Sentiments and Aggregate Fluctuations

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We try to capture the Keynesian notion that animal spirits or sentiments, unconnected to fundamentals, can drive employment and output fluctuations under rational expectations.

The key Keynesian feature of our model is that employment and production decisions are based on expectations of aggregate demand driven by consumer sentiments, while realized demand follows from the production and employment decisions of firms.
Nevertheless, in equilibrium all agents know correct distributions, all prices are flexible, all markets clear and consumer expectations about aggregate consumption, employment and real wages are correct each period.

So we have rational expectations equilibria.
Our work is inspired by Angeletos and Lao (2011), where sentiments can drive output, by the Lucas Island model.

It is related to Cass and Shell (1983), and to correlated equilibria via Aumann (1974, 1987), and to Maskin and Tirole (1987).

In the absence of sentiments, the models that we study have unique equilibria.

But sentiments and beliefs can amplify employment, production and consumption decisions, and lead to multiple rational expectations equilibria.
Consumers make consumption and labor supply plans based on their "sentiments" about aggregate demand, and real wages. Nominal wages are normalized to one.

Each firm must make optimal production decisions on the basis of signals about what its demand will be, before demand is realized.

The signals can be based on firms’ market research about their demand, early orders, initial inquiries, as well as public signals of aggregate demand/consumer sentiments.

Since the real wages and employment have not yet been determined, and production has not yet taken place, these signals capture consumer sentiment.
In the first simplest benchmark model, the signal is a weighted sum of the firm’s idiosyncratic demand shock and a shock to aggregate demand/sentiments, both of which enter the firm’s demand curve.

Later in the paper we provide the explicit microfoundations for how the signals are generated.

We also introduce a second noisy but public signal of aggregate demand. This signal may represent public forecasts of aggregate demand/sentiments.
Trades take place in centralized markets rather than bilaterally through random matching. At the end of each period all history can become public knowledge.

Firms optimally decide on how much to produce on the basis of their private but correlated signals about demand. Only then aggregate output is realized and prices clear all markets.

In equilibrium all agents "know" the correct distribution of the idiosyncratic and aggregate demand shocks.

The realized real wage is equal the wage that the consumers expected, given their sentiments.

Aggregate output equals to the households’ planed consumption. So households can in fact implement their consumption plans.

Thus we have rational expectations equilibria.
Model cont’d

- We show that in the simple benchmark model, there can be two distinct rational expectations equilibria: one with constant output and one with stochastic output driven by self-fulfilling sentiments.
- But note that the self-fulfilling stochastic equilibrium is not a randomization over multiple equilibria.
- When we discuss the microfoundations of signals, we will see that in fact we have a continuum of equilibria, parametrized by the variance of sentiment shocks.
Households make a consumption and labor supply plan:

$$\max E_0 \sum \beta^t [\log(C_t) - \psi N_t]$$

subject to

$$C_t \leq \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

where $W_t$ denotes nominal wage and $\Pi_t$ aggregate profit income from firms, all measured in final goods. The first-order conditions for labor imply

$$\frac{1}{C_t} \frac{W_t}{P_t} = \psi$$

Households have (so far) common point expectations/sentiments about aggregate $C_t$, so given a nominal wage $W_t = 1$, they can infer a price $P_t$ and a real wage $\frac{W_t}{P_t}$, consistent with (1) and supply labor.
The final-good firms (or a representative consumer) produce a final good according to

$$C_t = Y_t = \left[ \int \varepsilon_{jt}^{\frac{1}{\theta}} Y_{jt}^{\frac{\theta - 1}{\theta}} dj \right]^{\frac{\theta}{\theta - 1}}$$

where $\theta > 1$ and $\log \varepsilon_{jt}$ are iid zero mean firm-specific shocks, and maximizes profit

$$\max P_t \left[ \int \varepsilon_{jt}^{\frac{1}{\theta}} Y_{jt}^{\frac{\theta - 1}{\theta}} dj \right]^{\frac{\theta}{\theta - 1}} - \int p_{jt} Y_{jt} dj.$$  

The demand function depends on both $\varepsilon_{jt}$ and $Y_t$:

$$\frac{P_{jt}}{P_t} = Y_{jt}^{-\frac{1}{\theta}} (\varepsilon_{jt} Y_t)^{\frac{1}{\theta}}$$

$$Y_{jt} = \left( \frac{P_t}{P_{jt}} \right)^{\theta} \varepsilon_{jt} Y_t$$
Intermediate Goods

- Each intermediate firm produces good $Y_{jt}$ without perfect knowledge either of $\epsilon_{jt}$ or of aggregate demand $\tilde{Y}_t$, which could be random and sentiment-driven. Instead, as in the Lucas island model, firms have a noisy indication of what their demand will be from a signal

$$s_{jt} = \lambda \log \epsilon_{jt} + (1 - \lambda) \log \tilde{Y}_t$$

- The parameter $\lambda$ reflects the weights of the idiosyncratic and aggregate components of demand.

- This signal is our simplest signal. In the paper the simplest signal has iid shocks $v_{jt} \sim N(0, \sigma_v^2)$ as well.

- Signal microfoundations will determine $\lambda$ later.

- In a REE aggregate demand $\tilde{Y}_t$ that generates the signal will be equal to actual aggregate output $Y_t$. 
An intermediate goods producer \( j \) has the production function

\[
Y_{jt} = An_{jt}
\]

The firm maximizes expected nominal profits \( \Pi_{jt} = p_{jt} Y_{jt} - W_t \frac{Y_{jt}}{A} \):

\[
\max_{Y_{jt}} E_t \left[ \left( P_t Y_{jt}^{-\frac{1}{\theta}} (\epsilon_{jt} Y_t)^{\frac{1}{\theta}} \right) Y_{jt} - W_t \frac{Y_{jt}}{A} \right] s_{jt}.
\]

After simplifications using \( \frac{1}{C_t} = \frac{1}{Y_t} = \psi \frac{P_t}{W_t} \), and \( W_t = 1 \):

\[
Y_{jt} = \left\{ \left( 1 - \frac{1}{\theta} \right) \frac{A}{\psi} E_t \left[ (\epsilon_{jt})^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}-1} \right] \right\}^{\theta} s_{jt}.
\]

Strategic Substitutability: since \( \theta > 1 \), \( Y_{jt} \) is negatively related to \( Y_t \).
Certainty Equilibrium

- Information is perfect and sentiments have no role. The signal $s_j$ fully reveals the firm’s own idiosyncratic demand $\epsilon_{jt}$. ($\sigma_{\tilde{y}} = 0$)
- There exists a fundamental certainty equilibrium with constant aggregate output $C_t = Y_t = Y^*$ and $P_t = P$ and firm output is:

$$Y_{jt}^{\frac{1}{\theta}} = \left(1 - \frac{1}{\theta}\right) \frac{A}{\psi} \epsilon_{jt}^{\frac{1}{\theta}} Y_t^{\frac{1-\theta}{\theta}}$$

- Without loss of generality set $\left(1 - \frac{1}{\theta}\right) \frac{A}{\psi} = 1$. Final good output is:

$$Y_t = \left[ \int \epsilon_{jt}^{\frac{1}{\theta}} Y_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}$$

or, if $\epsilon_{jt} \equiv \log \epsilon_{jt}$ has zero mean and variance $\sigma_\epsilon^2$,

$$\tilde{\phi}_0 = \log Y_t = \frac{1}{\theta - 1} \log E \exp(\epsilon_{jt}) = \frac{1}{2(\theta - 1)} \sigma_\epsilon^2$$
We conjecture there exists an another equilibrium, such that aggregate output is not a constant. In particular all agents "know" output follows
\[ \log Y_t = \phi_0 + z_t, \]
The noisy signal received by each firm (now defined net of the constant term \( \phi_0 \)) is
\[ s_{jt} = \lambda \varepsilon_{jt} + (1 - \lambda) z_t \]
where \( z_t \sim \mathcal{N}(0, \sigma_z^2) \).
With fluctuations in aggregate output, the signal is not fully revealing.
In the self-fulfilling equilibrium:

- Each period the sentiment $z_t$ held by households in $\log Y_t = \phi_0 + z_t$ will be the realized $z_t$ in $\log Y_t$.
- So the distribution of the perceived sentiment $\{z\}$ will be consistent with the realized distribution of aggregate output $\{Y\}$.
- Prices will clear all markets each period.
Self-Fulfilling Equilibrium

Proposition

If \( \lambda \in (0, \frac{1}{2}) \), there exists a self-fulfilling rational expectations equilibrium with stochastic aggregate output \( Y_t = \phi_0 + z_t \). Furthermore \( \log Y_t \) is normally distributed with mean

\[
\phi_0 = \frac{(1 - \lambda) + (\theta - 1) \lambda}{\theta(1 - \lambda)} \bar{\phi}_0 < \bar{\phi}_0
\]

and variance

\[
\sigma_z^2 = \frac{\lambda (1 - 2\lambda)}{(1 - \lambda)^2 \theta} \sigma^2
\]

Welfare? Note that in this case (but not necessarily in models that follow) the mean of the constant output equilibrium \( \bar{\phi}_0 > \phi_0 \), so in this case it Pareto dominates the sentiment driven stochastic equilibrium.
In the Certainty Equilibrium with $\sigma_z^2 = 0$, $Y_{jt} = \epsilon_{jt} Y_t^{1-\sigma}$, and since $\sigma > 1$, equilibrium firm-level outputs depend negatively on aggregate output as in the case of strategic substitutability. Hence, this fundamental equilibrium is unique.

Given $\lambda$ in $s_{jt} = \lambda \epsilon_{jt} + (1 - \lambda) z_t$ and the variance of the idiosyncratic shock $\sigma_{\epsilon}^2$, for markets to clear for all possible realizations of the sentiment $z_t$, the variance $\sigma_z^2$ has to be precisely pinned down, as in the Proposition above.

If $\sigma_z^2$ is too high, output is too low relative to aggregate demand by consumers, and vice-versa.
Intuition

If firms believe that the signal contains information about changes in aggregate demand, \( z_t \), then this belief will induce all agents to amplify their output response, up or down, and sustain self-fulfilling fluctuations consistent with the agents’ beliefs about the distribution of output. Note that both the variance of the sentiment shock \( \sigma_z^2 \) and \( \lambda \) affect the firms’ optimal output responses through their signal extraction. Given \( \lambda \) and the variance of the idiosyncratic shock \( \sigma_\varepsilon^2 \), for markets to clear for all possible realizations of the sentiment \( z_t \), the variance \( \sigma_z^2 \) has to be precisely pinned down, as in the Proposition. If however the signal gives a low weight to aggregate as opposed to idiosyncratic demand, that is if \( \lambda \in [0.5, 1] \), then we cannot find a positive variance \( \sigma_z^2 \) that will clear the markets for every \( z_t \).
So far we assumed that firms can get an initial signal for the overall demand for their product, but cannot disaggregate it into its components arising from idiosyncratic and from aggregate demand. They only observe their sum.

Since the signals are based on early and initial demand indications for each of the firms, they may well contain additional firm-specific noise components. Suppose then that the signal takes the slightly more general form,

$$s_{jt} = \nu_{jt} + \lambda \varepsilon_{jt} + (1 - \lambda) z_t,$$

where $\nu_{jt}$ is a pure firm-specific iid noise with zero mean and variance $\sigma^2_{\nu}$.
Imperfect signal with firm-specific noise, Cont’d (SKIP)

The self-fulfilling equilibrium

We had defined

\[ \log Y_t = y_t = \phi_0 + z_t \]

**Proposition**

Let \( \lambda < \frac{1}{2} \), and \( \sigma_v^2 < \lambda (1 - 2\lambda) \sigma_\varepsilon^2 \). In addition to the certainty equilibrium, there also exists a self-fulfilling rational expectations equilibrium with stochastic aggregate output, \( \log Y_t = \phi_0 + z_t \) that has a mean

\[ \phi_0 = \frac{1}{2} \left( \frac{(1 - \lambda + (\theta - 1) \lambda)}{\theta (1 - \lambda)} \right) \frac{1}{(\theta - 1)} \sigma_\varepsilon^2 - \frac{(\theta - 1) \sigma_v^2}{2 \theta^2 (1 - \lambda)^2} \]

and variance

\[ \sigma_z^2 = \frac{\lambda (1 - 2\lambda)}{(1 - \lambda)^2 \theta} \sigma_\varepsilon^2 - \frac{1}{(1 - \lambda)^2 \theta} \sigma_v^2. \]
The government and public forecasting agencies as well as news media often release their own forecasts of the aggregate economy.

Suppose firms receive two independent signals, $s_{jt}$ and $s_{pt}$.

The firm-specific signal $s_{jt}$ is based on firm’s own preliminary information about its demand:

$$s_{jt} = v_{jt} + \lambda \varepsilon_{jt} + (1 - \lambda) z_t$$

The public signal is:

$$s_{pt} = z_t + e_t$$

where $e_t \sim N(0, \sigma^2_e)$ is noise in the public forecast of aggregate demand.

We can also model the noisy public signal with heterogenous but correlated sentiments across consumers, so observing a subset of consumers will reveal a noisy signal of the average sentiment.
To establish the existence of the certainty equilibrium, we also assume that $\sigma_e^2 = \gamma \sigma_z^2$, where $\gamma > 0$.

This assumption states that the variance of the forecast error of the public signal for aggregate demand is proportional to the variance of $z$, or of equilibrium output.

Then in the certainty equilibrium where output is constant over time, the public forecast of output is correct and constant as well.
Multiple signals, Cont’d

**Proposition**

If \( \lambda < \frac{1}{2} \), and \( \sigma_v^2 < \lambda \left( 1 - 2\lambda \right) \sigma^2_\varepsilon \), then there exists a self-fulfilling rational expectations equilibrium with stochastic aggregate output

\[
\log Y_t = y_t = z_t + \eta e_t + \phi_0
\]

which has mean \( \phi_0 = \frac{1}{2} \left( \frac{1 - \lambda + (\theta - 1)\lambda}{\theta(1-\lambda)} \right) \frac{1}{(\theta - 1)} \sigma^2_\varepsilon - \frac{(\theta - 1)\sigma^2_v}{2\theta^2(1-\lambda)^2} \) and variance

\[
\sigma_z^2 = \frac{\lambda \left( 1 - 2\lambda \right)}{(1-\lambda)^2\theta} \sigma^2_\varepsilon - \frac{1}{(1-\lambda)^2\theta} \sigma^2_v > 0,
\]

and where \( \eta = -\frac{\sigma^2_z}{\sigma^2_\varepsilon} = -\frac{1}{\gamma} \). In addition, there is a "certainty" equilibrium with constant output identical to the certainty equilibrium in the previous Proposition with a single signal.

- Note: the public forecast error \( e_t \) affects output.
A Simple Abstract Model

To set up the intuition consider the following simple model.

Assume for simplicity that the economy is log-linear, so optimal log output of firms coming from a linear quadratic objective, is given by the rule

$$y_{jt} = E_t\{[\beta_0 \epsilon_{jt} + \beta y_t] | s_{jt}\}$$

where $\epsilon_{jt}$ is zero mean, iid.

The coefficient $\beta$ can be either negative or positive, so we can have either strategic substitutability or strategic complementarity in firms' actions.

Market clearing requires

$$y_t = \int y_{jt} dj.$$
Abstract Model Cont’d:

- We have, from a linear quadratic objective, for $\beta < 1$,

  $$ y_{jt} = E_t \{ [\beta_0 \epsilon_{jt} + \beta y_t] | s_{jt} \} $$  \hspace{1cm} (3) 

  $$ s_{jt} = v_{jt} + \lambda \epsilon_{jt} + (1 - \lambda) y_t $$ \hspace{1cm} (4) 

  $$ y_t = \int y_{jt} \, dj $$ \hspace{1cm} (5) 

- Assume that $y_t$ is normally distributed with zero mean and variance $\sigma^2_y$. Based on (3), signal extraction gives:

  $$ y_{jt} = \frac{\lambda \beta_0 \sigma^2_{\epsilon} + (1 - \lambda) \beta \sigma^2_y}{\sigma^2_v + \lambda^2 \sigma^2_{\epsilon} + (1 - \lambda)^2 \sigma^2_y} \left[ v_{jt} + \lambda \epsilon_{jt} + (1 - \lambda) y_t \right] $$

- Market clearing implies:

  $$ y_t = \int y_{jt} \, dj = \left[ \frac{\lambda \beta_0 \sigma^2_{\epsilon} + (1 - \lambda) \beta \sigma^2_y}{\sigma^2_v + \lambda^2 \sigma^2_{\epsilon} + (1 - \lambda)^2 \sigma^2_y} \right] (1 - \lambda) y_t $$
Market clearing implies

\[
y_t = \int y_{jt} \, dj = \frac{\lambda \beta_0 \sigma_{\varepsilon}^2 + (1 - \lambda) \beta \sigma_y^2}{\sigma_v^2 + \lambda^2 \sigma_{\varepsilon}^2 + (1 - \lambda)^2 \sigma_y^2} (1 - \lambda)y_t.
\]

(6)

The Certainty Equilibrium is \(y_t \equiv 0\).

But (6) holds for all \(y_t\) if

\[
\sigma_y^2 = \frac{\lambda(\beta_0 - (1 + \beta_0)\lambda)\sigma_{\varepsilon}^2 - \sigma_v^2}{(1 - \lambda)^2(1 - \beta)}
\]

Thus, \(\sigma_y^2\) is pinned down uniquely and it defines the Self-Fulfilling Stochastic Equilibrium.

Note that if \(\beta < 1\), a necessary condition for \(\sigma_y^2\) to be positive is \(\lambda \in \left(0, \frac{\beta_0}{1+\beta_0}\right)\).
Our model is essentially static, but we can investigate whether the equilibria of the model are stable under adaptive learning.

For simplicity we will confine our attention to the simplified abstract model of section with $\sigma_v^2 = 0$, where without loss of generality we set $\beta_0 = 1$. So the model is

$$s_{jt} = \lambda \epsilon_{jt} + (1 - \lambda) \log Y_t$$

$$y_{jt} = E_t\{\epsilon_{jt} + \beta y_t | s_{jt}\}$$

$$y_t = \int y_{jt} \, dj$$
We can renormalize our model so that the sentiment or sunspot shock \( z_t \) has unit variance by redefining output as \( y_t = \log Y_t = \sigma_z z_t \). The variance of output \( y_t \) then is still \( \sigma_z^2 \).

Suppose that agents understand that equilibrium \( y_t \) is proportional to \( z_t \) and they try to learn \( \sigma_z \).

If agents conjecture at the beginning of the period \( t \) that the constant of proportionality is \( \sigma_{zt} = \frac{y_t}{z_t} \), then the realized output is

\[
y_t = \frac{\lambda \sigma_{\varepsilon}^2 + (1 - \lambda) \beta \sigma_{zt}^2}{\lambda^2 \sigma_{\varepsilon}^2 + (1 - \lambda)^2 \sigma_{zt}^2} (1 - \lambda) \sigma_{zt} z_t.
\]

Under adaptive learning with constant gains \( g = 1 - \alpha \), agents update \( \sigma_{zt} \):

\[
\sigma_{zt+1} = \sigma_{zt} + (1 - \alpha) \left( \frac{y_t}{z_t} - \sigma_{zt} \right) \equiv h(\sigma_{zt})
\]
Stability Under Learning Cont’d

- We have the adaptive learning updating rule

\[ \sigma_{zt+1} = \alpha \sigma_{zt} + (1 - \alpha) \left( \frac{y_t}{z_t} \right) = \sigma_{zt} + (1 - \alpha) \left( \frac{y_t}{z_t} - \sigma_{zt} \right) \equiv h(\sigma_{zt}) \]

- For any initial \( \sigma_{zt} > 0 \), we can show that \( \sigma_{zt} \) does not converge to 0, the certainty equilibrium.

- By contrast the sentiment-driven sunspot equilibrium is locally stable under learning provided the gain \( g = 1 - \alpha \) is not too large.

- In particular \( h'(0) > 1 \) and \( |h(\sigma_z)| < 1 \) at the sentiment driven equilibrium \( \sigma_z = \left( \frac{\lambda(1-2\lambda)\sigma_{\epsilon}^2}{(1-\lambda)^2(1-\beta)} \right)^{0.5} \).
So far we simply assumed that firms receive signals

\[ s_{jt} = v_{jt} + \lambda \varepsilon_{jt} + (1 - \lambda) y_t \]

based on their market research, market surveys, early orders, initial inquiries and advanced sales, to form such expectations.

In particular, we assumed that the signals weights \( \lambda \) and \( (1 - \lambda) \) are exogenous. It is therefore desirable to spell out in more detail the microfoundations for how firms obtain these signals.

If the signal reveals, except for iid noise, the precise weighted sum of the fundamental and the sentiment shocks that the firms need to forecast, then the signal extraction problem of the firm disappears.

This then excludes the possibility of sentiment-driven equilibria.
For example, suppose firm $j$ can post a hypothetical price $\tilde{p}_{jt}$ and ask a subset of consumers about their intended demand given this hypothetical price.

The firms then obtain a signal about the intercept of their demand curve, possibly with some noise $v_{jt}$ if consumers have heterogeneous sentiments, $s_{jt} = \varepsilon_{jt} + (1 - \theta) y_t + v_{jt}$.

To see this note that the demand curve of firm $j$ is given by

$$Y_{jt} = \left( \frac{p_{jt}}{P_t} \right)^{-\theta} Y_t \varepsilon_{jt}.$$ Since from the labor market first order conditions we have $P_t = \frac{1}{\psi Y_t}$, the logarithm of the demand curve, ignoring constants that can be filtered, becomes

$$y_{jt} = \varepsilon_{jt} + (1 - \theta) y_t - \theta p_{jt}.$$

Suppose firm $j$ can post a hypothetical price $\tilde{p}_{jt}$ and ask a subset of consumers about their intended demand given this hypothetical price.

The firms can then obtain a signal about the intercept of their demand curve, possibly with some noise $v_{jt}$ if consumers have heterogeneous sentiments, $s_{jt} = \varepsilon_{jt} + (1 - \theta) y_t + v_{jt}$.
The optimal output decision of firms is given by

\[ y_{jt} = E \left[ \varepsilon_{jt} + (1 - \theta) y_t \right] \], \quad s_{jt} = \frac{\sigma_s^2 - \sigma_v^2}{\sigma_s^2} \left[ \varepsilon_{jt} + (1 - \theta) y_t + v_{jt} \right] \]

where \( \sigma_s^2 \) is the variance of the signal.

Note that \( \sigma_s^2 - \sigma_v^2 > 0 \) is the variance of \( \varepsilon_{jt} + (1 - \theta) y_t \), or \( \text{Cov} (\varepsilon_{jt} + (1 - \theta) y_t, s_{jt}) \).

Integrating across firms, \( y_t = \int y_{jt} \, dj \), and equating coefficients of \( y_t \) yields

\[ 1 = \frac{\sigma_s^2 - \sigma_v^2}{\sigma_s^2} (1 - \theta) \].

This equality is impossible (even if \( \sigma_v^2 = 0 \)) since by construction \( \sigma_s^2 > \sigma_v^2 \) and \( \theta > 1 \).

In other words, a constant output with \( y_t = 0 \) is the only equilibrium.
To obtain the possibility of sentiment-driven equilibria, we can either slightly complicate the signal extraction problem of the firm by adding an extra source of uncertainty,

Or we can modify the signal so that it does not eliminate the signal extraction problem faced by the firm.

We provide microfoundations for both of these approaches below.
First we study a model with an additional source of uncertainty. We still allow a firm to post a hypothetical price $\tilde{p}_{jt}$ and ask a subset of consumers about their intended demand at that price. However at the time of the survey the preference shock is not yet realized with certainty: each consumer $i$ receives a signal for his/her preference shock $\epsilon_{jt}$: $s_{ht}^i = \epsilon_{jt} + h_{jt}^i$ which forms the basis of their response to the posted hypothetical price.

Let utility of consumers be $\frac{C_t^{1-\gamma}-1}{1-\gamma} - \psi N_t$, and let $\sigma_h^2$ be the variance of $h_{jt}^i$ and $\sigma_\epsilon^2$ be the variance of $\epsilon_{jt}$.

Then we can show the weight $\lambda$ in the firm’s signal

$s_{jt} = \nu_{jt} + \lambda \epsilon_{jt} + (1 - \lambda) z_t$ is uniquely determined if $1 - \theta \gamma > \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_h^2}$:

$$\lambda \equiv \frac{\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_h^2}}{\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_h^2} + (1 - \theta \gamma)} \in \left(0, \frac{1}{2}\right)$$
Instead of introducing additional sources of uncertainty to establish sentiment-driven equilibria, we instead modify the signals so that they do not eliminate the signal extraction problem faced by firms.

Suppose the intermediate-good firms receive a signal from consumers about the quantity of the demand $Y_{jt}$ for their good.

We drop the final-good sector and assume instead that the representative household purchases a variety of goods $C_{jt}$ to maximize utility

$$\log C_t - \psi N_t,$$

where

$$C_t = \left[ \int e^{\frac{1}{\theta}} C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}.$$
The households have utility utility log $C_t - \psi N_t$ and choose their demand $C_{jt}$ and labor supply $N_{jt}$ based on the sentiment shock $Z_t$ and the preference shocks $\epsilon_{jt}$.

In equilibrium household demand function for each variety is:

$$C_{jt} = \left( \frac{P_t}{P_{jt}} \right)^{\theta} \epsilon_{jt} C_t = \left( \frac{P_t}{P_{jt}} \right)^{\theta} \epsilon_{jt} Y_t.$$  

Since demand depends on the consumers’ conjectures about their price $P_{jt} = P_t(\epsilon_{jt}, Z_t)$, the demand $C_{jt}$ is a function of preference shocks and the sentiment, $C_{jt} = C(\epsilon_{jt}, Z_t)$.
The intermediate-good firms, based on the signal \( s_{jt} = c_{jt} \) choose their production according to the first order condition given by

\[
C_{jt} = Y_{jt} = \left\{ \left( 1 - \frac{1}{\theta} \right) \frac{A}{\psi} E_t \left[ \epsilon_{jt}^{\frac{1}{\theta}} Y_{jt}^{\frac{1}{\theta} - 1} \mid S_{jt} \right] \right\}^\theta,
\]

We conjecture that in equilibrium,

\[
c_{jt} = \phi \epsilon_{jt} + \phi_z z_t
\]

where \( \phi \) and \( \phi_z \) are undetermined coefficients.
Proposition

There is a continuum of sentiment-driven equilibria \( c_{jt} = \phi \varepsilon_t + \phi_z z_t \) with

i) \( \phi_z = 1 \)

ii) \( \phi \in [0, 1] \)

iii) \( \sigma_z^2 = \frac{\phi(1-\phi)}{1-\beta} \sigma^2 \varepsilon \in \left[ 0, \frac{1}{4(1-\beta)} \sigma^2 \varepsilon \right] \)

Proposition

The sentiment-driven equilibria of this model with signal \( s_{jt} = c_{jt} \) can be mapped one-to-one to the sentiment-driven equilibria of our benchmark model with the signal \( s_{jt} = \lambda \varepsilon_t + (1 - \lambda) y_t \) where

\[
\lambda = \frac{\phi}{\phi + 1} \in \left[ 0, \frac{1}{2} \right].
\]
Proposition

Under the signal $s_{jt} = c_{jt}$, there also exists another type of sentiment-driven equilibria with firm-level output driven not only by the fundamental shock $\varepsilon_{jt}$ and aggregate sentiment shock $z_t$, but also by a firm-specific iid shock $v_{jt}$ with zero mean and variance $\sigma_v^2$:

$$c_{jt} = \phi \varepsilon_{jt} + z_t + (1 + \phi) v_{jt}.$$  

Furthermore the signal $s_{jt} = c_{jt}$ is isomorphic to the signal $s_{jt} = \lambda \log \varepsilon_{jt} + (1 - \lambda) y_t + v_{jt}$. 


Utility is $C_t^{1-\gamma-1} - \psi N_t$ where $C_t = \left[ \int \epsilon_{jt} \frac{1}{\theta} C_{jt}^{\theta-1} \, dj \right]^{\frac{\theta}{\theta-1}}$.

Each firm $j$’s total production cost (or labor productivity) is affected by an idiosyncratic shock that is correlated with the demand shock $\epsilon_{jt}$.

For example, marketing costs may be lower under favorable demand conditions: for a higher amount of sales, labor becomes more productive so that labor demand $N_{jt} = Y_{jt} \epsilon_{jt}^{-\tau}$ is lower, where $\tau > 0$ is a parameter.

Alternatively, if marketing costs increase with sales and labor demand is higher, we may have $\tau < 0$.

Firm $j$’s optimal output, given consumer demands, is

$$y_{jt} = E_t \left[ (1 + \theta \tau) \epsilon_{jt} + (1 - \theta \gamma) y_t \right] | s_{jt}.$$
Let firms get a signal \( s_{jt} = \varepsilon_{jt} + (1 - \theta \gamma) y \): the intercept of their demand curve.

Optimal output, after taking logs, is

\[
y_{jt} = \frac{(1 + \theta \tau) \sigma_{\varepsilon}^2 + (1 - \theta \gamma)^2 \sigma_z^2}{\sigma_{\varepsilon}^2 + (1 - \theta \gamma)^2 \sigma_z^2} (\varepsilon_{jt} + (1 - \theta \gamma)y_t)
\]

Define \( \beta = (1 - \theta \gamma) < 1 \). Since in equilibrium \( C_{jt} = Y_{jt} \), integrating for market clearing and equating coefficients we get

\[
\sigma_z^2 = \frac{[(1 + \theta \tau) \beta - 1]}{\beta^2 (1 - \beta)} \sigma_{\varepsilon}^2
\]

For \( \sigma_z^2 \geq 0 \) we need \( (1 + \theta \tau) \beta > 1 \) so we must have \( \tau \neq 0 \).

If \( \beta > 0 \) (the case where firm output and aggregate output are complements) sentiment-driven equilibria exist if \( \tau > \frac{1 - \beta}{\beta \theta} \).

If \( \beta < 0 \) (the case where firm output and aggregate output are substitutes) we need \( \tau < \frac{1 - \beta}{\beta \theta} \) and \( \tau \) can be negative.
Sunspots can be idiosyncratic across consumers.

Price setting instead of quantity setting before demand is realized.

Persistence in output can be obtained with: a) Markov sunspots, b) Autocorrelated productivity shocks.
Conclusion

- When production decisions must be made under uncertain demand conditions, optimal decisions based on sentiments can generate self-fulfilling rational expectations equilibria in simple production economies without persistent informational frictions.