Private Information and Sunspots in Sequential Asset Markets[*]

Jess Benhabib[†] Pengfei Wang[‡]

(This version: September 7, 2014)

Abstract

We study a model where some agents have private information about risky asset returns and trade to obtain capital gains, while others acquire the risky asset and hold it to maturity, forming expectations of returns based on market prices. We show that under such a structure, in addition to fully revealing rational expectations equilibria, there exists a continuum of equilibrium prices consistent with rational expectations, where the asset prices are subject to sunspot shocks. Such sunspot shocks can generate persistent fluctuations in asset prices that look like a random walk in an efficient market.

Keywords: The Grossman-Stiglitz paradox, Sunspots

JEL codes: D82, D83, G12,G14

1 Introduction

The efficient markets hypothesis states that prices on traded assets reflect all publicly available information. In their classic work Grossman and Stiglitz (1980) discussed a model where agents can obtain private information about asset returns and can trade on the basis of that information. If, however, the rational expectations equilibrium price reveals the information about the asset, and if information collection is costly, then agents have no incentive to collect the information before they observe the price and trade. But then prices no longer reflect the information about the asset, and markets are no longer efficient. Since then a large empirical and theoretical literature has explored the informational efficiency of markets under private information.[1]

We study the possibility of multiple rational expectations sunspot equilibria driven by non-fundamentals in asset markets with private information by introducing a simple time dimension

[*]We are grateful to referees and the editor for comments and to Federico Filippini who caught an error in an early draft. Wang acknowledges the financial support from the Research Grants Council of Hong Kong under project number 693513.

[†]Department of Economics, New York University, 19 West 4th Street, New York, NY 10012, USA. Office: (212) 998-8971, Fax: (212) 995-4186. Email: jess.benhabib@nyu.edu.

[‡]Department of Economics, The Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong. Office: (+852) 2358 7612. Email: pfwang@ust.hk

[1]See for example Malkiel (2003).

to markets where agents trade sequentially. In our simplest benchmark model short-term traders have noisy information about the return or dividend yield of the asset, but hold and trade the asset before its return is realized at maturity. The returns to short-term traders consist of capital gains. Investors, on the other hand, who may not have private information about the returns or dividend yields, but can observe past and current prices, purchase and hold the asset for its final dividend return. While we do not impose constraints on borrowing, asset holdings, or short-selling,[2] we exclude traders that have private information on dividends from holding the risky asset from its inception at time 0 all the way to its maturity when terminal dividends are paid.[3] We show that under such a market structure, in addition to equilibria where equilibrium prices fully reveal asset returns as in Grossman and Stiglitz (1980), there also exists a continuum of equilibria with prices driven by sunspot shocks. These equilibria are fully consistent with rational expectations and they are not randomizations over multiple fundamental equilibria. Furthermore the sunspot or sentiment shocks generate persistent fluctuations in the price of the risky asset that look to the econometrician like a random walk in an efficient market driven by fundamentals.[4]

Multiple equilibria also arises in Cespa and Vives (2014) in a two period model of sequential trading with short-term traders, as well as noise or liquidity traders whose demands for assets are persistent, or correlated across periods. Risk-averse short-term traders limit their trading because of price uncertainty, but the persistence of asset demands by liquidity traders limits the information about fundamentals revealed by prices. This dampens the response of investor demand since prices become less informative about fundamentals (liquidation values), and thus tends to reduce price volatility. On the other hand faced with lower volatility, risk-averse short-term traders trade more aggressively based on their private information. Under these effects, driven by the persistence of the demand by liquidity traders across periods, we obtain two equilibria: a high information equilibrium with low volatility and prices informative about fundamentals, and a low information equilibrium with high volatility and prices that are less informative about fundamentals.

In the next two sections we start with a simple three-period model and derive results on the fundamentals and the sunspot equilibria of our model, and we discuss the intuition behind our results. In Section 4 we study more general information and signal structures to show that our results are robust to such generalizations. We relax the assumption that all short-term traders perfectly observe the same sunspot and allow them to observe private sunspot or sentiment signals that are correlated. We show that our results in the benchmark model carry over in this case. We

[2] Compare, for example, with Miller (1977) or Harrison and Kreps (1978) where traders hold heterogeneous beliefs about terminal returns, but where short-selling constraints rule out unbounded trades.

[3] This market structure or similar ones involving short-term traders and long-term investors have been widely used, for example in Cass and Shell (1983), Allen and Gorton (1993), Allen, Morris and Shin (2006), as well as in Angeletos, Lorenzoni and Pavan (2010) where entrepreneurs sell their investments to traders. See our discussion in Section 9.

[4] See Section 7. For a survey of the literature on asset prices driven by sentiments see Baker and Wurgler (2007).

then also allow the long-term investors to receive private signals on the dividends and on sunspot shocks that may be correlated with the signal of short-term traders. We show that the sunspot or sentiment driven equilibria are robust to this generalized information structure. In Section 5 we allow long-term investors to also trade in the initial period, and to obtain private signals on dividends and sunspots, which again can be correlated with the signals of short-term traders. We show that there can still be a continuum of sunspot equilibria in this context, and that our results are not driven by a market structure that excludes long-term investors from trading in period 0.

In Section 6 we introduce multiple assets and show that the co-movement of asset prices in excess of co-movements in fundamentals can be explained by our sunspot equilibria. In Section 7 we extend our model to multiple periods. We show that asset prices under the sunspot equilibria exhibit random walk behavior even though the asset prices are not purely driven by fundamentals.

Recently Vives (2014) questioned whether fully revealing rational expectation are implementable: if short-term traders choose ignore their private signals at fully revealing equilibrium prices, how in fact are such equilibrium prices incorporating private signals are realized in the market? Since Vives' critique applies in our model, we address the implementability of our equilibria in Section 8 along the lines suggests in Vives (2014), and we also consider the approach of Golosov, Lorenzoni and Tsyvinski (2014). Finally to put our results in context in Section 9 we briefly discuss some papers in the literature that are closely related to ours.

We should also emphasize that we have deliberately not introduced any noise traders or imperfectly observed stochastic asset supplies, often used in the literature on asset prices to prevent prices from being perfectly revealing[5]. Therefore the continuum of sunspot equilibria that we obtain in our model are not related to noise traders in any way.

2 The Model

We start with a three-period benchmark model with a continuum of short-term traders and long-term investors. We index the short-term trader by j and the long-term investor by i. In period 0 there is a continuum of short-term traders of unit mass endowed with 1 unit of an asset, a Lucas tree. This tree yields a dividend D in period 2. We assume that

$$\log D = \theta. \tag{1}$$

where θ is drawn from a normal distribution with a mean of $-\frac{1}{2}\sigma_\theta^2$ and variance of σ_θ^2. Each trader in period 0 is a short-term trader who receives utility in period 1 and therefore sells the asset in period 1 before D is realized in period 2. This short-term trader, perhaps due to his involvement

[5] Vives (2014) is a notable exception that provides a solution to the Grossman-Stiglitz Paradox without resorting to noise traders.

in creating and structuring the asset, receives a signal s_j

$$s_j = \theta + e_j \tag{2}$$

where e_j has a normal distribution with a mean of 0 and a variance of σ_e^2. We assume that e_j is independent of θ. So the short-term trader j in period 0 solves

$$\max_{x_{j0}, B_{j0}} \mathbb{E}[C_{j1}|P_0, s_j] \tag{3}$$

with the budget constraints

$$P_0 x_{j0} + B_{j0} = P_0 + w. \tag{4}$$

$$C_{j1} = P_1 x_{j0} + B_{j0.} \tag{5}$$

where w is his endowment or labor income, x_{j0} is the asset quantity and B_{j0} is a safe bond that he carries over to period 1. We assume that there is no restriction on B_{j0}. Therefore using the budget constraint we can rewrite the $j'th$ short-term trader's problem as[6]

$$\max_{x_{j0} \in (-\infty, +\infty)} \mathbb{E}[P_0 + w + (P_1 - P_0)x_{j0}|P_0, s_j] \tag{6}$$

There is a continuum of investors of unit mass in period 1 who trade with the short-term traders. Each of them is endowed with w and enjoys consumption in period 2 when the dividend D is realized. These investors solve a similar problem, but they have no direct information about the dividend of the Lucas tree, except through the prices they observe. Hence an investor i in period 1, solves

$$\max_{x_{i1}, B_{i1}} \mathbb{E}[C_{i2}|P_0, P_1] \tag{7}$$

with the budget constraints

$$P_1 x_{i1} + B_{i1} = w \tag{8}$$

$$C_{i2} = D x_{i1} + B_{i1.} \tag{9}$$

where w is his endowment, x_{i1} is his asset purchase, and B_{i1} is his bond holdings carried over to period 2. Similarly the objective function (7) can be written as

$$\max_{x_{i1} \in (-\infty, +\infty)} \mathbb{E}[w + (D - P_1)x_{i1}|P_0, P_1], \tag{10}$$

after substituting out B_{i1} from the budget constraints.

[6] Note that we are not restricting the domains of $x_{j0} \in (-\infty, +\infty)$ and $x_{i0} \in (-\infty, +\infty)$, so in principle traders and investors may choose unbounded trades in the risky asset. This will of course be impossible in equilibrium since the asset supply $x = 1$. Alternatively we could constrain trades so $x_{j0}, x_{i0} \in (-B_l, B_h)$, $B_l, B_h > 0$, with results unaffected for $B_l, B_h \geq 1$ for example.

3 Equilibrium

An equilibrium is a pair of prices $\{P_0, P_1\}$ such that x_{j0} solves problem (6) and x_{i1} solves problem (10), and markets clear. Formally we define our equilibrium concept below.

Definition 1 *An equilibrium is an individual portfolio choices $x_{j0} = x(P_0, s_j)$ for the short-term traders in period 0, $x_{i1} = y(P_0, P_1)$ for the long-term investors in period 1, and two price functions $\{P_0 = P_0(\theta), P_1 = P_1(\theta)\}$ that jointly satisfy market clearing and individual optimization,*

$$\int x_{j0} dj = 1 = \int x_{i1} di, \tag{11}$$

$$P_0 = \mathbb{E}[P_1 | P_0, s_j] \tag{12}$$

for all $s_j = \theta + e_j$, and

$$P_1 = \mathbb{E}\left[D | P_0, P_1\right]. \tag{13}$$

Equation (11) gives market clearing, (12) gives the first-order condition for an interior optimum for the short-term trader, and equation (13), the first-order condition for the long-term investors, says that the long-term-investors are willing to pay a price equal to their Bayesian updating of the dividend. Under these interior first-order conditions our risk-neutral agents are indifferent about the amount of the asset they carry over, so for simplicity we may assume a symmetric equilibrium with $x_{i0} = x = 1$, and $x_{i1} = x = 1$. Hence the market clearing condition (11) holds automatically. In what follows, we only need to check equations (12) and (13) to verify an equilibrium.

Proposition 1 $P_0 = P_1 = \exp(\theta)$ *is always an equilibrium.*

Proof. *The proof is straightforward. It is easy to check that both (12) and (13) are satisfied.* ■

In this case, the market price fully reveals the fundamental value. Whatever their individual signals, traders in period 0 will be happy to trade at $P_0 = \exp(\theta)$, which reveals the dividend to investors in period 1. Even though each trader j in period 0 receives a noisy private signal s_j about θ, which may be high or low, these traders choose to ignore their signal. In maximizing their utility they only care about the price at which they can sell next period. If each short-term trader believes the price in the next period depends on θ, competition in period 0 will then drive the market price exactly to $\exp(\theta)$. For a given market price, the expected payoff to holding one additional unit of the asset will be $\mathbb{E}\left\{[\exp(\theta) - P_0] | P_0, s_j\right\}$. As long as $\log P_0 \neq \theta$, those traders with low signals would want to short the risky asset while those traders with high signals would want to go long. An equilibrium can only be reached when the market price has efficiently aggregated all private information in such a way that idiosyncratic signals cannot increase profits any further based on private information: namely $\log P_0 = \int s_j dj = \theta$. Since price fully reveals the dividend, the long-term investors will be happy to pay $\log P_1 = \theta$ in the next period.

There is also a second equilibrium where the market price reveals no information about dividends.

Proposition 2 *$P_0 = P_1 = 1$ is always an equilibrium.*

Proof. *Both (12) and (13) are satisfied. It is clear that with $P_0 = P_1 = 1$, investors in period 1 obtain no information about the dividend as the prices simply reflect the unconditional expectation of the dividend in period 2.* ■

Again in the above equilibrium, the short-term traders "optimally" choose ignore their private signals. If the short-term traders believe that the price in the next period is independent of θ, then their private signal s_j is no longer relevant for their payoff.

3.1 Sentiment-driven Equilibria

We now assume the traders in period 0 also receive some sentiment or sunspot shock z which they believe will drive prices. We also assume that z has a standard normal distribution. We define an sentiment-driven equilibrium as follows.[7]

Definition 2 *An sentiment-driven equilibrium is given by optimal portfolio choices $x_{j0} = x(P_0, s_j, z)$ for the short-term trader in period 0, $x_{i1} = y(P_0, P_1)$ for the long-term investors in period 1, and two price functions $\{P_0 = P_0(\theta, z), P_1 = P_1(\theta, z)\}$ that jointly satisfy market clearing and individual optimization,*

$$\int x_{j0} dj = 1 = \int x_{i1} di, \tag{14}$$

$$P_0 = \mathbb{E}[P_1 | P_0, s_j, z] \tag{15}$$

for all $s_j = \theta + e_j$ and z, and

$$P_1 = \mathbb{E}\left[D | P_0, P_1\right], \tag{16}$$

[7] The equilibria may simply be defined in the context of a Bayesian game where players are the short-term traders $j \in J = [0,1]$ and long-term investors $i \in I = [0,1]$, and each player is endowed with $w > 0$. Each short term trader is also endowed with one unit of the risky asset $x = 1$. The action spaces can be taken as $x_{j0} \in [-B_l, B_h] = B$ for short-term traders and $x_{j0} \in [-B_l, B_h] = B$ for the long-term investors where $B_l, B_h > 0$, and where in the paper we take $B_l = B_h = \infty$. (Of course in equilibrium the aggregate asset supply must be $x = 1$ so unbounded trades are impossible.) The states of the world $S = \left(\theta, z, \{\varepsilon_j\}_{j\in[0,1]}\right)$ are realizations of $\left(\theta, z, \{\varepsilon_j\}_{j\in[0,1]}\right)$ according to the probability distributions. For prices P_0, P_1, payoffs for short-term traders j are defined by $P_0 + w + (P_1 - P_0)x_{j0}$ and payoffs for long-term investors i by $w + (D - P_1)x_{i1}$. Strategies for short-term traders j map (P_0, s_j, z), where $s_j = \theta + e_j$, into actions $x_{j0} \in B$ to optimize (6), and strategies for long-term traders map (P_0, P_1) into actions $x_{i1} \in B$ to optimize (10), both using optimal Bayesian updating in the forming of expectations. The equilibrium given by Definition 2 defines equilibrium price functions under market clearing and optimization by agents.

As in Section 3, equation (14) gives market clearing, (15) gives the first-order condition for an interior optimum for the short-term traders and equation (16), the first order condition for the long-term investors, says the long-term investors are willing to pay a price equal to their conditional expectation of the dividend.

Proposition 3 *There exists a continuum of sentiment driven equilibria indexed by* $0 \leq \sigma_z^2 \leq \frac{1}{4}\sigma_\theta^2$, *with* $x_{i0} = 1 = x_{j1}$ *and the the prices in two periods are given by*

$$\log P_1 = \log P_0 = \phi\theta + \sigma_z z, \tag{17}$$

where

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_\theta^2}} \leq 1. \tag{18}$$

Proof. Note that since the prices are the same in both periods, (15) is satisfied automatically. We only need to check if equation (16) is satisfied. Taking the log of equation (16) generates

$$\begin{aligned} \log P_1 &= \phi\theta + \sigma_z z \qquad (19) \\ &= \log \mathbb{E}\{\exp(\theta|\phi\theta + \sigma_z z)\}, \\ &= \mathbb{E}\left[\theta|\phi\theta + \sigma_z z\right] + \frac{1}{2} var(\theta|\phi\theta + \sigma_z z) \\ &= -\frac{1}{2}\sigma_\theta^2 + \frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\left[\phi\theta + \sigma_z z + \frac{\phi}{2}\sigma_\theta^2\right] \\ &\quad + \frac{1}{2}\left[\sigma_\theta^2 - \frac{\left(\phi\sigma_\theta^2\right)^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\right]. \end{aligned}$$

which follows from the property of the normal distribution. Comparing terms, coefficients of $\phi\theta + \sigma_z z$ yield

$$\frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2} = 1. \tag{20}$$

Solving equation (20) yields the expression of ϕ in equation (18). ■

In this case traders in period 0 receive a common sunspot shock z. The investors in period 1, in forming their expectation of D conditional on the prices, believe prices are affected by the sunspot z, as in equation (19). But now they face a signal extraction problem of distinguishing θ from z. Their first-order conditions will be satisfied in equilibrium provided the variance of z lies within the interval given in Proposition 3, generating a continuum of sunspot equilibria indexed by σ_z^2. For example, a low z will induce pessimistic expectations for the period 0 trader, who will pay a low price for the asset and expect a low price in the next period. The investor in period 1 will observe the period 0 price and infer that in part, this must be due to a low dividend yield. He will then also pay a low price in period 1, thus confirming the expectations of the period 0 trader. Of course for this to be possible for every realization of the sunspot z, the variance of z that enters the signal extraction problem of the investor in period 1 must lie in the interval given in the above proposition.

We can understand the intuition behind the multiplicity by analogy to the Keynesian Beauty Contest put in the context of informational asymmetries and correlated signals. Note that the multiplicity of equilibria in our model does not hinge on the precision of signal s_j. We may, for simplicity, assume that $s_j = \theta$, so the short-term investors are assumed to know the dividend for sure. The price revealing equilibrium in this case would be $\log P_0 = \log P_1 = \theta$. As in the Keynesian Beauty Contest however, even though short-term traders' own views of the true value of stock (the dividend) is equal to θ, this does not matter to them. If the other short-term traders think the price is different from θ, a short-term trader will still be willing to accept such a price as long as he can sell it in the next period at the same price. So any price can support an equilibrium from the short- term trader's point of view. However, a price can only be an equilibrium price if the long-term investors are willing to trade the asset at such a price. In order for a rational expectation sunspot equilibrium to exist, the price has to reflect the fundamental dividend value θ with noise in such a way that the Bayesian updating of θ exactly equals the market price. This gives a restriction on the coefficient ϕ and the variance of noise in the price rule (19). The role of sunspots then is to correlate and coordinate the behavior of short-term traders so the investors, knowing the variance of sunspots and fundamentals, can optimally update their expectation of the dividends that they will collect in equilibrium.

In our three-period model the assumption that short-term traders cannot participate in period 2 trades is important. If the short-term traders are allowed to trade in period 2, the multiplicity of equilibrium will disappear. In such a case if the price $\log P_0 = \log P_1 < \theta$, the short-term trader will opt to go long on the asset in period 1. The purchase of each additional unit of the risky asset will increase his utility by $\exp(\theta) - P_1 > 0$. Competition will then bring the price to $\log P_0 = \log P_1 = \theta$. Likewise any price (in logs) that is above θ will induce the short-term trader to short the asset, forcing $\log P_0 = \log P_1 = \theta$ in equilibrium.

For the same reason in a multi-period context with periods $t = 0, ...T + 1$, such as in Section 7, for sunspots to exist we have to rule out traders that can hold risky assets from period 0 all the way to maturity at $T + 1$: If the market price at any t differs from such traders' expectations of the terminal dividend θ, they will be able to arbitrage the difference by buying or short-selling the asset, unless we explicitly introduce borrowing or short-selling constraints to prevent arbitrage.

In the following sections we will relax the informational assumptions of our model and generalize it to multiple assets and periods.

4 Alternative Information Structures

We examine the robustness of our results to alternative information structures. We first relax the assumption that all short-term traders perfectly observe the same sunspot. Instead, we assume that they observe private sunspots or sentiments that are correlated. Thus their sentiments are heterogeneous but correlated. We show that our results in the benchmark model carry over in this case. We then also allow the investors to receive some private signal on the dividend and on sunspot shocks. We show that the sunspot-driven or sentiment-driven equilibria are robust to this generalized information structure. For expositional convenience, we denote the information sets of a particular short-term trader and the investor by Ω_0 and Ω respectively. The equilibrium conditions can then be written as

$$P_0 = \mathbb{E}[P_1|\Omega_0], \tag{21}$$

and

$$P_1 = \mathbb{E}\left[D|\Omega_1\right]. \tag{22}$$

We can now proceed to study various alternative specifications of the information sets Ω_0 and Ω_1.

4.1 Heterogenous but Correlated Sentiments

If each short-term trader receives a noisy sentiment or sunspot shock z, then the information set Ω_0 for a particular trader becomes $\Omega_0 = \{P_0, \theta + e_j, z + \varepsilon_j\}$, where ε_j are drawn from a normal distribution with a mean of 0 and a variance of σ_ε^2 and $cov(e_j, \varepsilon_j) = 0$. Note again that the sentiment or sunspot shocks are correlated across traders due to the common component z. Furthermore $\Omega_1 = \{P_0, P_1\}$ is the same as in our benchmark model of the previous section. In this case, equilibrium prices still take the form described in equation (17). Namely there exist an continuum of sunspot equilibria indexed by the sunspot's variance as in Proposition 3. It is easy to check that (21) holds for any realization of e_j and ε_j, hence the short-term trader's first order conditions hold. Since the information set Ω_1 is the same as in the benchmark model, equation (22) will be automatically satisfied. The market "efficiently" washes out the noise ε_j. The intuition is similar to that behind the fully revealing equilibrium in the benchmark model. The market clearing condition must be such that the agents choose ignore their private signal. Otherwise agents with a high realization of $\theta + e_j$ or $z + \varepsilon_j$ would go long on the assets while agents with low signals would keep shorting them, destroying the market equilibrium. An equilibrium can be reached only if no agent has an incentive to either short the asset or go long on it based on his own private information. In equilibrium the prices must incorporate all private information, so that given the market price no agent has an informational advantage based on his own signal.

4.2 The Investors and Market Signals on the Dividend and Sunspots

We first relax the assumption that only the short-term investor receives information about the dividend through a private signal, and instead allow both the short-term trader and the investor to receive private information on the dividend θ. We change the information sets to $\Omega_0 = \{P_0, \theta_0 + e_j, z + \varepsilon_j\}$ and $\Omega_1 = \{P_0, P_1, \theta_1 + v_i\}$. Here $s_{j0} = \theta_0 + e_j$ is the private signal on the dividend received by trader j in the first period, and $s_{i1} = \theta_1 + v_i$ is the signal of investor i in the second period. We assume that $cov(\theta_0, \theta) > 0$ and $cov(\theta_1, \theta) > 0$, but $cov(\theta_0, \theta_1) = 0$. For example, $\theta = \alpha\theta_0 + (1-\alpha)\theta_1$, with $0 < \alpha < 1$, but $cov(\theta_0, \theta_1) = 0$ satisfies these assumptions. Without loss of generality we assume $\theta = \theta_0 + \theta_1$. In addition, we assume that θ_0 (θ_1) is drawn from a normal distribution with a mean of $-\frac{1}{2}\sigma^2_{\theta_0}$ ($-\frac{1}{2}\sigma^2_{\theta_1}$) and a variance of $\sigma^2_{\theta_0}$ ($\sigma^2_{\theta_1}$) and v_i is normally distributed with a mean of 0 and a variance of σ^2_v. The equilibrium conditions are again given by (21) and (22). Proposition 4 specifies the equilibrium prices with such an information structure[8].

Proposition 4 *The exists a continuum of sunspot equilibria indexed by* $0 \le \sigma^2_z \le \frac{1}{4}\sigma^2_{\theta_0}$, *where equilibrium prices are given by*

$$\log P_0 = \phi\theta_0 + \sigma_z z, \tag{23}$$

$$\log P_1 = \phi\theta_0 + \sigma_z z + \theta_1, \tag{24}$$

where ϕ *is given by*

$$0 \le \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma^2_z}{\sigma^2_{\theta_0}}} \le 1. \tag{25}$$

Proof. *The proof is similar to that of Proposition 3. Plugging the expression of* $\log P_0$ *and* $\log P_1$ *into (21), we obtain*

$$\phi\theta_0 + \sigma_z z = \phi\theta_0 + \sigma_z z + \mathbb{E}(\theta_1|\Omega_0) + \frac{1}{2}var(\theta_1|\Omega_0). \tag{26}$$

Since θ_1 *is independent of* Ω_0, *we have* $\mathbb{E}(\theta_1|\Omega_0) + \frac{1}{2}var(\theta_1|\Omega_0) = -\frac{1}{2}\sigma^2_{\theta_1} + \frac{1}{2}\sigma^2_{\theta_1} = 0$. *Therefore equation (21) is satisfied for any trader* j *in period* 0. *We now turn to equation (22). Notice that by studying the prices in the two periods, the investor can now learn* θ_1 *with certainty. Hence we*

[8] In what follows we assume that correlation between two random variables, if not explicitly specified, is zero.

have

$$\begin{aligned}\log P_1 &= \mathbb{E}(\theta|\Omega_1) + \frac{1}{2}var(\theta|\Omega_1) \\ &= \theta_1 + \mathbb{E}(\theta_0|\phi\theta_0 + \sigma_z z) + \frac{1}{2}var(\theta_0|\phi\theta_0 + \sigma_z z) \\ &= \theta_1 + \frac{\phi\sigma_{\theta_0}^2}{\phi^2\sigma_{\theta_0}^2 + \sigma_z^2}\left[\phi\theta_0 + \sigma_z z\right], \end{aligned} \tag{27}$$

Since ϕ is given by (25), we have $\phi\sigma_{\theta_0}^2 = \phi^2\sigma_{\theta_0}^2 + \sigma_z^2$. Hence equation (22) holds also. ■

In this case, the market efficiently aggregates the private information of long-term investors. The price in period 1 has to incorporate all the private information of the long-term investors, otherwise some investors with high(low) signals on the underlying dividend would attempt to profit by shorting(going long on) the asset. However, the period 1 price only partially incorporates the private information of the short-term traders regarding the underlying dividend. These short-term traders benefit only from potential capital gains, and are risk neutral. As long as the expected return is equal to the risk-free rate, they will not care whether the prices are driven by sunspots or by fundamentals and sunspot equilibria will continue to exist.

We now further generalize the information structure by allowing investors as well as short-term traders to receive signals on sunspots and dividends. We assume that $\theta = \theta_0 + d + \theta_1$ and $z = z_0 + \xi + z_1$. The information set of short-term traders is given by $\Omega_0 = \{P_0, \theta_0 + d + e_j,\ z_0 + \xi + \varepsilon_j\}$, while the information set of the long-term investors becomes $\Omega_1 = \{P_0, P_1,\ \theta_1 + d + v_i,\ z_1 + \xi + \zeta_i\}$. In other words, the private of signals of the short-term traders and investors are correlated. In particular we are now allowing the investors, just like the short-term traders, to observe a noisy sunspot signal correlated with the sunspot signals received by short-term traders. Here d and ξ are known both to short-term traders and long-term investors. θ_0 and z_0, however, are in the private information sets of the short-term traders, and the investors can only learn about them from observing the market price. We assume z_0, ξ and z_1 are drawn from standard normal distributions and ζ_i are drawn from a normal distribution with a mean of 0 and a variance of σ_ζ^2. The equilibrium conditions are again given by (21) and (22). We can now show that there exists a continuum of sunspot equilibria indexed by $0 \leq \sigma_z^2 \leq \frac{1}{4}\sigma_{\theta_0}^2$, with the prices

$$\log P_0 = \phi\theta_0 + \sigma_z z_0 + d, \tag{28}$$

$$\log P_1 = \phi\theta_0 + \sigma_z z_0 + d + \theta_1, \tag{29}$$

where ϕ is given by

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_{\theta_0}^2}} \leq 1. \tag{30}$$

The proof is very similar to that of Proposition 4 and hence is omitted.[9] A conclusion we can draw is that the market is in general not fully efficient in aggregating the information of the short-term traders, even if the investors receive sunspot and dividend signals correlated with the private signals of short-term traders. As long as the short-term traders as a whole have some private information, there exists sunspot equilibria.

5 Alternative Market Structures

In our benchmark model, only short-term traders are present in the period 0 market. We first relax that assumption. Suppose now that both short-term traders and long-term investors are present in period 0, but only long-term investors are present in period 1. The short-term traders maximize their expected payoff in period 1 and the long-term investors maximize their expected payoff in period 2. Let Ω_0 denote the information set of short-term trader j. The short-term trader's utility maximization problem is given by

$$\max_{x_{j0} \in (-\infty, +\infty)} \mathbb{E}[(P_1 - P_0)\, x_{j0} + P_0 + w|\Omega_0].$$

and the first-order condition is

$$P_0 = \mathbb{E}[P_1|\Omega_0] \tag{31}$$

The long-term investors trade in both periods. Let x_{i0} and x_{i1} be the asset holdings of the long-term trader in period 0 and 1 respectively and let B_{i0} and B_{i1} be the bond holdings of long-term investors in periods 0 and 1 respectively. The long-term investors try to maximize their consumption in period 2. Then the budget constraints for investor i are

$$\begin{aligned} P_0 x_{i0} + B_{i0} &= w \\ P_1 x_{i1} + B_{i1} &= P_1 x_{i0} + B_{i0}, \\ C_{i2} &= D x_{i1} + B_{i1}, \end{aligned}$$

The long-term investor's problem can be solved recursively. Let Ω_0^* and Ω_1 denote the information sets of a particular investor i in periods 0 and 1, respectively. Note that $\Omega_0^* \subseteq \Omega_1$. Given x_{i0} and B_{i0}, the utility maximization problem of the long-term investor in period 1 then becomes:

$$\max_{x_{i1} \in (-\infty, +\infty)} \mathbb{E}\left\{[P_1 x_{i0} + B_{i0} + (D - P_1) x_{i1}]\, |\Omega_1\right\}.$$

[9] As d is commonly observed by both the investors and traders, its distribution does not matter. We can assume, for example, that $d \sim \mathcal{N}(-\frac{1}{2}\sigma_d^2, \sigma_d^2)$.

Applying the law of iterated expectations and substituting out B_{i0} using the budget constraint, we can write the period 0 problem as

$$\max_{x_{i0}\in(-\infty,+\infty)} \mathbb{E}\left\{\left[\left(P_1 - P_0\right)x_{i0} + w\right]|\Omega_0^*\right\}.$$

The first-order conditions for investor i are

$$P_1 = \mathbb{E}[D|\Omega_1], \tag{32}$$

$$P_0 = \mathbb{E}[P_1|\Omega_0^*], \tag{33}$$

The asset market clearing conditions require $\int x_{j0}dj + \int x_{i0}di = 1$ and $\int x_{i1}di = 1$. We discuss two cases below.

Case 1: In the first case, we assume that only the short-term trader has private information regarding D and the sunspots. Namely $\Omega_0 = [P_0, \theta + e_j, z + \varepsilon_j]$, $\Omega_0^* = P_0$ and $\Omega_1 = [P_0, P_1]$. In this case, Proposition 3 applies. It is easy to verify that $\log P_1 = \log P_0 = \phi\theta + \sigma_z z$ satisfies equations (31), (32) and (33).

Case 2: In this case, both the short-term traders and the investors receive private information on the dividends and the sunspot in period 0. As in section 4.2, we assume that $\log D = \theta_0 + \theta_1 + d$ and that $z = z_0 + \xi + z_1$. We also assume that θ_0 and z_0 are known only to the short-term traders, while d and ξ are known to both short-term traders and investors. In other words, the information set of the short-term trader in period 0 is $\Omega_0 = [P_0, \theta_0 + d + e_j, z_0 + \xi + \varepsilon_j]$, while the long-term investor's information sets in period 0 and 1 are $\Omega_0^* = \{P_0, d + v_i, \xi + \zeta_i\}$ and $\Omega_1 = \{P_1,\ \theta_1,\ z_1\} \cup \Omega_0^*$.[10] Again it is easy to show that there exists an continuum of sunspot equilibria indexed by $0 \le \sigma_z^2 \le \frac{1}{4}\sigma_{\theta_0}^2$, with the prices

$$\log P_0 = \phi\theta_0 + \sigma_z z_0 + d, \tag{34}$$

$$\log P_1 = \phi\theta_0 + \sigma_z z_0 + d + \theta_1, \tag{35}$$

where ϕ is given by

$$0 \le \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_{\theta_0}^2}} \le 1. \tag{36}$$

Notice that the price functions are exactly the same as in the section 4. In both cases, under equilibrium prices both the short-term traders and the long-term investors are indifferent between holding stocks or holding bonds in period 0, and the long-term investors are indifferent between

[10] Other information partitions for the long-term investors, for example, $\Omega_0^* = \{P_0, d, z_1 + \xi + \zeta_i\}$ and $\Omega_1 = \{P_1, \theta_1 + \nu_i\} \cup \Omega_0^*$, can support the same equilibrium. We can also allow noisy information in Ω_1, for example, $\Omega_0^* = \{P_0, d + v_i, \xi + \zeta_i\}$ and $\Omega_1 = \{P_1,\ \theta_1 + v_i^*\ z_1 + \zeta_i^*\}$, to support the same equilibrium prices.

holding stocks or holding bonds in period 1. We can then assume that $0 \leq x_{j0} = x \leq 1$, $x_{i0} = 1 - x$ and $x_{i1} = 1$ in a symmetric equilibrium. Our results are therefore robust to incorporating investors with private information in the early stages of trading.

6 Multiple Assets and Price Co-movements

It is widely known that the traditional asset pricing models cannot explain why the high covariance of asset prices relative to the covariance of their fundamentals. (see Pindyck and Rotemberg (1993), Barberis, Shleifer and Wurgler (2005), and Veldkamp (2006).[11]). The excessive co-movement of the asset prices may be explained by correlated liquidity or noise trading across markets. However such excessive co-movement is also present in markets dominated by institutional investors who are less likely to behave like noise traders. For example, Sutton (2000) documents excessive co-movement of 10-year government bond yields between the United States, Japan, Germany, the United Kingdom and Canada that are not fully explained by co-movements in economic activity, inflation or short-term interest rates. These excessive co-movements in the bond markets appear to be even stronger than those in the equity markets. Cappiello, Engle, and Sheppard (2006) find that the median bond-bond return correlation is 0.7276, and the median equity-equity return correlation is 0.4435 for bond and equity indices across 21 countries. These observations suggest that asset prices co-move excessively even in markets where arbitrage by institutional investors may dominate liquidity or noise trading.

In this section we show, without resorting to noise traders, that asset prices driven by the sentiment or sunspot shocks can exhibit high co-movements even if their underlying fundamentals are uncorrelated. The model is similar to the benchmark model above, but with multiple assets. For simplicity we consider two assets, a and b. The two assets yield final dividends in period 2 given by

$$\log D_{2\ell} = \theta_\ell, \text{for } \ell = a, b. \tag{37}$$

We assume that θ_ℓ, $\ell = a, b$ are drawn from the same normal distribution with a mean $-\frac{1}{2}\sigma_\theta^2$ and a variance of σ_θ^2. To highlight the co-movement, we assume that $cov(\theta_a, \theta_b) = 0$. For simplicity we consider representative agents in each period. The trader in period 0 solves

$$\max_{x_{0a}, x_{0b}} \sum_{\ell=a,b} \{\mathbb{E}[P_{1\ell}|\theta_a, \theta_b, P_{0a}, P_{0b}] - P_{0\ell}\} x_{0\ell}, \tag{38}$$

where x_{0a}, x_{0b} are the asset holdings of the trader for assets a and b, respectively. Here $P_{1\ell}$ and

[11] Veldkamp (2006) constructs a model with markets for information to explain asset price co-movements which we discuss in Section 9.

$P_{0\ell}$ are asset $\ell's$ price in periods 0 and 1. The investor in period 1 solves

$$\max_{\substack{x_{1a}\in(-\infty,+\infty),\\ x_{1b}\in(-\infty,+\infty)}} \sum_{\ell=a,b} \left\{\mathbb{E}[D_{2\ell}|P_{1a},P_{1b},P_{0a},P_{0b}] - P_{1\ell}\right\} x_{1\ell}, \tag{39}$$

where $x_{1\ell}$, for $\ell = a, b$ are the asset holdings of the investor in period 1. The first-order conditions are

$$\begin{aligned} P_{0\ell} &= \mathbb{E}[P_{1\ell}|\theta_a,\theta_b,P_{0a},P_{0b}], && (40)\\ P_{1\ell} &= \mathbb{E}[D_{2\ell}|P_{1a},P_{1b},P_{0a},P_{0b}]. && (41)\end{aligned}$$

Since the agents are risk neutral, they will be indifferent between buying the asset or or not buying. We will focus on the symmetric equilibrium again, namely an equilibrium with $x_{0\ell} = 1, x_{1\ell} = 1$ for $\ell = a$ and b.

Proposition 5 *There exists a fully revealing equilibrium with* $\log P_{0\ell} = \log P_{1\ell} = \theta_\ell$.
Proof. *The proof is straightforward.* ■

Notice that in the fully revealing equilibrium the correlation of asset prices is zero and there is no co-movement of asset prices.

Proposition 6 *There exists a continuum of equilibria with prices fully synchronized among assets. The asset prices take the form*

$$\log P_{0\ell} = \log P_{1\ell} = \phi(\theta_a + \theta_b) + \sigma_z z + \frac{1}{2}\phi\sigma_\theta^2, \tag{42}$$

for $\ell = a$ *and* b. *Here we have* $0 \le \phi \le \frac{1}{2}$ *and*

$$\sigma_z^2 = \phi(1-2\phi)\sigma_\theta^2. \tag{43}$$

Proof. *Asset prices do not change in periods 0 and 1. Hence equation (40) is satisfied automatically. We only need to ensure equation (41) holds. Due to symmetry, it is sufficient to prove that (41) holds for asset a. We need to show*

$$\begin{aligned} \phi(\theta_a+\theta_b) + \sigma_z z + \frac{1}{2}\phi\sigma_\theta^2 &= \log\exp\left\{\mathbb{E}[\theta_a|\phi(\theta_a+\theta_b)+\sigma_z z]\right\} \\ &= -\frac{1}{2}\sigma_\theta^2 + \frac{\phi\sigma_\theta^2}{2\phi^2\sigma_\theta^2+\sigma_z^2}\left[\phi(\theta_a+\theta_b)+\sigma_z z+\phi\sigma_\theta^2\right] + \frac{1}{2}\left[\sigma_\theta^2 - \frac{\left(\phi\sigma_\theta^2\right)^2}{2\phi^2\sigma_\theta^2+\sigma_z^2}\right] \\ &= \frac{\phi\sigma_\theta^2}{2\phi^2\sigma_\theta^2+\sigma_z^2}\left[\phi(\theta_a+\theta_b)+\sigma_z z\right] + \frac{1}{2}\frac{\left(\phi\sigma_\theta^2\right)^2}{2\phi^2\sigma_\theta^2+\sigma_z^2}. \end{aligned} \tag{44}$$

Comparing terms we obtain $\phi\sigma_\theta^2 = 2\phi^2\sigma_\theta^2 + \sigma_z^2$, *or*

$$\sigma_z^2 = \phi(1-2\phi)\sigma_\theta^2. \tag{45}$$

Then we have $\frac{1}{2}\frac{(\phi\sigma_\theta^2)^2}{2\phi^2\sigma_\theta^2+\sigma_z^2} = \frac{1}{2}\phi\sigma_\theta^2$. ■

The intuition for asset price co-movements is straightforward. Since the same traders trade the two assets, the asset prices will be determined by the same information set. If prices are not driven solely by fundamentals but also by sentiments, then the sentiment shocks of the traders will drive both asset prices. The sentiment-driven co-movements are also consistent with the findings of Sutton (2000) and of Cappiello, Engle, and Sheppard (2006).

It is straightforward to extend the information structure to allow any degree of co-movement. For example, we can assume that the total dividend of asset ℓ is given by $\theta_\ell + d_\ell$ and the information sets are $\Omega_0 = \{\theta_a + d_a, \theta_b + d_b, P_{0a}, P_{0b}, z\}$ and $\Omega_1 = \{d_a, d_b, P_{0a}, P_{0b}, P_{1a}, P_{2b}\}$. Here d_ℓ is the dividend information observed by both short-term traders and long-term investors. We assume that $cov(d_a, d_b) = 0$ and $\sigma_d^2 = cov(d_a, d_a)$. Hence the total dividends of these two assets are not correlated. Then we can construct equilibria with prices

$$\log P_{0\ell} = \log P_{1\ell} = \phi(\theta_a + \theta_b) + \sigma_z z + \frac{\phi\sigma_\theta^2}{2} + d_\ell. \tag{46}$$

where ϕ and σ_z are given by Proposition (6). If $\sigma_d^2 > 0$, then the asset prices do not co-move perfectly. When $\sigma_d^2/\sigma_\theta^2$ approaches infinity, the correlation between the asset prices becomes zero. Therefore we can always set the value of σ_d^2 to match the observed covariance of asset prices. Notice that if $\sigma_d^2/\sigma_\theta^2$ increases, the covariance of asset prices declines. This could be the result of a reduction in the cost of acquiring information faced by uninformed long-term investors. Legal reform on disclosure requirements, for example, can also produce more information for the uninformed investors. Fox, Durnew, Morck and Yeung (2003) show that the enactment of new disclosure requirements in December 1980 caused a decline in co-movements, consistent with our theory.

7 Multi-period Assets

We now extend our model to multiple periods to show that our sentiment driven equilibrium persists in a dynamic setting.[12] Along the lines initially demonstrated by Tirole (1982), we also want to show that the sentiment-driven asset price movements in our model look like an efficient market in

[12] It is important to study whether a sentiment driven equilibrium will emerge in a dynamic setting as it has been shown that in some dynamic settings (although not in general) learning from prices can generate a full information equilibrium in the limit (see chapter 7 of Vives (2008) for a literature review).

the following sense: if an econometrician studies the asset data generated by the sunspot equilibria, they will find that the asset price movements will follow a random walk and will not reject the efficient market hypothesis.

Suppose an asset created in period 0 yields a return or dividend only in period $T+1$. In each period between periods 0 and $T-1$, a continuum of short-term traders can trade the asset. Short-term traders in each period hold the asset only for capital gains only. As in section 3 where there were only three periods, given prices the private signals that traders receive do not matter, so we focus on a representative trader in each period. The final dividend is given by

$$\log D_{T+1} = \left(\sum_{t=0}^{T} \theta_t \right)$$

Again we assume that θ_t is independently drawn from a normal distribution with a mean of $-\frac{1}{2}\sigma_\theta^2$ and a variance of σ_θ^2. So the unconditional mean of D_{T+1} is given by 1. Denote the information set of traders in period $t = 0, 1, ..T$ by Ω_t, and their asset holding from period t to $t+1$ by x_t. Their maximization problem can be written as

$$\max_{x_t \in (-\infty, +\infty)} \left[\mathbb{E} \left(P_{t+1} | \Omega_t \right) - P_t \right] x_t, \tag{47}$$

for $t = 0, 1, ...T-1$. Investors who purchase the asset in period T solve

$$\max_{x_T \in (-\infty, +\infty)} \left[\mathbb{E} \left(D_{T+1} | \Omega_T \right) - P_T \right] x_T. \tag{48}$$

We assume short-term traders in period t know θ_t, which may be interpreted as trader t's private information regarding the final dividend of the underlying asset. Since all the agents observe the past and current prices, their information Ω_t is given by $\Omega_t = \{\theta_t, z_t\} \cup \{\cup_{\tau=0}^{t} P_\tau\}$, where z_t are i.i.d. draws from the standard normal distribution representing the sentiment shocks of traders in period t, as in our benchmark model.

An equilibrium is a set of price functions $\{P_t\}_{t=0}^{T}$ such that $x_t = 1$ solves (47) for $\tau = 0, 1, ...T-1$ and $x_T = 1$ solves (48), where the equilibrium conditions for an individual optimization are given by

$$\mathbb{E} \left(P_{t+1} | \Omega_t \right) = P_t, \text{for } t = 0, 1, ..T-1 \tag{49}$$

and

$$\mathbb{E} \left(D_{T+1} | \Omega_T \right) = P_T. \tag{50}$$

Proposition 7 $P_t = \exp(\sum_{\tau=0}^{t} \theta_\tau)$ *for* $t = 0, 1, ...T$ *is always an equilibrium.*

Proof. *The proof is straightforward. The past prices reveal the history of* $\{\theta_\tau\}_{\tau=0}^{t-1}$. *It is easy to check that*

$$\begin{aligned}\exp(\sum_{\tau=0}^{t}\theta_\tau) &= \mathbb{E}_t\left[\exp(\sum_{\tau=0}^{t}\theta_\tau+\theta_{t+1})\right] \\ &= \exp(\sum_{\tau=0}^{t}\theta_\tau). \end{aligned} \tag{51}$$

So (49) is satisfied for $\tau = 0, 1, ...T-1$, *where we used the that fact* $\mathbb{E}_t \exp(\theta_{t+1}) = 1$. *Finally by construction, equation (50) is automatically satisfied.* ■

In the above equilibrium, the price will eventually converge to the true fundamental price. The market is dynamically efficient in the sense that all private information is revealed sequentially by the market prices. However, as in the benchmark model, the above equilibrium is not the only one. Assume that traders at each t condition their expectations of the price P_t on the sentiment or sunspot shock z_t that they receive. We assume that θ_t, z_t are only observed by the traders at t. We have the following proposition regarding the equilibrium price.

Proposition 8 *There exists a continuum of sentiment-driven or sunspot equilibria indexed by* $0 \leq \sigma_z \leq \frac{1}{4}\sigma_\theta^2$, *with the price in period* t *given by*

$$\log P_t = \sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau), \tag{52}$$

for $t = 0, 1, 2, ...T-1$ *and*

$$\log P_T = \theta_T + \sum_{\tau=0}^{T-1}(\phi\theta_\tau + \sigma_z z_\tau). \tag{53}$$

and

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_\theta^2}} \leq 1. \tag{54}$$

Proof. *We first prove that (53) holds. For the investor in period* $T-1$, *equation (49) requires*

$$\begin{aligned}\sum_{\tau=0}^{T-1}(\phi\theta_\tau + \sigma_z z_\tau) &= \log \mathbb{E}\left[\exp\left(\log P_T\right)|\Omega_{T-1}\right] \\ &= \sum_{\tau=0}^{T-1}(\phi\theta_\tau + \sigma_z z_\tau) + \log \mathbb{E}\left[\exp(\theta_T)|\Omega_{T-1}\right]. \end{aligned} \tag{55}$$

Notice that $\mathbb{E}\left[\exp(\theta_T)|\Omega_{T-1}\right] = 1$. *So the above requirement is satisfied. We then prove that equation (49) holds for* $t = 0, 1, 2, ...T-2$. *Given the price structure, the information set* Ω_t *is now equivalent to* $\tilde{\Omega}_t = \{z_t\} \cup \{\theta_t\} \cup \{\phi\theta_\tau + \sigma_z z_\tau\}_{\tau=0}^{t-1}$. *Equation (49) can then be rewritten as*

$$\log P_t = \log \mathbb{E}[\exp\left(\log P_{t+1}\right)|\tilde{\Omega}_t]. \tag{56}$$

Plugging in the expression of $\log P_{t+1}$, *we obtain*

$$\begin{aligned}
\log P_t &= \mathbb{E}\left\{\left[\sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau) + \phi\theta_{t+1} + \sigma_z z_{t+1}\right]|\tilde{\Omega}_t\right\} + \frac{1}{2}\left(\phi^2\sigma_\theta^2 + \sigma_z^2\right) \\
&= \sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau) - \frac{\phi}{2}\sigma_\theta^2 + \frac{1}{2}\left(\phi^2\sigma_\theta^2 + \sigma_z^2\right) \\
&= \sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau). \qquad (57)
\end{aligned}$$

where the third line comes from the fact that $\phi^2\sigma_\theta^2 + \sigma_z^2 = \phi\sigma_\theta^2$ *by exploiting equation (54). So Equation (49) holds for* $t = 0, 1, 2, ..T-2$. *Finally for the investor of period* T, *we have*

$$P_T = \mathbb{E}\left[D_{T+1}|\Omega_T\right], \tag{58}$$

where Ω_T *is equivalent to* $\tilde{\Omega}_T = \{\theta_T,\ z_T\} \cup \{\phi\theta_\tau + \sigma_z z_\tau\}_{\tau=0}^{T-1}$, *learned from observing past prices. Notice that* $\tilde{\Omega}_T$ *does not contain any past realization of* θ_t *or* z_t *separately, assumed to be private information of the trader in period* $t \leq T-1$. *The above equation then yields*

$$\begin{aligned}
\log P_T &= \theta_T + \sum_{t=0}^{T-1}\left\{\mathbb{E}\left[\theta_t|\phi\theta_t + \sigma_z z_t\right] + \frac{1}{2}var(\theta_t|\phi\theta_t + \sigma_z z_t)\right\} \\
&= \theta_T + \sum_{t=0}^{T-1}\frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\left(\phi\theta_t + \sigma_z z_t + \frac{\phi}{2}\sigma_\theta^2\right) - \frac{T}{2}\sigma_\theta^2 + \frac{T}{2}\left[\sigma_\theta^2 - \frac{\left(\phi\sigma_\theta^2\right)^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\right] \qquad (59) \\
&= \theta_T + \sum_{t=0}^{T-1}\frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\left(\phi\theta_t + \sigma_z z_t\right).
\end{aligned}$$

We can then simplify the above equation to obtain equation (53). ■

Prices are driven by the sentiments of the short-term traders. Note that equation (52) implies that the asset price follows a random walk in the sentiment-driven equilibria. Although the efficient market hypothesis and the random walk of asset prices are not identical concepts, most tests of the efficient market hypothesis focus on the predictability of asset prices: if the market is efficient then rational investors will immediately react to informational advantages so that the profit opportunities are eliminated. As a result, information will be fully revealed by asset prices, and all subsequent price changes will only reflect new information. In other words, future asset prices are unpredictable. Therefore market efficiency and unpredictability are not equivalent. If an econometrician studies the asset price driven by the sentiment shocks in our model, he will conclude that asset prices are unpredictable. Yet, the sentiment shocks can generate permanent deviations of asset prices from their fundamental value.

8 Implementing the Equilibria

So far we have relied on rational expectations to characterize equilibrium prices and showed that fully revealing rational expectations equilibria (REE) exist. Vives (2014), however, raises the issue that fully revealing rational expectations equilibria may not be implementable: if short-term traders choose ignore their private signals at fully revealing equilibrium prices, how in fact are such equilibrium prices incorporating private signals are realized in the market? We therefore study implementability of our equilibria in our model. For simplicity we focus on our baseline three-period model in Section 3.1; the equilibria in other extended models can be implemented in the same way.

We first note that the sentiment driven equilibrium with $\log P_1 = \log P_0 = \phi\theta + \sigma_z z$ exists even if the signals are perfect, namely when $\sigma_e^2 = 0$ and $\sigma_\varepsilon^2 = 0$. To fix ideas, we first consider this simple special case. As the fully revealing equilibrium is a special case of the sentiment-driven equilibrium, we focus on the implementability of the latter. We follow Vives (2014) to implement the equilibria in our model by constructing demand functions for short-term traders and the investors. To construct well-defined demand functions with risk neutral agents, we assume that the trading is limited by $0 \leq x_{j0} \leq \bar{x}$[13], where $\bar{x} > 1$ is the maximum asset holding for all agents. The equilibrium can then be implemented by constructing demand functions as follows. Every short-term trader believes that the price in the next period is given by $\log P_1 = \phi\theta + \sigma_z z$, and submits his demand function to a market auctioneer,

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_0 < \phi\theta + \sigma_z z \\ 1 & \text{if } \log P_0 = \phi\theta + \sigma_z z \\ 0 & \text{if } \log P_0 > \phi\theta + \sigma_z z \end{array} \right\}. \tag{60}$$

The only price that can clear the market is $\log P_0 = \phi\theta + \sigma_z z$. Note that since the agents know θ and z perfectly, the equilibrium price is implementable. After observing the price $\log P_0 = \phi\theta + \sigma_z z$,

[13] It is easy to see that our results are not affected if we assume $x_{j0} \geq -x_L$, where $-x_L > -\infty$ is the minimum asset holding for all agents.

the long-term investors submit their demand functions to the market auctioneer according to:

$$x_{i1} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_1 < \log P_0 = \phi\theta + \sigma_z z \\ 1 & \text{if } \log P_1 = \log P_0 = \phi\theta + \sigma_z z \\ 0 & \text{if } \log P_1 > \log P_0 = \phi\theta + \sigma_z z \end{array} \right\} \tag{61}$$

Again the only price that will clear the market, consistent with (16) and (19) is $\log P_1 = \phi\theta + \sigma_z z$. Since the long-term trader can observe the first period price, his demand function is well-defined. This establishes that the equilibrium with $\log P_1 = \log P_0 = \phi\theta + \sigma_z z$ is indeed an implementable rational expectation equilibrium.

Now we turn to the case with $\sigma_e^2 > 0$. We assume that z is drawn from a normal distribution with mean 0 and variance σ_z. Note that a fully revealing equilibrium with $\log P_0 = \log P_1 = 0$ that does not depend on sentiments is always implementable. In this fully revealing equilibrium, as well as in the sentiment-driven equilibria, the short-term traders ignore their private signals. As noted by Vives (2014), this raises the question of how equilibrium prices that aggregate private signals are in fact implemented. Vives (2014) assumes that the valuation of assets by each trader has a common as well as a private component. Each trader receives a private signal that bundles the common and private valuations together. In that case, although the price fully reveals the common component, private signals are still useful for providing information on private valuations. As a result, the demand functions for the traders depend on private signals. The demand functions and information contained in the private signals can then be aggregated to obtain the equilibrium price.

We now show how this mechanism proposed by Vives (2014) can also implement the equilibria in our economy. Suppose that the asset also yields a random idiosyncratic utility u_j drawn from a normal distribution with mean 0 and variance σ_u^2 for the short-term trader in period 1. One can imagine, for example, that the asset is a firm generating profits in both periods, and that the profits depend on the ability of the firm owner. If the short-term traders' abilities are idiosyncratic and the long-term investors' abilities are homogenous, then we can generate payoffs for the short-term traders that depend on idiosyncratic components. The short-term traders receive signals $s_j = \theta + u_j + e_j$, and they solve the following problem:

$$\max_{x_{j0}} \mathbb{E}[(P_1 - P_0 + u_j)\, x_{j0} | \theta + u_j + e_j, P_0, z], \tag{62}$$

The problem of the long-term investors is the same as in section 3.1 and therefore their first-order condition is given by (13). Let us define $p_0 = \log(P_0 - \bar{u})$, where $\bar{u}$ is to be determined by market clearing in (66) below. Suppose all short-term investors believe that $p_0 = \log P_1 = \phi\theta + \sigma_z z$. Under this belief $\frac{p_0 - \sigma_z z}{\phi} = \theta$ and $u_j + e_j = s_j - \frac{p_0 - \sigma_z z}{\phi}$. Therefore their demand function depends on their estimate of u from the signal $s_j - \frac{p_0 - \sigma_z z}{\phi}$ relative to the threshold $\bar{u}$:

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2}(s_j - \frac{p_0-\sigma_z z}{\phi}) > \bar{u} \\ 1 & \text{if } \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2}(s_j - \frac{p_0-\sigma_z z}{\phi}) = \bar{u} \\ 0 & \text{if } \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2}(s_j - \frac{p_0-\sigma_z z}{\phi}) < \bar{u} \end{array} \right\} \tag{63}$$

Denote by $\Phi(\cdot)$ the cumulative distribution function of the standard normal distribution. The aggregate demand function is therefore given by

$$x_0(p_0) = \bar{x} \int_{s_j > \bar{u}\frac{\sigma_u^2+\sigma_e^2}{\sigma_u^2} + \frac{p_0-\sigma_z z}{\phi}} dj = \bar{x} \left[1 - \Phi\left(\frac{\bar{u}\frac{\sigma_u^2+\sigma_e^2}{\sigma_u^2} + \frac{p_0-\sigma_z z}{\phi} - \theta}{\sqrt{\sigma_u^2+\sigma_e^2}} \right) \right], \tag{64}$$

where the second equality relies on the fact that $s_j \sim N(\theta, \sigma_u^2 + \sigma_e^2)$ for a given realization of θ. We can solve for p_0 and $\bar{u}$ from the market clearing condition, namely $x_0(p_0) = 1$. Then the price function

$$p_0 = \phi\theta + \sigma_z z \tag{65}$$

and

$$1 - \Phi(\bar{u}\frac{\sqrt{\sigma_u^2+\sigma_e^2}}{\sigma_u^2}) = \frac{1}{\bar{x}} \tag{66}$$

clear the market for any realization of θ and z.

Finally the demand functions for the long-term traders are

$$x_{i1} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_1 < p_0 = \phi\theta + \sigma_z z \\ 1 & \text{if } \log P_1 = p_0 = \phi\theta + \sigma_z z \\ 0 & \text{if } \log P_1 > p_0 = \phi\theta + \sigma_z z \end{array} \right\} \tag{67}$$

Note that in the second period the market will clear if $\log P_1 = p_0 = \phi\theta + \sigma_z z$ and the first-order condition (13) will be satisfied. Thus as in Proposition 3 we have a continuum of sentiment-driven rational expectations equilibria that are implementable. However, if σ_u^2 becomes zero the equilibrium is difficult to implement. In that case $\bar{u} = 0$ and $\frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2} = 0$, so the demand for each trader is simply $x_{j0} = 1$ regardless of their private signals. In this case, it is difficult to see how the equilibrium prices would be implemented or would incorporate the information in the private signals.

Alternatively, to implement the sentiment-driven equilibria with a noisy signal, we can also follow the approach of Golosov, Lorenzoni and Tsyvinski (2014) by assuming decentralized trading in period 0. In particular, we can divide the first period into N sub-periods and allow $N \to \infty$, in which case the short-term traders would randomly meet and trade. In so doing they obtain information from other traders through their trading history. Eventually at the end of first period, all short-term traders become perfectly informed about θ. We have already shown that the equilibrium can be implemented in this case. Finally following Vives (2008)[14] and Golosov, Lorenzoni and Tsyvinski (2014), we construct an approach based on learning to implement the fully revealing equilibrium of our model in the Appendix. In this extended model, the first period is divided into N sub-periods, with $N \to \infty$. The short-term trader with index $j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ is assumed to trade only twice, first in the τth sub-period of period 0 and then in period 1. We assume that all short-term traders with index $j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ receive the same private signal about θ, $s_{j0} = \theta + e_{\tau,0}$. In each sub-period τ of the initial period 0, the signal s_{j0} and the past history of prices map to the price in that sub-period, P_{j0}. As the equilibrium price in each sub-period reflects the common information of the short-term traders, it will be implementable. So the sequence of prices in period 0 will eventually reveal θ. In the Appendix we show that the sentiment driven equilibrium can also be implemented similarly.

9 Some Related Literature

To put our results in context we now briefly discuss some papers based on informational asymmetries or restricted participation that are related to ours.

The sunspot equilibria that we have considered are not randomizations over multiple fundamental equilibria. Instead they are related to the early sunspot results of Cass and Shell (1983).[15] As in our model, Cass and Shell (1983) have a finite overlapping generations economy with a unique fundamental equilibrium. There are two periods, uniform endowments, and the agents have separable utility functions defined over the two commodities. The consumers in the initial period are born before the commonly observed sunspot activity is revealed, and can trade securities with payoffs contingent on the outcome of the extrinsic random variable determined by sunspot activity. There are also consumers born in the second period, after the sunspot is realized. The two generations of traders can then trade commodities on the spot market. In addition to the unique certainty equilibrium, Cass and Shell (1983) show the existence of a sunspot equilibrium with the relative commodity prices driven by extrinsic uncertainty. This rational expectations equilibrium arises from state-contingent trades based on sunspot probabilities that create wealth effects. This mechanism differs from ours where a continuum of sunspot equilibria arise under private signals from the signal extraction problem, as agents optimally disentangle the price signal into the fundamental

[14] Chapter 7 of Vives (2008) presents an excellent summary on learning and convergence to a full information equilibrium.

[15] See their appendix.

and sunspot components.[16]

Allen, Morris and Shin (2006) also study a model structure similar to ours, with overlapping generations of traders who live for two periods. A new generation of traders of unit measure is born at each date t. When the traders are young, they receive a noisy signal about the liquidation value of the asset at terminal time T, and they trade the asset to build up a position in the asset, but do not consume. In the next period when they are old, they unwind their asset position to acquire the consumption good, consume, and die. The asset supply each period is stochastic and unobserved, which prevents the equilibrium prices from revealing the terminal liquidation value. Allen, Morris and Shin (2006) show that the law of iterated expectations for average expectations can fail, so that market prices may be systematically lower than average expectations. This only happens if the traders are risk averse and their signals are imprecise. With risk-neutral short-term traders, prices again become fully revealing. Unlike our model where short-term traders condition their portfolio decisions on both fundamental and sunspot signals that gives rise to correlated actions, in Allen, Morris and Shin (2006) agents condition their trades on fundamentals alone, so the issue of multiple sunspot equilibrium does not arise.

Cespa and Vives (2014), however, show that the presence of liquidity or noise traders with correlated demands across periods tend to decrease price volatility, and induce risk-averse short-term traders to trade more aggressively based on fundamentals, as discussed in the Introduction. Cespa and Vives (2014) find that, in their high information equilibrium, this effect can moderate the bias in prices arising from the failure of the law of iterated expectations noted by Allen, Morris and Shinn (2006), and can bring equilibrium prices closer to fundamentals.

Our results are closely related to those of Angeletos, Lorenzoni and Pavan (2010), who explore a related market structure where entrepreneurs are receive noisy private signals about the ultimate return to their investments. Entrepreneurs are also aware that they are collectively subject to correlated sentiments of market optimism or pessimism about investment returns. These sentiments are embodied in a second correlated noisy signal that introduces non-fundamental noise into entrepreneurs' investment decisions. The traders buy the assets from entrepreneurs without observing their signals, but they do observe the aggregate level of investment. This observation induces a signal extraction problem for the traders as aggregate investment is now driven by the fundamentals of investment returns as well as non-fundamentals. Angeletos, Lorenzoni and Pavan (2010) show that the resulting correlated market sentiments can amplify the noise of fundamentals and introduce self-fulfilling multiple equilibria. In such equilibria traders are willing to purchase

[16] Peck and Shell (1991) also show the existence of sunspot equilibria in a finite economy with a unique fundamental equilibrium by allowing non-Walrasian trades prior to trading on the post-sunspot spot markets. Spear (1989) shows the possibility of sunspots with a unique fundamental equilibrium for an OLG model with two islands where prices in one island act as sunspots for the other and vice versa. Finally, in OLG models the non-monetary equilibrium may be considered **the unique** fundamental equilibrium while monetary equilibria may be viewed as non-fundamental bubbles as in Tirole (1982).

assets at "speculative" prices consistent with the price expectations of the entrepreneurs that differ from the expectations on fundamentals. The correlation in entrepreneurial investment decisions is similar to the correlated decisions of the short-term and long-term traders in our model. They are induced by the sunspot-driven prices and give rise to sunspot equilibria distinct from the price-revealing Grossman-Stiglitz equilibrium. An essential component of the multiplicity therefore stems from the correlated actions induced by non-fundamentals, yielding additional "correlated equilibria" as discussed earlier by Aumann (1987), Aumann Peck and Shell (1988) and Maskin and Tirole (1987).[17]

Other recent papers in the literature have also explored the role of informational asymmetries and costly information in generating price movements that diverge from fundamentals in order to explain market data, without incorporating non-fundamentals into the information structures of markets that generate multiple rational expectations equilibria and sunspot fluctuations.

For example, in Albagli, Hellwig and Tsyvinski (2011), prices diverge from expected dividends from the perspective of an outside observer. Their model has noise traders as well as risk neutral informed and uninformed traders facing limits on their asset positions. Under risk neutrality and heterogeneous beliefs driven by private signals, market clearing prices are determined by the marginal trader whose noisy private signal makes him indifferent between trading or not trading. Thus fluctuations in demand coming from realizations of fundamentals, or from noise traders, alter the identity of the marginal investor. This drives a wedge between prices and expected returns from the perspective of an outsider, and generates excess price volatility relative to fundamentals. The equilibrium is nevertheless unique since, unlike in our model, price expectations and investment decisions are not conditioned on non-fundamentals or sunspots.

Our results on the co-movement of asset prices in excess of the fundamentals are also related to those of Veldkamp (2006), who introduces multiple assets with correlated payoffs and information markets into the model of Grossman and Stiglitz (1980). The stochastic supplies of risky assets are unobserved and prevent prices from being fully revealing, as would also be the case in the presence of noise traders. Introducing information markets creates strategic complementarities in information acquisition. Since information is produced with both fixed and variable costs, as more agents purchase the same information, the average cost of information is reduced. In such an economy with increasing returns to information production, information producers will supply and investors will purchase the signals that yield information on multiple assets. The co-movement is created as investors use a common subset of signals to predict the prices of different assets. Unlike in our model, in Veldkamp (2006) there are no short-term traders and the information obtained

[17] Maskin and Tirole (1987) study a simple finite two period endowment economy with a unique equilibrium that can yield additional sunspot equilibria under correlated private signals, provided one of the goods is inferior. Such inferiority is not present in the models we consider. Benhabib, Wang and Wen (2012) also use the idea of correlated signals arising from sunspots to induce a continuum of equilibria in a Keynesian macroeconomic model.

by traders pertains only to fundamentals, so additional equilibria that can emerge when a trader's price expectations can also depend on non-fundamental sunspots do not arise.

10 Conclusion

We study a market where sequential short-term traders possess private information and make capital gains by trading a risky asset before it yields dividends, while uninformed investors purchase the asset for its dividend yield, forming expectations based on observed prices. In a rational expectation equilibrium, prices based on fundamentals can reveal private traders' information. However we show that there are also rational expectations equilibria driven by sunspots. This result is robust to a wide range of informational assumptions and market structures. If an econometrician studies the asset data generated from these sunspot equilibria, they will find that the asset prices follow a random walk and look as if they are generated by an efficient market reflecting fundamental values.

Our sentiment-driven asset prices under informational frictions are closely related to the recent literature examining sentiment-driven business cycles (see, for example, Angeletos and La'O (2012) and Benhabib, Wang and Wen (2012)). The recent financial crisis suggests that asset price movements can have considerable impact on macroeconomic fluctuations. Future work may explore more closely the connections between sentiment-driven asset prices and macroeconomic fluctuations.

A Appendices

In this appendix, we show in more detail how the fully revealing equilibrium can be implemented through a sequence of trades in the first period, as in Vives (2008) and Golosov, Lorenzoni and Tsyvinski (2014). The first period is divided into N sub-periods[18]. The short-term traders trade the asset sequentially. Let $\tau = 1, 2, ..., N$ be the τth sub-period in period 0. If we think of a period as a month or year, then the sub-periods may be trading days,or hours. The short-term trader with index $j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ is assumed to trade only twice, first in the $\tau' th$ sub-period of period 0, and then in period 1. We assume that each short-term trader with index $i \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ receives the same private signal about θ:

$$s_{j0} = [\theta + e_{\tau,0}, z + \varepsilon_{\tau,0}] \tag{A.1}$$

Finally we assume that the short-term trader with index i can observe prices $(P_{1,0,}P_{2,0}, ...P_{i,0})$. The basic idea is that through trading in these sub-periods, private information will gradually be incorporated into prices. As the short-term traders in the same sub-period have common information, equilibrium prices are implementable if they are functions of this common information.

Each short-term trader solves

$$\max_{x_{j0}} E[(P_1 - P_{\tau,0})x_{j0}|P_{1,0,}P_{2,0}, ...P_{\tau,0}, \theta + e_{\tau,0}, z + e_{\tau,0}], \text{ for } j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]. \tag{A.2}$$

The first-order condition is then given by

$$P_{\tau,0} = E[P_1|P_{1,0,}P_{2,0}, ...P_{\tau,0}, \theta + e_{\tau,0}, z + \varepsilon_{\tau,0}]. \tag{A.3}$$

The long-term investor solves

$$\max_{x_{i1}} E[(\exp(\theta) - P_1)x_{i,1}|P_{1,0,}P_{2,0}, ...P_{N,0}, P_1], \tag{A.4}$$

which yields the interior first-order condition:

$$E[\exp(\theta)|P_{1,0,}P_{2,0}, ...P_{N,0}, P_1] = P_1. \tag{A.5}$$

The market clears in each sub-period and in period 2, so we also have

$$\int_{j\in[\frac{\tau-1}{N},\frac{\tau}{N}]} x_{j0}dj = \frac{1}{N} \text{ and } \int x_{i1}di = 1. \tag{A.6}$$

Proposition 9 $P_{\tau,0} = P_1 = 1 = E\exp(\theta)$ *for* $\tau = 1, 2, .., N$ *is always an equilibrium.*

Proof. *The proof is straightforward.* ■

[18] We can also divide the second period into N sub-periods. The results will be the same since long-term traders have the same information. It does not matter whether they trade sequentially or not.

The equilibrium is implementable since all agents know the distribution of θ.

We now consider the fully revealing equilibrium. For simplicity, we will only consider the limiting case $N \to \infty$. We have the following proposition.

Proposition 10 *There is another equilibrium with* $\log P_1 = \theta$ *and*

$$\log P_{\tau,0} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{0,\ell}), \tag{A.7}$$

where $\bar{p}_{0\tau}$ *is a constant. In other words, prices become fully revealing asymptotically.*

Proof. *First we notice that*

$$E\left[\theta|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}\right] = -\frac{1}{2}\sigma_\theta^2 + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} - \frac{1}{2}\sigma_\theta^2),$$

$$var(\theta|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}) = \sigma_\theta^2 - \frac{\tau\sigma_\theta^2}{\tau\sigma_\theta^2 + \sigma_e^2}\sigma_\theta^2 = \sigma_\theta^2 - \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}\sigma_\theta^2.$$

So

$$\log E[\exp(\theta)|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}).$$

Second, under the belief $\log P_1 = \theta$*,*

$$\log P_{1,0} = \log E[\exp(\theta)|\theta + e_{1,0}] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_e^2}\left(\theta + e_{1,0}\right).$$

By observing $P_{1,0}$ *and* $\theta + e_{2,0}$*, the short-term trader in the second sub-period of period 0 effectively observes* $\theta + e_{1,0}$ *and* $\theta + e_{2,0}$*, so we have*

$$\log P_{2,0} = \log E[\exp(\theta)|\theta + e_1, \theta + e_2] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_e^2}{2}}(\theta + \frac{e_{1,0} + e_{2,0}}{2}).$$

By induction, for $\tau = 1, 2, ..., N$

$$\log P_{\tau,0} = \log E[\exp(\theta)|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}).$$

Finally when $N \to \infty$ *we then have* $\lim_{N\to\infty} \log P_{N,0} = \theta$ *with probability 1. There is no uncertainty for the long-term traders in period 1, so they will be willing to pay* θ *in that period.* ■

The equilibrium above is also implementable by constructing demand functions

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_{\tau,0} < \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) \\ 1 & \text{if } \log P_{\tau,0} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) \\ 0 & \text{if } \log P_{\tau,0} > \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) \end{array} \right\}, \text{ for } j \in [\frac{\tau - 1}{N}, \frac{\tau}{N}] \tag{A.8}$$

for the short-term trader in sub-period τ where $x_{i1} = \bar{x}$ if $\log P_1 < \theta$, $x_{i1} = 1$ if $\log P_1 = \theta$ and $x_{i1} = 0$ if $\log P_1 > \theta$.

Finally we look at the implementability of the sentiment-driven equilibria. We will focus on the equilibrium with $\log P_1 = \phi\theta + z$. Notice that since z has variance σ_z^2, we can rewrite it as $\log P_1 = \phi\theta + \sigma_z \hat{z}$ as in Proposition 3, relabeling $\hat{z}$ as a random sentiment shock drawn from the standard normal distribution.

Proposition 11 *If $\sigma_\varepsilon^2 < \frac{1}{4}\sigma_e^2$, there exists a sentiment-driven equilibrium with $\log P_1 = \phi\theta + z$, where z is the sentiment shock whose variance is endogenously given by $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$, and where $\phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_\varepsilon^2}{\sigma_e^2}}$. In addition,*

$$\log P_{\tau,0} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}} \left[\phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau} \varepsilon_{\ell,0} \right]. \tag{A.9}$$

Proof. *We first prove that under the assumption $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$, the effective signals for the short-term traders are $\phi(\theta + e_{\tau,0}) + (z + \varepsilon_{\tau,0})$ if they believe $\log P_1 = \phi\theta + z$. To see that, note that $[\theta + e_{0,\tau}, z + \varepsilon_{0,\tau}]$ are equivalent to two orthogonal signals $[\phi(\theta + e_{\tau,0}) + (z + \varepsilon_{\tau,0}), \frac{\sigma_\varepsilon^2}{\phi\sigma_e^2}(\theta + e_{\tau,0}) - (z + \varepsilon_{\tau,0})] \equiv [s_\tau^1, s_\tau^2]$. We show, under the assumption $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$,*

$$cov(s_\tau^1, s_\tau^2) = \frac{\sigma_\varepsilon^2}{\sigma_e^2}(\sigma_\theta^2 + \sigma_e^2) - (\sigma_z^2 + \sigma_\varepsilon^2) = 0,$$

$$cov(s_\tau^2, \phi\theta + z) = \phi\frac{\sigma_\varepsilon^2}{\phi\sigma_e^2}\sigma_\theta^2 - \sigma_z^2 = 0,$$

and for any $\ell \neq \tau$

$$cov(s_\tau^2, s_\ell^1) = \phi\frac{\sigma_\varepsilon^2}{\phi\sigma_e^2}\sigma_\theta^2 - \sigma_z^2 = 0.$$

It is then easy to show that

$$E[\phi\theta + z|\phi(\theta + e_{1,0}) + z + \varepsilon_{1,0}, ..., \phi(\theta + e_{\tau,0}) + z + \varepsilon_{\tau,0}]$$

$$= -\frac{1}{2}\phi\sigma_\theta^2 + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left[\phi(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) + (z + \frac{1}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}) + \frac{1}{2}\phi\sigma_\theta^2\right],$$

and

$$Var[\phi\theta + z|\phi(\theta + e_{1,0}) + z + \varepsilon_{1,0}, ..., \phi(\theta + e_{\tau,0}) + z + \varepsilon_{\tau,0}]$$

$$= \phi^2\sigma_\theta^2 + \sigma_z^2 - \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left(\phi^2\sigma_\theta^2 + \sigma_z^2\right).$$

It follows that

$$\log E[\exp(\phi\theta + \sigma_z z)|\phi(\theta + e_{1,0}) + \sigma_z(z + \varepsilon_{1,0}), ..., \phi(\theta + e_{\tau,0}) + \sigma_z(z + \varepsilon_{\tau,0})]$$

$$= \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left[\phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{1}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}\right].$$

Finally we note

$$\log P_{1,0} = \log E[\exp(\phi\theta + z)|\phi(\theta + e_{1,0}) + (z + \varepsilon_{1,0})] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left[\phi\theta + z + \phi e_{1,0} + \varepsilon_{1,0}\right].$$

So by observing $P_{1,0}$ and $\phi(\theta + e_{2,0}) + (z + \varepsilon_{2,0})$, the short-term trader effectively observes $\{\phi(\theta + e_{\ell,0}) + (z + \varepsilon_{\ell,0})\}_{\ell=1,2}$. By induction the effective information set for short-term trader in sub-period τ is $\{\phi(\theta + e_{\ell,0}) + (z + \varepsilon_{\ell,0})\}_{\ell=1,2,..,\tau}$. We obtain (A.9) from (A.3). Finally when $\tau \to N \to \infty$, the price becomes $\lim_{\tau\to\infty} \log P_{\tau,0} = \phi\theta + z$. The long-term traders can therefore infer $\phi\theta + \sigma z$ from the history of prices. They can also infer the combination of noisy shocks $\{\phi e_{\ell,0} + \varepsilon_{\ell,0}\}_{\ell=1,2,...}$ from the prices, although this information is useless to them. They still need to solve the signal extraction problem defined by (A.5). $E[\exp(\theta)|\phi\theta + z] = \phi\theta + z$ holds if $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$ and $\phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_\varepsilon^2}{\sigma_e^2}}$. ■

Notice that unlike the results in Section 4.1, we now put a further restriction that pins down the variance of sentiment shocks σ_z^2 at the sentiment-driven equilibrium. As the long-term traders can observe more than one past price, they can potentially infer the fundamental shocks and sentiment shocks from these prices. However if two signals are the equally precise, namely $\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_e^2} = \frac{\sigma_z^2}{\sigma_z^2+\sigma_e^2}$ or $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$, the mapping between prices and the signals becomes non-invertible. As a result the long-term trader will still not be able to distinguish the fundamental shocks and sentiments, even if they observe an infinite history of past prices. Since in each sub-period the short-term traders know $\phi\theta+\sigma_z z+\frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}+\frac{1}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}$ and the long-term trader knows $\phi\theta+\sigma_z z$, the equilibrium can be implemented in a straightforward manner by the demand functions

$$x_{j0} = \left\{ \begin{array}{lll} \bar{x} & \text{if } \log P_{\tau,0} < \frac{\sigma_\theta^2}{\sigma_\theta^2+\frac{\sigma_\varepsilon^2}{\tau}}(\phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}) \\ 1 & \quad\text{if } \log P_{\tau,0} = \phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0} \\ 0 & \quad\text{if } \log P_{\tau,0} > \phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0} \end{array} \right\} \text{ for } j \in [\frac{\tau-1}{N}, \frac{\tau}{N}] \tag{A.10}$$

of the short-term traders, and the demand functions of long-term investors, $x_{i1} = \bar{x}$ if $\log P_1 < \phi\theta + z$, $x_{i1} = 1$ if $\log P_1 = \phi\theta + z$, and $x_{i1} = 0$ if $\log P_1 > \phi\theta + z$.

References

[1] Allen, F., and G. Gorton, (1993), "Churning Bubbles," *Review of Economic Studies,* 60, 813–836.

[2] Allen, F., Morris, S., and H. Shin, (2006), "Beauty Contests and Iterated Expectations in Asset Markets," *Review of Financial Studies*, 19, 719-752.

[3] Angeletos, G-M., Lorenzoni, G. and Pavan, A. (2010). "Beauty Contests and Irrational Exuberance: A Neoclassical Approach", NBER working paper 15883.

[4] Angeletos, G-M. and J. La'O. (2012). "Sentiments", *Econometrica*, 81(2), 739-779.

[5] Albagli, E., Hellwig, C. and Tsyvinski, A. (2011), "A Theory of Asset Pricing Based on Heterogeneous Information", NBER working paper 17548.

[6] Aumann, R. J.,(1987), "Correlated Equilibrium as an Expression of Bayesian Rationality", *Econometrica, 55, 1-18.*

[7] Aumann, Robert J., Peck, James and Karl Shell,(1988), "Asymmetric Information and Sunspot Equilibria: A Family of Simple Examples", CAE Working Paper #88-34

[8] Baker, M. and Wurgler, J., (2007), "Investor Sentiment in the Stock Market," *Journal of Economic Perspectives*, 21(2), 129-152.

[9] Barberis, Nicholas & Shleifer, Andrei & Wurgler, Jeffrey, (2005), "Co-movement," *Journal of Financial Economics*, Elsevier, 75(2), pages 283-317, February

[10] Benhabib, J., Wang, P. and Wen, Y., (2012). "Sentiments and Aggregate Demand Fluctuations", NBER working paper 18413.

[11] Cappiello, L., Engle, R. F. and Sheppard, K., (2006). "Asymmetric Dynamics in the Correlations of Global Equity and Bond Returns," *Journal of Financial Econometrics*, 4, 537-572.

[12] Cass, D. and Shell, K., (1983), "Do Sunspots Matter?", *Journal of Political Economy,* 91, 193-227.

[13] Cespa and Vives (2014), "Expectations, Liquidity, and Short-term Trading", CEPR DP 8303.

[14] Fox, M., Durnew, A., Morck, R. and Yeung, B. (2003), "Law, Share Price Accuracy and Economic Performance: The New Evidence", *Michigan Law Review*, 102(3), 331-386.

[15] Golosov, M., Lorenzoni, G., and Tsyvinski, A. (2014). "Decentralized Trading with Private Information", forthcoming, Econometrica.

[16] Grossman, Sanford J. and Stiglitz, Joseph, (1980), "On the Impossibility of Informationally Efficient Markets", *American Economic Review*, 70, 393-408.

[17] Harrison, Michael and David Kreps (1978), "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quarterly Journal of Economics,* 92, 323-336.

[18] Malkiel, B. G. (2003). "The Efficient Market Hypothesis and Its Critics", *Journal of Economic Perspectives*, 17, 59–82.

[19] Maskin, E. and Tirole, J., (1987), "Correlated Equilibria and Sunspots", *Journal of Economic Theory* 43, 364-373.

[20] Miller, Edward (1977), "Risk, Uncertainty and Divergence of Opinion", *Journal of Finance* 32, 1151-1168.

[21] Peck, J. and Shell, K., (1991), "Market Uncertainty: Correlated and Sunspot Equilibria in Imperfectly Competitive Economies", *Review of Economic Studies*, 58, 1011-29.

[22] Pindyck, R. S. and & Rotemberg, J. J., (1993), "The Co-movement of Stock Prices," *Quarterly Journal of Economics*, (108), 1073-1104.

[23] Spear, S. E., (1989). "Are Sunspots Necessary?", *Journal of Political Economy*, 97, 965-973.

[24] Sutton, G. D. (2000). "Is There Excess Co-movement of Bond Yields Between Countries?", *Journal of International Money and Finance* 19, 363-376.

[25] Tirole, J. (1982). "On the Possibility of Speculation under Rational Expectations", *Econometrica*, 50, 1163-1182

[26] Veldkamp, L. (2006), "Information Markets and the Co-movement of Asset Prices," *Review of Economic Studies*, 73(3), 823-845.

[27] Vives, X. (2008), "Information and Learning in Markets: The Impact of Market Microstructure," *Princeton University Press*

[28] Vives, X. (2014), "On the Possibility of Informationally Efficient Markets", Schumpeter lecture EEA, forthcoming, *Journal of the European Economic Association.*