth is
deposited in the bequest account.
Imperfect Insurance

2. Consider a dynastic infinite horizon economy. Agents face Poisson probability $p$ that their wealth is wiped out unless invested in a protected insurance account. Wealth held at $t$ by an agent born at $s$ in the protected insurance account is $\omega(s, t)$, $r$ is the interest rate. Assume the insurance account pays $r - \delta$, where $\delta - p \geq 0$ is a measure of market imperfection. The agent’s problem is

$$V(w(s, t)) = \max \int_t^\infty e^{(\theta+p)(t-v)}(\ln c(s, v) + pV(\omega(s, v)))dv$$

$$\frac{dw(s, t)}{dt} = rw(s, t) - \delta \omega - c(s, t)$$

Then

$$c = \theta w, \quad \omega = \frac{p}{\delta} w$$

So our analysis extends to the dynastic economy if $\chi$ is taken as endogenous and $1 - \mu$ is appropriately redefined. If $p = \delta$
and insurance is frictionless, all wealth is deposited in the insurance account.