Tornell-Velasco

$$Max \int_0^\infty \left(\frac{\sigma}{\sigma - 1}\right) c_{it}^{\left(\frac{\sigma}{\sigma-1}\right)} e^{-\delta t} dt$$

ST:

$$\dot{k} = ak - z_{it} - (n-1)\beta k_t$$

$$\dot{f}_{it} = rf_{it} + z_{it} - c_{it}$$

$$\underline{\theta} k_t \leq z_{it} \leq \overline{\theta} k_t$$

$$\frac{a-r}{n} < \underline{\theta} < \frac{a-r}{n-1} < \overline{\theta}$$

$$H = \left(\frac{\sigma}{\sigma - 1}\right) c_{it}^{\left(\frac{\sigma}{\sigma-1}\right)} + \lambda[ak - z_{it} - (n-1)\beta k_t]$$

$$+ \phi[rf_{it} + z_{it} - c_{it}] + \underline{\mu}[z_{it} - \underline{\theta} k_t] - \overline{\mu}[\overline{\theta} k_t - z_{it}]$$

$$H = \left(\frac{\sigma}{\sigma - 1}\right) c_{it}^{\left(\frac{\sigma}{\sigma-1}\right)} + \lambda[ak - z_{it} - (n-1)\beta k_t] + \phi[rf_{it} + z_{it} - c_{it}] + \underline{\mu}[z_{it} - \underline{\theta} k_t] - \overline{\mu}[\overline{\theta} k_t - z_{it}]$$

FOC:

$$c_{it}^{-\frac{1}{\sigma}} = \phi$$

$$\underline{\mu} - \overline{\mu} = \lambda_t - \phi_t$$

$$\dot{\lambda} = \lambda(\delta - a - (n-1)\beta) + \underline{\mu}\,\underline{\theta} - \overline{\mu}\,\overline{\theta}$$

$$\dot{\phi} = \phi(\delta - r)$$

$$\lim_{t\to\infty} e^{-\delta t}\lambda_t k_t = 0$$

$$\lim_{t\to\infty} e^{-\delta t}\phi_t f_{it} = 0$$

Solving:

$$\phi(t) = \phi(0)e^{(\delta - r)t}$$

$$c_{it} = c_{i0}e^{\sigma(r-\delta)t}$$

Conjecture:

$$z_{it} = \beta k_t$$

$$\dot{k}_t = (a - \beta n) k_t$$

$$k_t = k_0 e^{(a-\beta n)t}$$

$$\dot{f}_{it} = r f_{it} + \beta k_0 e^{(a-\beta n)t} - c_{i0} e^{\sigma(r-\delta)t}$$

$$\dot{f}_{i0} = r f_{i0} + \beta k_0 - c_{i0}$$

$$f_{it} = \left(qk_0 - \frac{c_{i0}}{[r(1-\sigma)+\delta\sigma]} \right) e^{rt} - qk_0 e^{(a-\beta n)t}$$

$$+ \left(\frac{c_{i0}}{[r(1-\sigma)+\delta\sigma]} \right) e^{\sigma(r-\delta)t}$$

$$\dot{f}_{it} = r\left\{ qk_0 - \frac{c_{i0}}{r(1-\sigma)+\delta\sigma} \right\} e^{rt} - (an-\beta) qk_0 e^{(a-\beta n)t}$$

$$+ \sigma(r-\delta) \left(\frac{c_{i0}}{[r(1-\sigma)+\delta\sigma]} \right) e^{\sigma(r-\delta)t}$$

But also, plugging f_{it} in original diff. eq.

$$\dot{f}_{it} = r\left\{ \begin{array}{c} \left(qk_0 - \frac{c}{[r(1-\sigma)+\delta\sigma]} \right) e^{rt} \\ -qk_0 e^{(a-\beta n)t} + \left(\frac{c}{[r(1-\sigma)+\delta\sigma]} \right) e^{\sigma(r-\delta)t} \end{array} \right\}$$

$$+ \beta k_0 e^{(a-\beta n)t} - c_{i0} e^{\sigma(r-\delta)t}$$

Equating coefficients:

$$(a - \beta n) = r - \frac{\beta}{q}; \quad q = \frac{\beta}{r - a + \beta n}$$

$$\theta(r-\delta) = r - [r(1-\sigma)+\delta\sigma] = \theta(r-\delta)$$

Interior solution:

$$\underline{\mu} = \overline{\mu} = 0; \quad \phi_t = \lambda_t$$

$$\lambda_t = \lambda_0 e^{(\delta - a - (n-1)\beta)t} = \phi_0 e^{(\delta - r)t}$$

$$r = a - (n-1)\beta$$

$$\beta = \frac{a-r}{n-1}; \quad q = 1 : \quad r = a - \beta(n-1)$$

NOTE:

$$\frac{dr}{d\beta} = -(n-1) < 0$$

higher r, less appropriation in equilibrium.

Stable?

higher a, higher β

returns are being equated.

Corner:

$$z = \underline{\theta} k_t \quad or \quad z = \overline{\theta} k_t$$

Transversality:

Let

$$\lambda_t = q_t \phi_t$$

$$\dot{q} = \frac{\dot{\lambda}}{\lambda} - \frac{\dot{\phi}}{\phi}$$

$$= (\delta - a - (n-1)\beta) + \underline{\mu}\,\underline{\theta} - \overline{\mu}\,\overline{\theta} - (\delta - r)$$

$$= 0 \quad if \quad \underline{\mu} = \overline{\mu} = 0 \ (interior) : \quad q = 1$$

$$\lim_{t\to\infty} e^{-\delta t}\lambda_t k_t = \lim_{t\to\infty} e^{-\delta t}\phi_t q_t k_t$$

$$= \lim_{t\to\infty} e^{-\delta t}\phi_0 e^{(\delta - r)t} q_t k_0 e^{(\alpha - \beta n)t}$$

(using $r = a - (n-1)\beta$ from interiority)

$$= \lim_{t\to\infty} e^{-\delta t}\phi_0 q k_0 e^{(\delta - a + (n-1)\beta + a - \beta n)t}$$

$$= \lim_{t\to\infty} \phi_0 q k_0 e^{-(\beta)t} = \lim_{t\to\infty} \phi_0 q k_0 e^{-(\frac{a-r}{n-1})t}$$

$$More \quad generally$$

$$\lim_{t\to\infty} e^{-\delta t}\lambda_t k_t = \lim_{t\to\infty} e^{-\delta t}\phi_t q_t k_t$$

$$= \lim_{t\to\infty} e^{-\delta t}\phi_0 q k_0 e^{(\delta - r + a - \beta n)}$$

$$So \ a - \beta n < r; \qquad \frac{a-r}{n} < \beta$$

$$So \quad a > r$$

$\beta = \underline{\theta}$ or $\overline{\theta}$ means q_t can also be consant if

$$\dot{q} = r - a - (n-1)\underline{\theta} + \underline{\mu}\,\underline{\theta} \ : \quad now \ pick \ \underline{\mu}$$

Second Transversality:

$$\lim_{t\to\infty}\phi_0 e^{(\delta-r)t}e^{-\delta t}\left[\begin{array}{c}\left(qk_0-\frac{c}{[r(1-\sigma)+\delta\sigma]}\right)e^{rt}\\ -qk_0e^{(a-\beta n)t}+\left(\frac{c}{[r(1-\sigma)+\delta\sigma]}\right)e^{\sigma(r-\delta)t}\end{array}\right.$$

$$=\phi_0\left(qk_0-\frac{c}{[r(1-\sigma)+\delta\sigma]}\right)-qk_0e^{(a-\beta n-r)t}$$

$$+\left(\frac{c}{[r(1-\sigma)+\delta\sigma]}\right)e^{(\sigma(r-\delta)-r)t}$$

$$So\ :\ r(1-\sigma)+\delta\sigma>0;\qquad \frac{a-r}{n}<\underline{\theta}<\frac{a-r}{n-1}<\overline{\theta}$$

$$0=\left(qk_0-\frac{c}{[r(1-\sigma)+\delta\sigma]}\right)$$

$$\rightarrow c_{0i}=[r(1-\sigma)+\delta\sigma]qk_0$$