Survey of Consumer Finances, 2004

Graph showing the distribution of the ratio of individual wealth to aggregate wealth.
Pareto distribution:

\[
\frac{k}{x_m} \left( \frac{x}{x_m} \right)^{-k-1} = k(x_m)^k x^{-k-1}
\]
On old mechanisms possibly underlying a Pareto distribution of wealth (Cantelli-1921)

Suppose a variable determining wealth (e.g., talent, age), which we denote $\alpha$, is exponentially distributed.

$$N(\alpha) = pe^{-p\alpha}$$

Suppose wealth increases exponentially with $\alpha$:

$$w = e^{g\alpha}, \quad g \geq 0$$

Therefore, we can solve for $\alpha = g^{-1} \ln w$, operate a change of variables and express the distribution of wealth as

$$N(w) = N(g^{-1} \ln w) \frac{d\alpha}{dw}$$

that is,

$$N(w) = \frac{P}{g} w^{-\left(\frac{p}{g}+1\right)}$$

It is a Pareto distribution with exponent $\frac{p}{g}$.
A similar mechanism which makes wealth Pareto distributed is one in which the factor \( \alpha \) is represented by age. At any time \( t \), in this economy, the distribution of the population by age \( t - s \) implied by the demographic structure of the economy is in fact

\[
N(t - s) = pe^{-p(t-s)}
\]

Moreover, abstracting from the complications of inheritance, each optimal consumption-savings choices imply a wealth accumulation process results in wealth increasing exponentially with age.
Pareto in Blanchard’s Model

\[ \dot{w} = (r + p)w + y - c \]

\[ c(s, t) = (p + \theta)(w + h) \]

\[ h = \int_t^\infty y(s, v)e^{\int_t^v (r(u)+p)du} \ dv \]

\[ h = y \int_t^\infty e^{-(r+p)(v-t)} dv = -y(r + p)^{-1} e^{-(r+p)(v-t)} \bigg|_t^\infty \]

\[ = y(r + p)^{-1} \]

\[ \dot{w} = (r + p)w + y - (p + \theta)(w + h) \]

\[ = (r - \theta)w + y - (p + \theta)h \]

\[ = (r - \theta)w + y - (p + \theta)y(r + p)^{-1} \]

\[ = (r - \theta)w + y \left( \frac{r - \theta}{r + p} \right) \]

\[ w(t) = \left( w(0) + \frac{y}{r + p} \right) e^{(r-\theta)t} - \frac{y}{r + p} \]

If \( w(0) = 0, \) \( t \) is age:
\[ w(t) = \left(\frac{y}{r+p}\right)e^{(r-\theta)t} - \frac{y}{r+p} = \frac{y}{r+p} \left(e^{(r-\theta)t} - 1\right) \]

\[ w(t) + \frac{y}{r+p} = \left(\frac{y}{r+p}\right)e^{(r-\theta)t} \]

\[ \ln\left(\frac{w(t)(r+p)}{y} + 1\right) = (r-\theta)t \]

\[ t = \frac{\ln\left(\frac{(r+p)w}{y} + 1\right)}{r-\theta} \]

Transform variables:

\[ \frac{dt}{dw} = (r-\theta)^{-1}\left(\frac{(r+p)w}{y} + 1\right)^{-1} \frac{(r+p)}{y} \]

\[ N(t) = pe^{-pt}, \quad \int_0^\infty pe^{-pt} = 1 \]

\[ N(w) = N(t(w)) \frac{dt}{dw} = pe^{-p} \frac{\ln\left(\frac{(r+p)w}{y} + 1\right)}{r-\theta} \left[ \left(\frac{(r+p)w}{y} + 1\right)^{-1} \frac{(r+p)}{(r-\theta)y} \right] \]
\[ N(w) = p e^{\ln \left( \frac{(r+p)w}{y} + 1 \right)^{-\frac{p}{r-\theta}}} \cdot \left[ \left( \frac{(r+p)w}{y} + 1 \right)^{-1} \frac{(r+p)}{(r-\theta)y} \right] \]

\[ N(w) = p \left( \frac{(r+p)w}{y} + 1 \right)^{-\frac{p}{r-\theta}} \cdot \left[ \left( \frac{(r+p)w}{y} + 1 \right)^{-1} \frac{(r+p)}{(r-\theta)y} \right] \]
Pareto density in $W = \frac{(r+p)w}{y} + 1$:

$$N(w) = \left[ \frac{p(r+p)}{(r-\theta)y} \right] \left( \frac{(r+p)w}{y} + 1 \right)^{-p \frac{1}{r-\theta}}$$

$$N(0) = \left[ \frac{p(r+p)}{(r-\theta)y} \right]$$

Check that population integrates to 1:

$$\int_0^\infty N(w)dw$$

$$= \left[ \frac{p(r+p)}{(r-\theta)y} \right] \left[ \left( -\frac{r-\theta}{p} \right) \left( \frac{y}{r+p} \right) \left( \frac{(r+p)w}{y} + 1 \right) \right]$$

$$= \left[ \frac{p(r+p)}{(r-\theta)y} \right] \left( \frac{y}{r+p} \right) \left( \frac{r-\theta}{p} \right) = 1$$
Multiplicative Talent (Roy-1950)

Incomes $w$ are linearly dependent on talent $S$;

$$w = aS$$

Talent is the product of attributes, $s_i$ where each attribute is drawn from an iid distribution.

$$w = a s_1 s_2 \ldots s_n$$

$$\ln w = \ln a + \ln s_1 + \ln s_2 \ldots + \ln s_n$$

So incomes are log linear. You can also study skewness when the $s_i$ are not independent.
Kalecki (1945)

Idea: Kill random walk or variance exploding: make random return depend on firm size.

Let deviation of firm size, \( X_{t+1}^i = R_t^i X_t^i \) where \( R_t^i \) is a random variable. In logs

\[
\ln X_{t+1}^i = \ln X_t^i + \ln R_t^i = \ln X_0^i + \sum_{j=0}^{t} \ln R_j^i
\]

so \( \ln X_{t+1}^i \) is approximately normal. Assume variance of firm size remains constant.

\[
\frac{1}{n} \sum_i (\ln X_t^i + \ln R_t^i)^2 = \frac{1}{n} \sum_i (\ln X_t^i)^2 = M
\]

\[
2 \sum_i (\ln X_t^i \ln R_t^i) = - \sum_i (\ln R_t^i)^2
\]

and we may assume a negative linear relation
\[ \ln R_t = -\alpha \ln X_t + z, \]
\[ \alpha = \frac{\sum (\ln R^i_t)^2}{2 \sum (\ln X^i_t)^2}, \text{ } z \text{ iid} \]

\[ \sum_i (\ln X^i_t \ln R^i_t) = -\alpha \sum_i (\ln X^i_t)^2 \]

\[ = -\frac{\sum (\ln R^i_t)^2}{2 \sum (\ln X^i_t)^2} \sum_i (\ln X^i_t)^2 \]

\[ \sum_i (\ln X^i_t \ln R^i_t) = -\frac{1}{2} \sum (\ln R^i_t)^2 \]

Then the limiting distribution is normal:

\[ \ln X^i_{t+1} = (1 - \alpha) \ln X^i_{t+1} + z; \text{ } X_\infty = \alpha^{-1} z \]
Infinite horizon, progressive taxation
Small agents max CRRA utility, linear stochastic technology, \( r \) is log normal, take tax rate \( \tau \) as given as a function of time, save fraction \( (1 - \lambda_t) = \beta \frac{1}{\sigma} (r(1 - \tau_t))^{\frac{1-\sigma}{\sigma}} \). Take \( k_0 > 0 \). Taxes are progressive:
\[
\tau_t = 1 - (k_t)^{-a} \geq 0.
\]
\[
(1 - \lambda_t) = \beta \frac{1}{\sigma} (r(1 - \tau_t))^{\frac{1-\sigma}{\sigma}}
\]
\[
(1 - \lambda_t) = \beta \frac{1}{\sigma} (r(k_t)^{-a})^{\frac{1-\sigma}{\sigma}}
\]
Note: \( a \) has to be appropriately constrained so that \( 0 < \lambda_t < 1 \).
\[ \tau_t = 1 - (k_t)^{-a} \]

\[ k_{t+1} = \beta \frac{1}{\sigma} (r((k_t)^{-a})) \frac{1-\sigma}{\sigma} (1 - \tau_t)rk_t \]

\[ x_{t+1} = \ln (1 - \tau_t) + \ln \beta \frac{1}{\sigma} (r((k_t)^{-a})) \frac{1-\sigma}{\sigma} r + x_t \]

\[ x_{t+1} = \ln (k_t)^{-a} + \ln (r\beta) \frac{1}{\sigma} + \ln(k_t)^{-a} \left( \frac{1-\sigma}{\sigma} \right) + x_t \]

\[ x_{t+1} = \ln (r\beta) \frac{1}{\sigma} + (1 - a)x_t - a \left( \frac{1-\sigma}{\sigma} \right)x_t \]

\[ x_{t+1} = \ln (r\beta) \frac{1}{\sigma} + (1 - \frac{a}{\sigma})x_t \]

\[ x = \frac{\ln (r\beta) \frac{1}{\sigma}}{1 - a\sigma^{-1}} \]

Note: zero taxes, that is \( a = 0 \), makes \( x_t \) a random walk and variance blows up.
Mincer (1958)

Schooling increases income, but there is an opportunity cost in terms of lost time. Arbitrage implies where $s$ is years of schooling, $y$ is income without schooling, $\tilde{y}$ is income with schooling, $V$ is lifetime income to infinity.

\[
V = \frac{y}{1 + r} = e^{-rs} \frac{\tilde{y}}{1 + r}
\]

\[
\ln \tilde{y} = \ln y + rs
\]

If years of schooling is normally (?) distributed (truncated at zero?) (due to self-selection based on the distribution of ability) around a positive mean, log of incomes are normally distributed.
Distribution of profits with uniformly distributed talent-Span of Control

\( x \) is Talent, \( w \) is wage, \( n \) is labor (elastic), \( \pi \) is profit. Output is \( xn^a \)

\[
\begin{align*}
\max_x \ xn^a - wn \\
axn^{a-1} - w = 0 \\
n = \left( \frac{w}{ax} \right)^{\frac{1}{a-1}}
\end{align*}
\]

Profit

\[
\begin{align*}
\pi &= x \left( \frac{w}{ax} \right)^{\frac{a}{a-1}} - w \left( \frac{w}{ax} \right)^{\frac{1}{a-1}} \\
\pi &= Ax^{\frac{1}{1-a}}, \quad x = \left( \frac{\pi}{A} \right)^{1-\alpha}
\end{align*}
\]

Distribution of talent is uniform

\[
f(x) = b, \quad 0 \leq x \leq b^{-1}
\]

\[
f(x(\pi)) = f(x) \frac{dx}{d\pi} = (bA^{\alpha-1}(1 - \alpha))\pi^{-\alpha}
\]

So profits are a power law over

\[
0 \leq \left( \frac{\pi}{A} \right)^{1-\alpha} \leq b^{-1}, \text{ even if talent is uniform.}
\]

The stochastic multiplicative wealth accumulation process is given by

\[ W_i(t + 1) = \tilde{\lambda} W_i(t) \]

where \( W_i(t) \) is the wealth of investor \( i \) at time \( t \) and \( \tilde{\lambda} \) represents the total return, which is a random variable drawn from some distribution \( f(\lambda) \).
For people at the high-wealth range, changes in wealth are mainly due to financial investment, and are therefore typically multiplicative. For people at the lower wealth range, changes in wealth are mainly due to labor income and consumption, which are basically additive rather than multiplicative. Here we are only interested in modeling wealth dynamics in the high-wealth range.

There are many ways one could model the boundary between these two regions. We start by considering the most simple model in which there is a sharp boundary between the two regions.

As the stochastic multiplicative process describes the dynamics only at the higher wealth range, we introduce a threshold wealth level, $W_0$, above which the dynamics are multiplicative.

$$W_i(t) \geq W_0$$
A natural way to define the lower bound is in terms of the average wealth. We define the lower bound,

$$W_0 = \omega N^{-1} \sum_{i=1}^{N} W_i(t)$$

where $N$ is the number of investors and $\omega < 1$ is a threshold in absolute terms.

When individuals’ wealth changes they may cross the boundary between the upper and lower wealth regions. As we do not model the dynamics at the lower wealth range, and we assume that the market has reached an equilibrium in which the flow of people across the boundary is equal in both directions, i.e. the number of people participating at the process remains constant. The above assumption simplifies the analysis, but the results presented here are robust to the relaxation of this assumption.
Master equation:

\[ P(W_{t+1}) = P(W_t) + \int_{-\infty}^{\infty} P(W_t/\lambda)f(\lambda)d\lambda \]

\[ -\int_{-\infty}^{\infty} P(W_t)f(\lambda)d\lambda \]

But \( \int_{-\infty}^{\infty} P(W_t)f(\lambda)d\lambda = P(W_t) \) so

\[ P(W_{t+1}) = \int_{-\infty}^{\infty} P(W_t/\lambda)f(\lambda)d\lambda \]

\[ P(W) = CW^{-\alpha-1} \] as a stationary distribution:

\[ CW^{-\alpha-1} = \int_{-\infty}^{\infty} CW^{-\alpha-1}\lambda^{\alpha+1}f(\lambda)d\lambda \]

where \( \alpha \) solves

\[ 1 = \int_{-\infty}^{\infty} \lambda^{\alpha+1}f(\lambda)d\lambda \]

Note the problem with the lower bound. In range \([W_0, \lambda W_0)\), \( W/\lambda < W_0 \) for \( \lambda > 1 \) in the support of \( f(\lambda) \). So if \( \lambda_M = \text{Max} \ \lambda \) in the support of \( f \), Pareto works for \([\lambda_M W_0, \infty)\)?

What’s the economics? Notice that we must have \( W_0 > 0 \) for the distribution to be well defined. What if \( f \) has infinite support?