Regime Switching, Monetary Policy and Multiple Equilibria*

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1 Introduction

In simple settings the conditions under which monetary policy can lead indeterminacy are well understood: active Taylor rules generate determinacy and passive rules generate indeterminacy. When monetary policy is subject to regime switches, presumably because monetary policy has to shift randomly with changes in some underlying economic conditions, like output growth or employment, the situation becomes more complex, especially if policy is active in some regimes and passive in others.[1] It is natural then to expect that some average over the regimes, possibly weighted by transition probabilities, would allow the characterization of determinacy vs. indeterminacy, once indeterminacy is appropriately defined. The question has been studied by Davig and Leeper (2007) and then by Farmer, Waggoner and Zha (2009a, 2009b). We hope to further clarify the conditions for indeterminacy by characterizing the moments of the stationary distribution of inflation when monetary policy can swich across active and passive regimes acccording to a Markov process.

2 A simple model

We start with the simplest possible model, and leave the extensions for later. The simplest model has flexible prices where π_t is the inflation rate, r_t is the

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[1] We have in mind simple Taylor rules in simple settings where the a policy is active if the central bank changes the nominal rate by more than the change in the inflation rate, and passive otherwise. One possibility is that output growth follows a Markov chain, and policy is active or passive depending on whether output growth is above a treshold or not.

real rate, and R_t is the nominal rate at time t. The Fisher equation is satisfied, that is

$$R_t = E(\pi_{t+1}) + r_t \tag{1}$$

and the monetary authority sets the nominal rate according to the Taylor rule:

$$R_t = \tilde{R} + \phi_t (\pi_t - \tilde{\pi}) \tag{2}$$

We assume that $\{r_t\}_t$ is a bounded *iid* random variable with mean $\tilde{r}$, that $\{\phi_t\}_t$ is an irreducible, aperiodic, stationary Markov chain over state space $\Phi = (\bar{\phi}_1, ...\bar{\phi}_s)$ with transition matrix P and stationary distribution $\nu = (\nu_1, ...\nu_s)$, and that the target inflation rate is $\tilde{\pi} = \tilde{R} - \tilde{r}$. Then, substituting (2) into (1) and subtracting $\tilde{r}$ from both sides, we have:

$$\tilde{R} - \tilde{r} + \phi_t (\pi_t - \tilde{\pi}) = E(\pi_{t+1}) + r_t - \tilde{r}$$

$$\phi_t (\pi_t - \tilde{\pi}) = E(\pi_{t+1}) - \left(\left(\tilde{R} - \tilde{r} \right) - (r_t - \tilde{r}) \right)$$

$$\phi_t (\pi_t - \tilde{\pi}) = E(\pi_{t+1}) - (\tilde{\pi} - (r_t - \tilde{r}))$$

$$\phi_t (\pi_t - \tilde{\pi}) = E(\pi_{t+1} - \tilde{\pi}) + (r_t - \tilde{r})$$

If we set $q_t = \pi_t - \tilde{\pi}$, and we define $\varepsilon_t = r_t - \tilde{r}$ so that $E(\varepsilon_t) = 0$, we get:

$$\phi_t q_t = E(q_{t+1}) + \varepsilon_t \tag{3}$$

We can then explore additional solutions of (3) that satisfy

$$q_{t+1} = \phi_t q_t + \varepsilon_t \tag{4}$$

By repeated substitution we obtain

$$q_{t+N} = \left(\prod_{l=0}^{N-1} \phi_{t+l} \right) q_t + \sum_{l=0}^{N-1} \varepsilon_t \prod_{m=l+1}^{N-1} \phi_{t+m} \tag{5}$$

It is clear that if $\bar{\phi}_i > 1$ for $i = 1, ...s$, the only solution satisfying (3) that is bounded or that has finite moments is the Minimum State Variable solution (MSV) (see McCallum (1983)),

$$q_t = \frac{\varepsilon_t}{\phi_t} \tag{6}$$

When $\bar{\phi}_s < 1$ for one or more values of s, indeterminacy can become an issue and solutions of (3) other than (6) may emerge. For any initial q_0 and bounded *iid* sunspot process $\{\gamma_t\}_t$ with $E_t(\gamma_{t+1}) = 0$ for all t, there may be other ergodic solutions of (3) satisfying

$$q_{t+1} = \phi_t q_t - \varepsilon_t + \gamma_{t+1} \tag{7}$$

that are bounded or have finite moments. It may therefore be useful to consider what the set of admissible solutions to (3) are.

Typically, transversality conditions associated with underlying optimization problems are given in terms of the expected discounted value of assets in the limit as time goes to infinity. If for example the supply of nominal bonds or nominal balances are fixed, fast unbounded deflations may generate real asset levels that go to infinity, violating transversality conditions. Fast unbounded inflations that drive the real value of money to zero may also be inefficient or infeasible if money is essential for the functioning of the economy, so it is indeed reasonable from the perspective of optimizing agents to impose conditions assuring that at least the mean of the stationary distribution of $\{q_t\}_t$ exists. Other more stringent criteria may only require the existence of second or even higher moments.

3 Indeterminacy

If ϕ_t were fixed, it is well known that a standard condition for indeterminacy, or a multiplicity of bounded solutions that would satisfy underying transversality conditions of the agents, is $\phi < 1$. When ϕ is stochastic, or is a Markov chain, we may surmise that a condition for indeterminacy, admitting solutions to (3) other than (6), is given by $E(\phi) < 1$, where the expectation is taken with respect to the stationary distribution of ϕ. This however is not necessary: we will show that even when $E(\phi) < 1$, that is when the Taylor Rule is passive on average, solutions of (3) other than the (6) will exist but may not have first, second or higher moments, so that transversality conditions for the agents may fail. Therefore determinacy or uniqueness may be assured even if the Taylor rule is passive on average.

Let us first start with the existence of stationary solutions of (7). Since $\{\varepsilon_t\}_t$ and $\{\gamma_t\}_t$ zero mean *iid* processes, and $\{\phi\}_t$ has a stationary distribution, we can immediately apply a theorem of Brandt (1986). Recall that ν is the stationary probability induced by the transition matrix P. Brandt (1986) shows that if the condition $\nu \ln |\Phi'| < 0$ holds, that is if the expected value of $\ln\ |\phi|$ taken with respect to the stationary probabilities induced by the transition matrix P is negative, then (7) has a unique ergodic stationary distribution. Thus we see that the existence of stationary solutions requires not that $|\bar{\phi}_i| < 1$ for every i, but that the average over $\ln |\Phi'|$, computed using stationary probabilities for the Taylor coefficient ϕ, is negative. Clearly, the condition $\nu \ln |\Phi'| < 0$ cannot be satisfied if $|\bar{\phi}_i| > 1$ for all i.(See footnote 4.)

But this is not much help since a stationary distribution need not have finite moments, let alone be bounded. In fact it is precisely the finiteness of moments that will be the focus next. For this we invoke a recent Theorem of Saporta (2005).[2] Let Q be the diagonal matrix with diagonal entries $\bar{\phi}_i$.

[2] In a very different context Benhabib, Bisin and Zhu (2011) use similar techniques to study wealth distribution with stochastic returns to capital as well as stochastic earnings.

Theorem 1 *(Saporta (2005),Thm 2) Let*

$$q_{t+1} = \phi_t q_t - \varepsilon_t + \gamma_{t+1}$$

Assume: (i) $\nu \ln |\Phi'| < 0$,[3] *and (ii)* $\ln \phi_i$ $i = 1, ..s$ *are not integral multiples of the same number.*[4] *Then for* $x = \{-1, 1\}$, *the tails of the stationary distribution of* q_n, $P_{.>}(q_n > q)$, *are asymptotic to a power law:*

$$\Pr{}_{>}(xq_n > q) \sim L(x)\, q^{-\mu}$$

with $L(1) + L(-1) > 0$, *where* $\mu > 0$ *satisfies*

$$\lambda(Q^{\mu} P') = 1$$

and where $\lambda(Q^{\mu} P')$ *is the dominant root of* $Q^{\mu} P'$.

Remark 2 *The stationary distribution of* $\{q_t\}_t$ *is two-tailed because realizations of* ε_t *and* γ_t *as well as* $\bar{\phi}_i$ *may be positive or negative.*[5]

Remark 3 *Note that the* $i'th$ *the column sum of the matrix* QP' *gives the expected value of the Taylor coefficient conditional on starting at state* i.

Remark 4 *Most importantly, it follows from power law tails that if the solution of* $\mu = \hat{\mu}$, *then the stationary distribution has only moments* $m < \hat{\mu}$.

The above result is still not sharp enough because it does not sufficiently restrict the range of μ. Suppose for example, on grounds of microfoundations, we wanted to make sure that $\hat{\mu} > m$ for some m. To assure that the first moment of the stationary distribution of $\{q_t\}_t$ exists, we would want $\hat{\mu} > 1$, or if we wanted the variance to exist (mean square stability) would want $\hat{\mu} > 2$. The assumptions to guarantee this however are easy to obtain and trivial to check, given the transition matrix P and the state space Φ.

Define $\Phi^m = ((\phi_1)^m, ... (\phi_1)^m)$ for some positive integer m that we choose.

Assumption 1 *(a) Let the column sums of* $Q^m P'$ *be less than unity, that is* $P(\Phi^m)' < \mathbf{1}$, *where* $\mathbf{1}$ *is a vector with elements equal to* 1, *(b) Let* $P_{ii} > 0$ *for all* i, *and (c) Assume that there exists some* i *for which* $\bar{\phi}_i > 1$.

Remark 5 *In Assumption 1, (a) implies, for* $m = 1$, *that the expected value of the Taylor coefficient* ϕ_t *conditional on any realization of* ϕ_{t-1}, *is less than* 1, *that is that the policy is passive in expectation. (b) implies that there is a positive probability that the Taylor coefficient does not change from one period to the next, and (c) implies that there exists a state in which the Taylor rule is active.*

[3] Condition (i) may be viewed as a passive logarithmic Taylor rule in expectation. We will also use an expected passive Taylor rule in Assumption 1 and Proposition 1 but not in logarithms.

[4] Condition (ii) is a non-degeneracy condition often used to avoid lattice disatributions in renewal theory, and that will hold generically.

[5] The distribution would only have a right tail if we had $-\varepsilon_t + \gamma_{t+1} > 0$, and $\bar{\phi}_i > 0$ for all i, that is we would have $L(-1) = 0$. See Saporta (2005),Thm 1.

We now turn to our result on the conditions for indeterminacy.

Proposition 1 *Let assumption 1 hold. The stationary distribution of inflation exists and has moments of order m or lower.*

Proof. We have to show that there exists a solution $\hat{\mu} > m$ of $\lambda \left(Q^{\mu} P' \right) = \mathbf{1}$. Saporta shows that $\mu = 0$ is a solution for $\lambda \left(Q^{\mu} P' \right) = \mathbf{1}$, or equivalently for $\ln \left(\lambda \left(Q^{\mu} P' \right) \right) = 0$. This follows because $Q^0 = I$ and P is a stochastic matrix with a unit dominant root. Let $E \ln q$ denote the expected value of $\ln q$ evaluated at its stationary distribution. Saporta, under the assumption $E \ln q < 0$, shows that $\frac{d \ln \lambda \left(A^{\mu} P' \right)}{\delta \mu} < 0$ at $\mu = 0$, and that $ln \left(\lambda \left(A^{\mu} P' \right) \right)$ is a convex function of μ.[6] Therefore, if there exists another solution $\mu > 0$ for $\ln \left(\lambda \left(A^{\mu} P' \right) \right) = 0$, it is positive and unique. To assure that $\hat{\mu} > m$ we replace the condition $E \ln q < 0$ with $P \left(\Phi^m \right)' < \mathbf{1}$. . Since $Q^m P'$ is positive and irreducible, its dominant root is smaller than the maximum column sum. Therefore for $\mu = m$, $\lambda \left(Q^{\mu} P' \right) < 1$. Now note that if $P_{ii} > 0$ and $\overline{\phi}_i > 1$ for some i, the trace of $Q^{\mu} P'$ goes to infinity if μ $\phi_t$does (see also Saporta (2004) Proposition 2.7). But the trace is the sum of the roots so that the dominant root of $Q^{\mu} P'$, $\lambda \left(Q^{\mu} P' \right)$, goes to infinity with μ. It follows that the solution of $ln \left(\lambda \left(Q^{\mu} P' \right) \right) = 0$, $\hat{\mu} > m$. ■

Remark 6 *It follows from the Proposition that if admissible solutions of (7) require the mean of the stationary distribution of q to exist, we can apply the assumptions of the Proposition with $m = 1$; if we require both the mean and the variance to exist, we invoke the assumptions with $m = 2$. Certainly if Assumption 1 holds for $m = 1$, that is if the expectation that the Taylor rule ϕ_t is passive conditional on any ϕ_{t-1}, then the long-run mean of $\{q_t\}$ exists and constitutes a stationary soultion for 3 in addition to the MSV solution. This corresponds to indeterminacy.*

Remark 7 *If $P \left(\Phi \right)' > \mathbf{1}$, so that from every ϕ_{t-1} the expected value of $\phi_t > 1$, then from the proof of Proposition 1 the stationary solutions to (7) for inflation other than the MSV will not have a first moment[7], and would be inadmissible. It follows that if $P \left(\Phi \right)' > \mathbf{1}$, the only solution of (3) with a finite mean for $\{q_t\}$ is the MSV solution. This corresponds to determinacy.*

Remark 8 *However it is possible that the overall expected value of Taylor coefficient is passive at the stationary distribution, $E \left(\phi \right) < 1$ instead of passive in expectation at any t from every state $\phi_{t-1,}$ that is $P \left(\Phi^m \right)' < \mathbf{1}$, but that $\hat{\mu}$ in Theorem 1 is still less than 1. In such a case even if the Taylor Rule is passive on average, the stationary solution for 3 other than the MSV, as well as solutions converging to it, have infinite means, and can be discarded, so the MSV is the unique solution.*

The following Corollary follows immediately since it implies $\lambda \left(Q^m P' \right) > 1$.

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[6] This follows because $\lim_{n \to \infty} \frac{1}{n} \ln E \left(q_0 q_{-1} ... q_{n-1} \right)^{\mu} = \ln \left(\lambda \left(Q^{\mu} P' \right) \right)$ and the log-convexity of the moments of non-negative random variables (see Loeve(1977), p. 158).

[7] This is because if a positive μ exists it will have to be less than 1.

Corollary 1 *If $P(\Phi^m)' > \mathbf{1}$, then the stationary distribution of inflation, which exists if $\nu \ln|\Phi| < 0$, has no moments of order m or higher.*

Remark 9 *If we have a Markov chain for ϕ_t and we want it to be iid, then the rows of P must be identical: transition probabilities must be independent of the state. The dominant root $\lambda(Q^\mu P')$ is simply the trace of $Q^\mu P'$ since the other roots are zero, and column sums $\Sigma_i (\bar{\phi}_i)^\mu P_{ji}$ are identical for any j.*

Remark 10 *Comparative statics for μ can be obtained easily since the dominant root is an increasing function of the elements of $Q^\mu P'$. Since $\lambda(Q^\mu P')$ is a log-convex function of μ, the effect of mean preserving spreads the random variable $\lim_{N\to\infty}\left(\prod_{n=0}^{N-1}(\phi_{-n})^\mu\right)^{\frac{1}{N}}$ can be studied though second order dominance to show that they will decrease μ.*

The results above are also consistent with Proposition 1 of Davig and Leeper (2007). First note that as long as there is a state for the Taylor coefficient, $\bar{\phi}_i > 1$ with $P_{ii} > 0$, and $\gamma_{t+1} - \varepsilon_t$ is *iid* with zero mean, then a stationary distribution of inflation that solves (7) will unbounded even if $\gamma_{t+1} - \varepsilon_t$ has bounded support : there will always be a positive probability of a sufficiently long run of $\bar{\phi}_i > 1$ coupled with non-negative shocks, to reach any level of inflation. Therefore we may seek to obtain bounded solutions of (7) with $0 < \bar{\phi}_i < 1$, all i. In that case, the matrix given by Davig and Leeper (2007), $M = Q^{-1}P$ will have elements larger than those of P. But the dominant root of P, larger in modulus than other roots, is 1, and as is well known, an increasing function of its elements. So M must have a root larger than 1 the condition for determinacy given by Davig and Leeper (2007) fails. Conversely, if $\bar{\phi}_i > 1$ for all i, the dominant root, as well as other roots of $M = Q^{-1}P$ will be within the unit circle and satisfy the condition of Davig and Leeper (2007) for determinacy.

However, as shown by Farmer, Waggoner and Zha (2009b) in an example with a two state Markov chain, bounded sunspot solutions that satisy (3) may still exist. With regime-switching we may allow the sunspot variable γ_{t+1} to be proportional to $\phi_t q_t$ for all transitions to the active regime, and thereby to dampen the realization of the multiplicative effect the Taylor coefficient. This effectively transforms the system into one that behaves as if the policies were passive. The reason that this is compatible with a zero mean sunspot variable is that the dampening of the active policy can be offset by a value γ_{t+1} for all transitions to the passive regime, again proportional to the value of $\phi_t q_t$, to preserve the zero mean of γ. Therefore given transition probabilities, the random switching model makes it possible maintain the zero mean of the sunspot variable, as long as we allow a correlation between the sunspot variable and the contemporaneous realization of the Taylor coefficient ϕ. Boundedness follows because this scheme effectively delivers a stochastic difference equation with random switching between Taylor coefficients that are below one in each regime. Even more generally, in a New Keynesian model, Farmer, Waggoner and Zha (2009a) construct examples of bounded solutions without sunspots that depend not only on the fundamental shocks of the Minimum State Variable solution, but also on additional autoregressive shocks driven by fundamental shocks. The coefficients of the autoregressive structure have to depend on the transitions between the regimes as well as the transition probabilities in order to satisfy the analogue of (3). Markov switching across regimes allows the construction of such solutions. The autoregressive structure constructed in this manner however must also be non-explosive to allow bounded solutions. Farmer, Waggoner and Zha (2009a) show that this can be accomplished if at least one of the regimes is passive,and would permit indeterminacy operating on its own. A key element of the construction is the dependence of the additional shocks on the transitions between states and transition probabilities.

4 Extensions

1. The results can be extended to the case where $\{\varepsilon_t\}_t$ is not *iid*. We can define a Markov modulated process where we have a Markov chain on $\left\{\phi_t, \varepsilon_t, \gamma_{t+1}\right\}_t$ with the restriction that

$$\Pr\left(\phi_t, \varepsilon_t, \gamma_{t+1} | \phi_{t-1}, \varepsilon_{t-1}, \gamma_t\right) = \Pr\left(\phi_t, \varepsilon_t, \gamma_{t+1} | \phi_{t-1}\right)$$

The idea is that a single Markov process, here for simplicity $\{\phi_t\}_t$, drives the distributions of ε_t and γ_t, so that the parameters of the distribution of ε_t and γ_t depend on ϕ_{t-1} but not on past realizations of ε and γ. (See Saporta (2005) in remarks following Thm. 2). A pertinent example of such conditional independence is where the mean of interest rate deviations ε_t and the sunspot variable γ_t remain at zero irrespective of the realizations of ϕ_{t-1}, but other parameters of their distribution may be affected by ϕ_{t-1}. With an additional technical assumption the results of the previous sections go through unchanged.[8] Furthermore, the finite state Markov chain assumptions can also be relaxed. (See Roitershtein (2007).)

2. We may also want to study higher order systems of the type $q_{t+1} = A_t q_t + b_t$, where A_t are random d-dimensional square matrices with $\Pr(A_t \geq 0) = 1$, $\Pr(A_t \text{ has a zero row}) = 0$, b_t is a d-dimensional random vector with $\Pr(b_1 = 0) < 1$, $\Pr(b_1 \geq 0) = 1$, and $\left\{A_n, \acute{b}_n\right\}_n$ is a stationary *iid* Markov process. Such a structure arises for the sticky price new Keynesian models with regime-switching policies in two dimensions (as in Davig and Leeper (2007), Farmer, Waggoner and Zha (2009a)), and may be studied using the results of Kesten (1973, Theorems A and B). (See also Saporta (2004a, sections 4 and 5) and Saporta, Guivarc'h, and Le Page (2004b)). While the results concerning power tails in the one-dimensional case generalize at least for the case of *iid* transitions,[9] the technical

[8] The technical assumption is

$$\Pr\left(\phi_i q + \varepsilon_i + \gamma_{i+1} = q\right) < 1 \text{ for any } i \text{ and } q.$$

This prevents a degenerate stochastic process to get stuck at a particular value of q.

[9] For example, when the rows of the transition matrix are identical so transition probabilities are independent of the current state.

conditions that must be verified, although similar to the the one dimensional case, are more complex. Define $|x| = \left(\sum_{i=1}^{d} (x_i)^2\right)^{\frac{1}{2}}$ and $||A|| = \max_{|x|=1} xA$, and assume $E \ln ||A_1|| \ln^+ ||A_1|| < \infty$, $E \ln |b_1| < \infty$, $E|b_1|^\beta < \infty$ for some $\beta > 0$ (where $x^+ = \max(0, x)$). We must first make sure that an easy to check technical conditon, which holds generically and is analogous to (ii) in Theorem 1, holds for the higher dimensions. If $\delta(A)$ is the dominant root of A, assume that the group generated by $\{\ln \delta(\pi) : \pi = A_1 ... A_n$ for some n and $A_i \in supp\,(v)\ \ for\ \pi > 0\}$ is dense in R. Now we turn to the analogue of condition (i), $\nu \ln \phi_t < 0$, in Theorem 1: in higher dimensions we assume $\alpha = lim_{n \to \infty} \frac{1}{n} \ln ||A_1 ... A_n|| < 0$. This condition may seem hard to check, but an easily verified sufficient condition for it is $E \ln ||A_1|| < 0$. To assure that we have $\alpha < 0$, we may also use a stronger condition, that the expected value of the dominant root of $(A_1 \otimes^t A_1)$,that is $E\delta(A_1 \otimes^t A_1) < 1$, where $\otimes$ is the Kroenecker product. However this condition is not only strong enough to guarantee that $\alpha < 0$, but also that both the first and second moments of the stationary distribution of $\{q_t\}_t$ exist, yielding the desirable "mean square stability" results. (See Saporta (2004) Proposition 4.1 and its proof as well as Farmer, Waggoner and Zha (2009c).)) Let us stick with the weaker condition $\alpha < 0$, guaranteed by $E \ln ||A_1|| < 0$, and, following Kesten (2003), let the expected value of the minimum row sum of A_1,raised to some σ, be larger than or equal to $d^{\frac{\sigma}{2}}$, where d is the dimension of A_1. This assures that there exists $0 < \mu \leq \sigma$ such that the power law and moment results in the one dimensional case generalize.[10] The power law will apply to xq, with x any normalized non-negative unit row vector of the same dimension as q: $\lim_{t \to \infty} \Pr(xq \geq t) = Ct^{-\mu}$ where C is a positive constant. Note for example that if $\sigma < 1$ the stationary distribution of inflation has no mean, if $\sigma < 2$, it has no variance. If σ is not finite, all moments of $\{q_t\}_t$ will exist. It follows, as in the one dimensional case, that only the moments of order $m < \mu$ of the stationary distributions of the variables of the vector q_t, as well as xq_t, will exist.[11] Note of course that these multidimensional results will immediately apply to random coefficient $AR(q)$ models transformed into matrix format.

3. To simplify matters, with some additional assumptions we can introduce

[10] For significant extensions and relaxation of the technical assumptions, see Saporta (2004), Theorems 10, 11, 13 in section 4 and 5.1 in section 5. In particular in Theorem 13 Saporta(2004) also reports a condition to replace the minimum row sum condition of Kesten (1973): If we require the expected value of smallest root of $\left((A^T A)^{\frac{1}{2}}\right)^{\sigma}$ to be ≥ 1 for some σ, this can replace the minimum row sum condition to assure, as in the one dimensional case, the existence of a finite $\mu < \sigma$ that defines the power law tails for the stationary distribution of q, provided $||A_1||$, $||B_1||$ are finite, and the column sums of A_1 are positive. If there is no finite μ, all moments of the stationary distribution of q may exist.

[11] We may also inquire as to whether $\alpha > 0$ rules out the existence of a stationary distribution for the solution of $q_{t+1} = A_t q_t + b_t$. Bougerol and Picard (1992) prove that this is indeed the case under the assumptions that (i) $\{A_s, b_s\}$ is independent of q_n for $n < s$, and (ii) that for $\{A_0, b_0\}$ if there is an invariant affine subspace $H \in \mathbf{R}^d$ such that $\{A_0 q + b_0 | q \in H\}$ is contained in H,then H is $\mathbf{R}^d$. Condition (ii), which the authors call irreducibility, eliminates for example cases where $b_t = 0$ for all t, so that $q_t = 0$ is a stationary solution for all t irrespective of $\{A_s\}_s$.

a Phillips curve in a simplified model while still remaining in one dimension. Let the simple Phillips curve be given by $q_t = kx_t$ where x_t is output and q_t is inflation, and let the IS curve be $x_t = -m\left[R_t - Eq_{t+1}\right] + E_t x_{t+1}$ where R_t is the nominal interest rate. Let the Taylor rule be given by $i_t = \phi_t q_t$. Then after substitutions the system can be written as

$$Eq_{t+1} = \left(\frac{\phi_t mk + 1}{mk + 1}\right) q_t = \chi_t q_t$$

where $\chi_t = \chi > 1$ (< 1) if $\phi_t = \phi > 1$ (< 1). There is always a bounded solution given by $q_t = 0$ where inflation is always at its target steady state.. However, if ϕ_t is generated by a Markov chain, there may also be sunspot solutions given by

$$q_{t+1} = \chi_t q_t + \gamma_{t+1}$$

where γ_{t+1} is a sunspot variable. This equation may then be analyzed by the same methods used above.

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