The distribution of wealth in the Blanchard-Yaari model *

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Abstract

We study the dynamics of the distribution of wealth in an economy with infinitely lived agents, inter-generational transmission of wealth, and redistributive fiscal policy. We show that wealth accumulation with idiosyncratic investment risk and uncertain lifetimes can generate a double Pareto wealth distribution.

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1 Introduction

The wealth distribution in the United States has a fat tail. Wolff (2006), using the 2001 Survey of Consumer Finances, finds that the top 1% of households hold 33.4% of the wealth in the United States. Investigating a sample of the richest individuals in the United States, the Forbes 400 data during 1988-2003, Klass et al. (2006) find that the top end of the wealth distribution obeys a Pareto law with an average exponent of 1.49.

In this paper we study a model of wealth accumulation with idiosyncratic investment risk and uncertain lifetime and show that it can generate a double Pareto wealth distribution displaying a Pareto upper tail.[1]

Our model is a continuous time OLG heterogeneous agents model. There is a continuum of agents with measure 1 in the economy. Agents have uncertain lifetimes with constant probability of death at each point. The agents have "joy of giving" bequest motives and allocate their wealth among current consumption, a risky asset, a riskless asset, and the purchase of life insurance. The risky asset is a private investment project whose value follows a geometric Brownian motion. Returns of the private investment projects are subject to idiosyncratic risk. The returns from riskless assets and life insurance are the same for all agents. The government taxes capital income and redistributes the proceeds as means-tested subsidies.

The agent's optimal wealth accumulation process follows a geometric Brownian mo-

[1]A double Pareto distribution is a distribution exhibiting power-law behaviour in both tails; the name is due to William Reed; see Reed (2001).

tion and we can calculate the growth rate of aggregate wealth. The ratio of individual wealth to aggregate wealth also follows a geometric Brownian motion. The combination of the geometric Brownian motion for accumulation, a means-tested government subsidy policy, and an exponentially distributed age profile (induced by the constant death rate) leads to a stationary distribution for the ratio of individual wealth to aggregate wealth which is a double Pareto distribution.

Our analysis is related to Benhabib, Bisin, and Zhu (2011) and Zhu (2010), who exploit idiosyncratic investment risk to generate fat-tailed wealth distributions in an OLG model where agents certain finite lifetimes. With respect to previous literature, we provide closed-form solutions for the stationary distribution of wealth, rather than using simulation methods as, e.g., in Aiyagari (1994) and Castaneda et al. (2003).

The rest of this paper is organized as follows. Section 2 contains a continuous time OLG heterogeneous agents model with investment risk and lifetime uncertainty. In section 3 we characterize the wealth distribution of this economy. Section 4 discusses an alternative government policy for redistribution.

2 An OLG economy with capital risk and bequests

The economy we study is an extension of Yaari (1965) and Blachard (1985), in which lives are finite and end probabilistically. Blanchard (1985) and Yaari (1965) study only aggregate variables, but not the distribution of wealth. Richard (1975) studies consumption choice and portfolio selection in the same environment with a risky asset, also without characterizing the distribution of wealth.

We study the distribution of wealth in a model with uncertain lifetimes and a constant probability of death, where agents have a portfolio choice of risky and riskless assets.The duration of an agent's life is uncertain. Death is governed by a Poisson distribution with rate p.[2] Consequently, the density function of death at any time $t \in [0, +\infty)$ is $\pi(t) = pe^{-pt}$. When the agent dies, the agent's child is born. Each agent has one child.

Let $W(s,t)$ be the wealth at time t of an agent born at time $s \leq t$. An agent allocates individual wealth among current consumption, a risky asset, a riskless asset, and the purchase of life insurance.

Risky asset. The risky asset is the source of idiosyncratic capital income risk, e.g., household-owned housing risk and private business risk. We assume that every agent invests his/her wealth in his/her own risky asset. Risk sharing on the return of the risky asset is not possible. The stochastic processes for the agents' idiosyncratic risk are independent, but they follow the same process. For an agent born at time s, the value of the idiosyncratic risky asset at time $t \geq s$, $S(s,t)$, follows a geometric Brownian motion:

$$dS(s,t) = \alpha S(s,t)dt + \sigma S(s,t)dB(s,t)$$

where $B(s,t)$ is a standard Brownian motion, α is the instantaneous conditional ex-

[2] An agent alive at t dies with probability $p\Delta t$ in the time interval $(t, t + \Delta t)$.

pected percentage change in value per unit of time and σ is the instantaneous conditional standard deviation per unit of time. The value of the risky asset is then log-normally distributed and its rate of return is independent of its value.

Riskless asset. The value of the riskless asset, $Q(t)$, grows exponentially,

$$dQ(t) = rQ(t)dt$$

where r is the rate of return. Consistently with non-arbitrage, we assume $r < \alpha$.

Life insurance. For a price μ an agent buys life insurance, that is, the right to bequeath to his/her child $\frac{P(s,t)}{\mu}$ if he/she dies at time t. Negative life insurance should be interpreted as an annuity. Life insurance companies are assumed to earn zero profits, and hence $\mu = p$.Let $Z(s,t)$ denote the bequest that an agent born at time s would leave at death at time t. Then[3]

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{p}$$

2.1 Individual wealth accumulation

Agents derive utility from consumption, while alive, and also have a bequest motive of the "joy of giving" form: bequests enter directly the parents' utility function. Both the consumption and the bequest utility indices are assumed CRRA. Let θ denote the time discount rate and χ the strength of the bequest motive. Let $C(s,t)$ denote consumption

[3] In the presence of perfect life insurance markets there are no accidental bequests.

at time t of an agent born at time s and $\omega(s,t)$ the share of wealth the agent invests in the risky asset at the same time.

The agent's utility maximization problem is:

$$\max_{C,\ \omega,\ P} E_t \int_t^{+\infty} e^{-(\theta+p)(v-t)} \left(\frac{(C(s,v))^{1-\gamma}}{1-\gamma} + p\chi \frac{(Z(s,v))^{1-\gamma}}{1-\gamma} \right) dv \tag{1}$$

subject to

$$\begin{aligned} dW(s,v) &= (rW(s,v) + (\alpha - r)\omega(s,v)W(s,v) - C(s,v) - P(s,v))\, dv \\ &\quad + \sigma\omega(s,v)W(s,v)dB(s,v) \end{aligned} \tag{2}$$

and the transversality condition.[4,5]

Proposition 1 *The agent's optimal policies are characterized by*

$$\begin{aligned} C(s,t) &= A^{-\frac{1}{\gamma}} W(s,t) \\ \omega(s,t) &= \frac{\alpha - r}{\gamma\sigma^2} \\ Z(s,t) &= \rho W(s,t) \end{aligned}$$

[4]The transversality condition is

$$\lim_{t \to +\infty} e^{-(\theta+p)(t-s)} E\left[J(W(s,t))\right] = 0$$

where $J(W(s,t))$ is the agent's optimal value and $E[\cdot]$ the expectation operator; see Merton (1992).

[5]For a generalization to bequest functions for altruistic agents facing redistributive policies see Appendix A of Benhabib and Bisin (2007) or Footnote 15 in Benhabib and Zhu (2008). Also, Section 5 in Benhabib and Bisin (2007) shows that the Blanchard model can be mapped into a dynastic model with Poisson shocks to individual wealth.

with $A = \left(\frac{\theta+p-(1-\gamma)\left(r+p+\frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)}{\gamma\left(1+p\chi^{\frac{1}{\gamma}}\right)}\right)^{-\gamma}$ *and* $\rho = (\frac{\chi}{A})^{\frac{1}{\gamma}}$*. Furthermore,*

$$dW(s,t) = gW(s,t)dt + \kappa W(s,t)dB(s,t). \tag{3}$$

with $g = \frac{r-\theta}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}$ *and* $\kappa = \frac{\alpha-r}{\gamma\sigma}$*.*

Several properties of the solution, using the CRRA form for the utility function and the complete insurance markets, deserve notice. First of all, the mean growth rate of the agent's wealth, $g = \frac{r-\theta}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}$, is independent of the bequest parameter χ. Also, the share of risky asset, $\omega(s,t) = \frac{\alpha-r}{\gamma\sigma^2}$, is only influenced by the risk premium of the risky asset, the degree of risk aversion, and the volatility of the return of the risky asset. The volatility of the growth rate of the agent's wealth, $\kappa = \frac{\alpha-r}{\gamma\sigma}$, does not depend on the bequest motive parameter, χ, but is negatively related to the standard deviation of the price of the risky asset, σ.

Equation (3) means that individual wealth follows a geometric Brownian motion.[6] Note that $dB(s,t)$ represents a positive shock to the return of the risky asset. It is these shocks, as well as mortality, that induce wealth inequality in our economy.

[6]The growth rate is independent of wealth and so individual wealth follows Gibrat's law.

2.2 Redistribution policies

At time t the size of cohort born at time s is $pe^{p(s-t)}$. The mean wealth of cohort born at time s is denoted by $E_s W(s,t)$, where the expectation is calculated with respect to the cross-section wealth distribution of agents born at time s who are still alive at time t. $E_s W(s,t)$ grows at a rate of g,[7]

$$E_s W(s,t) = E_s W(s,s) e^{g(t-s)}. \tag{4}$$

Aggregate wealth $W(t)$ can be calculated as[8]

$$W(t) = \int_{-\infty}^{t} E_s W(s,t) p e^{p(s-t)} ds. \tag{5}$$

We assume that government subsidies are distributed so as to guarantee all newborns a threshold level of initial wealth proportional to aggregate wealth. Any newborn receiving an inheritance $Z(s,t) = \rho W(s,t)$ at time t higher than the threshold level does not obtain any subsidy. Let $x^* W(t)$ be such a threshold level.

Government subsidies are financed by a capital income tax. The interest rate on the riskless asset, r, is net of the tax rate τ: $r = \tilde{r} - \tau$, where $\tilde{r}$ is the before-tax interest rate on the riskless asset. The mean return on risky asset, α is also net of the tax rate τ:

[7]This follows easily from the fact that the stochastic return shock is idiosyncratic and the stochastic growth rate is independent of the individual wealth level. We thank Zheng Yang for this point.

[8]The aggregate wealth equals the mean wealth in our model since the measure of agents is 1.

$\alpha = \tilde{\alpha} - \tau$, where $\tilde{\alpha}$ is the before-tax mean return on risky asset. The government collects capital income taxes and pays subsidies to newborns. A balanced budget is maintained at all times. Thus

$$\tau W(t) = p \int_0^{\frac{x^*}{\rho} W(t)} (x^* W(t) - \rho W) h(W, t) dW$$

where $h(W, t)$ is the wealth distribution at time t.

2.3 Aggregate wealth

The aggregate wealth growth equation is derived from equations (5) and (4)

$$\dot{W}(t) = gW(t) - pW(t) + pE_t W(t, t)$$

The first term of $gW(t)$, is due to individual wealth growth. The second term of $-pW(t)$ is due to death. And the third term of $pE_t W(t, t)$ is the reinjection of wealth through the starting wealth of newborns.

Aggregate starting wealth of the newborns at time t, $pE_t W(t, t)$, is the sum of private bequest and a public subsidy. By Proposition 1, private bequests are, in the aggregate, $p\rho W(t)$. And from section 2.2, aggregate subsidies are equal to total tax revenue $\tau W(t)$. Then

$$pE_t W(t, t) = (p\rho + \tau) W(t).$$

Thus the aggregate wealth has a grow rate $\tilde{g}$:

$$dW(t) = \tilde{g}W(t)dt = (g + p\rho + \tau - p)\,W(t)dt. \tag{6}$$

3 The distribution of wealth

We now investigate the cross-sectional distribution of wealth in our economy. It is in fact convenient to study the ratio of individual to aggregate wealth,

$$X(s,t) = \frac{W(s,t)}{W(t)} \tag{7}$$

which displays a stationary distribution.[9] It is straightforward to see that $X(s,t)$ also follows a geometric Brownian motion,[10]

$$dX(s,t) = (g - \tilde{g})X(s,t)dt + \kappa X(s,t)dB(s,t).$$

To investigate the cross-sectional distribution of $X(s,t)$, we need to know not only the evolution function of $X(s,t)$ during an agent's lifetime, but also the change of $X(s,t)$

[9] Note that, under our normalization that total population is 1, $X(s,t)$ represents also the ratio of individual to mean wealth.

[10] Then $\frac{X(s,t)}{X(s,s)}$ is lognormally distributed, and

$$X(s,t) = X(s,s)\exp\left(\left(g - \tilde{g} - \frac{1}{2}\kappa^2\right)(t-s) + \kappa(B(s,t) - B(s,s))\right).$$

between two consecutive generations. The change of $X(s,t)$ between two consecutive generations reflects the role of inheritance and subsidies. Let the cross-sectional distribution of $X(\cdot,t)$ at time t, be denoted by $f(x,t)$. In Section 6.2 of Appendix we derive the forward Kolmogorov equation for $f(x,t)$:

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t) + pf(\frac{x}{\rho},t)\frac{1}{\rho}, \quad x > x^* \quad (8)$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t), \qquad x < x^*. \quad (9)$$

Partial differential equations (8) and (9) do not hold at $x = x^*$.[11] But $f(x^*,t)$ is determined by the boundary conditions

$$\int_0^{+\infty} f(x,t)dx = 1, \text{ and } \int_0^{+\infty} xf(x,t)dx = 1, \ \forall t \geq 0.$$

In turn, x^* is determined by government budget balance, given the capital income tax rate τ:

$$\tau = p\int_0^{\frac{x^*}{\rho}} (x^* - \rho x)f(x,t)dx.$$

It is difficult to solve the partial differential equations with an arbitrary initial dis-

[11]For this point we greatly benefited from discussions with Matthias Kredler and Professor Henry P. McKean.

tribution. Instead, we investigate the behavior of the equations in the long run, the stationary wealth distribution. In a stationary distribution, we have $\frac{\partial f(x,t)}{\partial t} = 0$. A stationary distribution $f(x)$ satisfies then the following ordinary differential equations:

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))xf'(x) + (\kappa^2 - (g - \tilde{g}) - p)f(x) + pf(\frac{x}{\rho})\frac{1}{\rho} = 0, \ x > x^* \quad (10)$$

$$\frac{1}{2}\kappa^2 x^2 f''(x) + (2\kappa^2 - (g - \tilde{g}))xf'(x) + (\kappa^2 - (g - \tilde{g}) - p)f(x) = 0, \ x < x^*, \quad (11)$$

as well as the boundary conditions

$$\int_0^{+\infty} f(x)dx = 1, \text{ and } \int_0^{+\infty} xf(x)dx = 1.$$

We are now ready to state the main result of this paper.

Proposition 2 *The stationary distribution $f(x)$ has the following form*

$$f(x) = \begin{cases} C_1 x^{-\beta_1} & \textit{when } x \leq x^* \\ C_2 x^{-\beta_2} & \textit{when } x \geq x^* \end{cases} \quad (12)$$

with $C_1 = \left(1 - \frac{1-\beta_2}{2-\beta_2}x^*\right)(x^*)^{\beta_1 - 2}\frac{(2-\beta_1)(2-\beta_2)(1-\beta_1)}{\beta_2 - \beta_1}$
and $C_2 = \left(1 - \frac{1-\beta_1}{2-\beta_1}x^*\right)(x^*)^{\beta_2 - 2}\frac{(2-\beta_1)(2-\beta_2)(1-\beta_2)}{\beta_2 - \beta_1}$.

Furthermore, $\beta_1 < 1$ is the smaller root of the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - \left(\frac{3}{2}\kappa^2 - (g - \tilde{g})\right)\beta + \kappa^2 - p - (g - \tilde{g}) = 0, \tag{13}$$

$\beta_2 > 2$ is the larger root of the characteristic equation

$$\frac{\kappa^2}{2}\beta^2 - \left(\frac{3}{2}\kappa^2 - (g - \tilde{g})\right)\beta + \kappa^2 - p - (g - \tilde{g}) + p\rho^{\beta-1} = 0, \tag{14}$$

and

$$x^* = \frac{\frac{\tau}{p} + \rho - \frac{2-\beta_1}{\beta_2-\beta_1}\rho^{\beta_2-1}}{1 - \frac{1-\beta_1}{\beta_2-\beta_1}\rho^{\beta_2-1}}. \tag{15}$$

The distribution $f(x)$ is a double Pareto distribution. The parameter β_2 controls the tail of this distribution: the smaller is β_2, the fatter is the tail. The integrability of $f(x)$ and $xf(x)$ on $(0, +\infty)$ is implied by $\beta_1 < 1$ and $\beta_2 > 2$. This assures that $f(x)$ is a distribution function with a finite mean, but its variance does not necessarily exist.

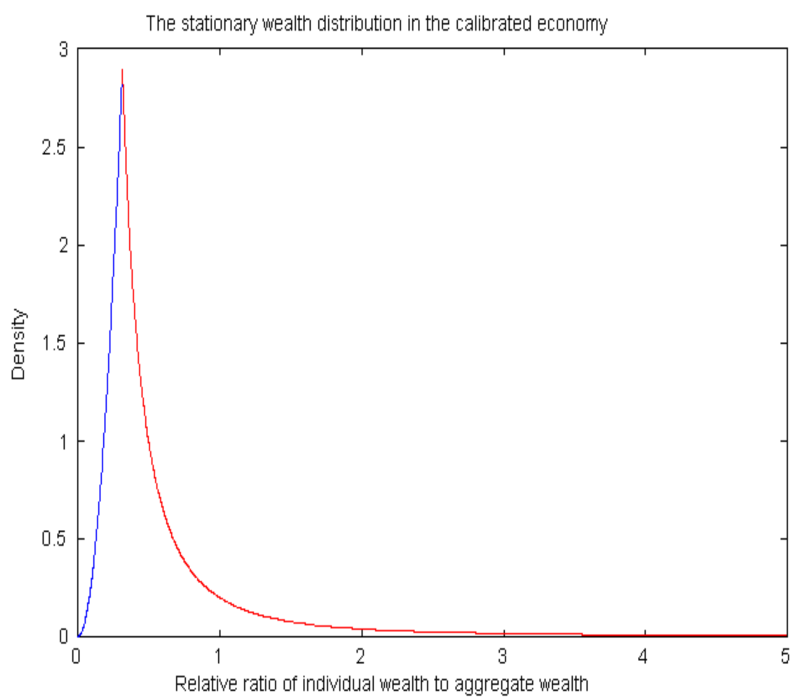
Finally, we show that $f(x)$ in equation (12) is the unique stationary distribution and starting from any initial distribution, $f(x, 0)$, the stochastic process $X(\cdot, t)$ converges to the stationary distribution.

Proposition 3 *The stochastic process, $X(\cdot, t)$, is ergodic.*

Even though β_1 has a closed form solution, β_2 does not, generally.[12] We can however

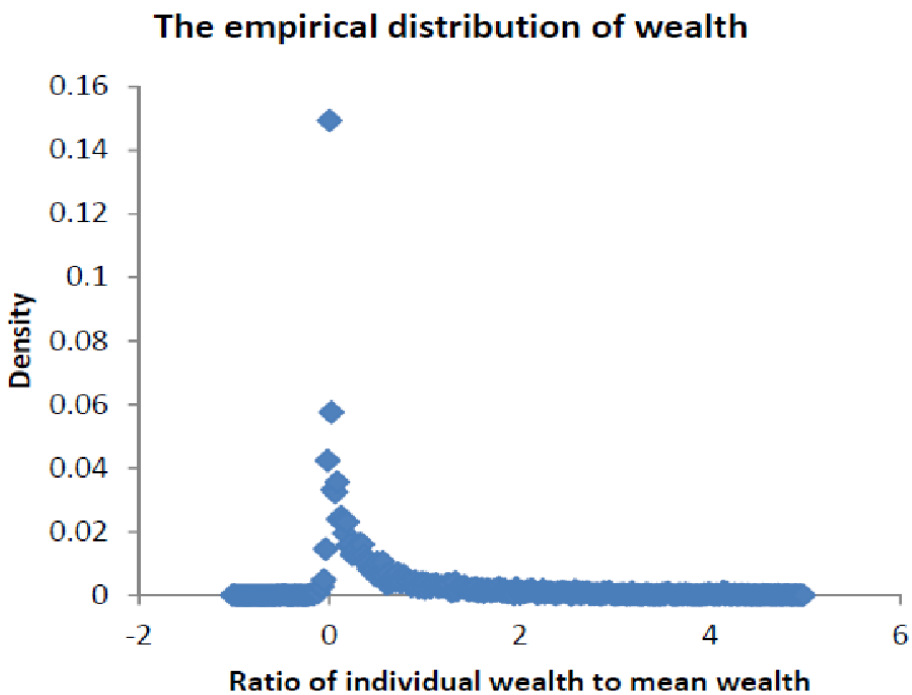
[12] A closed-form solution for β_2 can be obtained, however, when there is no bequest motive, i.e. $\chi = 0$ (note that $\chi = 0$ implies $\rho = 0$). The mathematical result of the double Pareto distribution in this special case is also in Reed (2001), though in a different economic environment.

solve equation (14) numerically. A calibration exercise of this economy is performed in Benhabib and Zhu (2008) with the following parameters: $\theta = 0.03$, $\chi = 15$, $\sigma = 0.26$, $\gamma = 3$, $r = 1.8\%$, $\alpha = 8.8\%$, $p = 0.016$, $q = 0.012$, and $\tau = 0.004$. In that parametrization two more parameters are introduced, q and ζ, which, for simplicity, we omit in the model above. These two additional features represent the fraction of agents having a bequest motive, $\frac{q}{p} = 0.75$, and a calibrated estate tax rate $\zeta = 0.19$. The estate taxes then add to government revenues, and are incorporated into the balanced budget each period. The stationary wealth distribution $f(x)$ for this calibration is:[13]



[13]The horizontal axis represents the ratio of individual wealth to aggregate wealth, or mean wealth in our normalization.

We can compare this to the wealth distribution for the U.S., using data from the 2004 Survey of Consumer Finances:[14]



Benhabib and Zhu (2008) also investigate the impact of fiscal policy on wealth inequality. Their calibration exercise numerically shows that a higher capital income tax rate implies a lower Gini coefficient of the wealth distribution and a higher estate tax rate implies a lower Gini coefficient: redistribution policies tend to reduce wealth inequality.

[14]Note that the densities on the vertical axis are not directly comparable across figures as the calibrated distribution is obtained from the continuous model while the distribution in the data is naturally obtained from discrete bins. Furthermore, note that in the U.S. data a small fraction of agents have negative physical wealth, while in our simulated data wealth is positive for all agents. In Benhabib and Zhu (2008) the wealth of agents is defined to include discounted future labor earnings (human capital) as well as physical wealth, which can be a factor that makes aggregate wealth non-negative.

3.1 No investment risk

The special case of our economy in which agents do not face any investment risk is worth considering explicitly because it can be solved more completely. In this economy, an agent allocates individual wealth among current consumption, a riskless asset, and the purchase of life insurance. The only risk that agents face is mortality. For simplicity, we also assume in the following analysis that utility indices are logarithmic, that is, $\gamma = 1$. Each agent's optimal policies are easily characterized and induce the following equation for the dynamics of individual wealth:

$$dW(s,t) = gW(s,t)dt$$

with $g = r - \theta$. The aggregate growth rate of wealth is $\tilde{g} = g + p\rho + \tau - p$. The stochastic process $X(s,t)$ then follows

$$dX(s,t) = (g - \tilde{g})X(s,t)dt.$$

Assuming $g - \tilde{g} > 0$, we proceed to obtain the forward Kolmogorov equation for the distribution $f(x,t)$ and show that a stationary distribution $f(x)$ satisfies:

$$-(p + g - \tilde{g}) f(x) + pf\left(\frac{x}{\rho}\right)\frac{1}{\rho} - (g - \tilde{g}) x \frac{df(x)}{dx} = 0. \tag{16}$$

We proceed then by guessing a Pareto distribution for $f(x)$:

$$f(x) = (\lambda - 1)(x^*)^{\lambda - 1} x^{-\lambda} \tag{17}$$

where $\lambda > 1$. Plugging equation (17) into equation (16), we find that λ must solve

$$(g - \tilde{g})\lambda - (p + g - \tilde{g}) + p(\rho)^{\lambda - 1} = 0. \tag{18}$$

Benhabib and Bisin (2007) show then that this economy has a unique stationary distribution $f(x)$ which is Pareto with λ corresponding to the (unique) root of (18) which is greater than 1, and with $x^* = \frac{\frac{\tau}{p}}{1 - \frac{\lambda-1}{\lambda-2}\rho + \frac{1}{\lambda-2}(\rho)^{\lambda-1}}$.[15] Furthermore, Benhabib and Bisin (2007) show that the stochastic process $X(\cdot, t)$ is ergodic.

Consider now the case in which agents have no preferences for bequests: $\chi = 0$ and hence $\rho = 0$. In this case equation (18) has a closed form solution, $\lambda = \frac{p}{p-\tau} + 1$; and the stationary distribution of wealth is

$$f(x) = \frac{p}{p - \tau}\left(\frac{\tau}{p}\right)^{\frac{p}{p-\tau}} x^{-\left(\frac{p}{p-\tau}+1\right)}.$$

Furthermore, in this case, Benhabib and Bisin (2007) show that the Kolmogorov equation implies that, for any initial distribution of x, $h(x) = f(x, 0)$, the distribution $f(x, t)$ is a

[15] A different method to obtain the Pareto distribution, via a "change of variable," can also be applied to this economy and dates back to Cantelli (1921).

truncated Pareto distribution in the range $\left(x^*, x^* e^{(p-\tau)t}\right)$:

$$f(x,t) = \begin{cases} \frac{p}{p-\tau}\, \underline{w}^{\frac{p}{p-\tau}}\; z^{-\left(\frac{p}{p-\tau}+1\right)} & \text{for } x \in \left(x^*, x^* e^{(p-\tau)t}\right) \\ e^{-(p+p-\tau)t} h\left(x e^{-(p-\tau)}\right) & \text{for } x \geq x^* e^{(p-\tau)t}. \end{cases}$$

In the economy with no investment risk, the stationary wealth distribution is a Pareto distribution, not a double Pareto distribution as in our general economy. It is the negative shocks to investment returns that send wealth below the threshold for a fraction of the agents in the economy, generating the left Pareto tail. With no investment risk, individual wealth increases for all agents while alive, so that the ratio of individual wealth to aggregate wealth is greater than x^* for any agent.

4 A lump-sum redistribution policy

In this section we discuss the effects on the wealth distribution of an alternative redistribution policy, lump-sum redistribution, under which all newborns receive the same subsidy:[16] the government collects $\tau W(t)$ of capital income tax and each newborn at t receives the same subsidy, $b(t) = \frac{\tau}{p} W(t)$. The individual wealth accumulation equation and the growth of aggregate wealth are unchanged under lump-sum redistribution, but

[16] See Huggett (1996) for a calibrated model where accidental bequests are distributed equally to everyone, not just newborns. De Nardi (2004) also has some specifications of calibrated models where accidental bequests are equally distributed to the population.

the stationary distribution $f(x)$ must satisfy the following Kolmogorov equation

$$\frac{1}{2}\kappa^2 x^2 f''(x) + \left(2\kappa^2 - (g - \tilde{g})\right) x f'(x) + \left(\kappa^2 - (g - \tilde{g}) - p\right) f(x)$$
$$+ pf\left(\frac{x - x^*}{\rho}\right)\frac{1}{\rho} = 0 \quad \text{when } x > \frac{x^*}{1-\rho} \tag{19}$$

Equation (19) differs from equation (10) of Section 3 only in that the last term is $pf\left(\frac{x-x^*}{\rho}\right)\frac{1}{\rho}$ rather than $pf\left(\frac{x}{\rho}\right)\frac{1}{\rho}$. However, for large x, the influence of the shift term, x^*, can be ignored and the stationary wealth distribution under lump-sum redistribution has an asymptotic Pareto tail, which admits the same Pareto exponent as in the distribution associated with the means-tested redistribution used in in Proposition 2. We summarize this result as follows.[17]

Proposition 4 *The stationary distribution is $f(x) \sim x^{-\beta_2}$ as $x \to +\infty$, where β_2 is the larger solution of the characteristic equation*

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g - \tilde{g}))\beta + \kappa^2 - p - (g - \tilde{g}) + p\rho^{\beta-1} = 0.$$

[17] A formal proof using the theory of Kesten processes is available from the authors on request. Note that in this economy, the government subsidy plays the role of reflecting barrier which pushes the wealth accumulation process away from zero. In Benhabib, Bisin, and Zhu (2011) and Zhu (2010) it is instead labor income that operates as a reflecting barrier.

5 Conclusion

We set up a continuous time OLG heterogeneous agents model to show that investment risk and uncertain lifetimes, plus a specific government subsidy policy, can generate a double Pareto wealth distribution.[18] We also show that uncertain lifetimes and the specific government subsidy policies can technically generate a Pareto wealth distribution. Finally, we show that our model is robust to the government subsidy policy. An alternative government policy, that is a lump-sump subsidy policy, can produce an asymptotic Pareto tail which admits the same Pareto exponent as in our benchmark model.

References

Aiyagari, S. R. (1994), "Uninsured idiosyncratic risk and aggregate saving," *Quarterly Journal of Economics*, 109, 659-684.

Benhabib, J. and A. Bisin, (2007), "The distribution of wealth: Intergenerational transmission and redistributive policies," Mimeo, NYU.

Benhabib, J. and S. Zhu, (2008), "Age, luck and inheritance," NBER working paper 14128.

Benhabib, J., A. Bisin, and S. Zhu (2011), "The distribution of wealth and fiscal policy in economies with finitely lived agents," *Econometrica*, 79, 123-157.

[18] Government subsidies essentially play the role of a reflecting barrier for the inheritance process. Without government subsidies, we are not able to obtain a non-tivial stationary distribution of wealth. Thus death also plays an important role since government subsidies only occur when an agent dies.

Blanchard, O. J. (1985), "Debt, deficits, and finite horizons," *Journal of Political Economy*, 93, 223-247.

Cantelli, F. P. (1921), "Sulla Deduzione delle Leggi di Frequenza da Considerazioni di Probabilita'," *Metron*, 83-91.

Castaneda, A., J. Diaz-Gimenza, and J. Rios-Rull (2003), "Accounting for the U.S. earnings and wealth inequality," *Journal of Political Economy*, 111, 818-857.

De Nardi, M. (2004), "Wealth inequality and intergenerational links," *Review of Economic Studies*, 71, 743-768.

Huggett, M. (1996), "Wealth distribution in life-cycle economies:, *Journal of Monetary Economics*, 38, 469-494.

Kamien, M. and Schwartz, N. (1991), *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, Second Edition, New York: Elsevier Science Publishing Co.

Karlin, S. and H. Taylor (1981), *A Second Course in Stochastic Processes*, New York: Academic Press.

Klass, O.S., O. Biham, M. Levy, O. Malcai, and S. Solomon (2006), "The Forbes 400 and the Pareto wealth distribution," *Economics Letters*, 90, 290-295.

Merton, R. (1992), *Continuous-Time Finance*, Cambridge, MA: Blackwell Publishers.

Meyn, S. P. and R. L. Tweedie (1993), *Markov Chains and Stochastic Stability*, Berlin: Springer.

Reed, W. (2001), "The Pareto, Zipf and other power laws", *Economics Letters*, 74, 15-19.

Richard, S. (1975), "Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model", *Journal of Financial Economics*, 2, 187-203.

Ross, S. M. (1996), *Stochastic Processes*, Second Edition, New York: John Wiley & Sons, Inc.

Yaari, M. (1965), "Uncertain lifetime, life insurance, and the theory of the consumer," *Review of Economic Studies*, 32, 137-150.

Wolff, E. N. (2006), "Changes in household wealth in the 1980s and 1990s in the United States," In E. N. Wolff ed. *International Perspectives on Household Wealth*, Cheltenham, UK: Edward Elgar.

Zhu, S. (2010), "Wealth distribution under idiosyncratic investment risk," Mimeo, NYU.

6 Appendix

We collect most of the technical proofs here.

6.1 Proof of Proposition 1

Proof: Let $J(W(s,t))$ be the optimal value function of the agent with wealth $W(s,t)$. Following Merton (1992) and Kamien and Schwartz (1991), we set up the Hamilton-Jacobi-Bellman equation of the maximization problem

$$\begin{aligned} &(\theta + p)J(W(s,t)) \\ = &\max_{C,\omega,P} \begin{pmatrix} \frac{(C(s,v))^{1-\gamma}}{1-\gamma} + p\chi\frac{(Z(s,v))^{1-\gamma}}{1-\gamma} \\ +J_W(W(s,t))\left(rW(s,t) + (\alpha - r)\omega(s,t)W(s,t) - C(s,t) - P(s,t)\right) \\ +\frac{1}{2}J_{WW}(W(s,t))\sigma^2\omega^2(s,t)W^2(s,t) \end{pmatrix} \end{aligned}$$

Using the relationship

$$Z(s,t) = W(s,t) + \frac{P(s,t)}{p}$$

we find the first order conditions:

$$C(s,t)^{-\gamma} = J_W$$

$$\chi Z(s,t)^{-\gamma} = J_W$$

and

$$(\alpha - r)J_W W(s,t) = -J_{WW}\sigma^2\omega(s,t)W^2(s,t).$$

We guess the value function

$$J(W(s,t)) = \frac{A}{1-\gamma}W(s,t)^{1-\gamma}$$

where A is an undetermined constant. Then we find the expressions of $C(s,t)$, $Z(s,t)$, $P(s,t)$, and $\omega(s,t)$ from the first order conditions

$$C(s,t) = A^{-\frac{1}{\gamma}}W(s,t)$$

$$Z(s,t) = \left(\frac{\chi}{A}\right)^{\frac{1}{\gamma}} W(s,t)$$

$$P(s,t) = \left(p\left(\frac{\chi}{A}\right)^{\frac{1}{\gamma}} - p\right) W(s,t)$$

$$\omega(s,t) = \frac{\alpha - r}{\gamma\sigma^2}.$$

Plugging these equations into the Hamilton-Jacobi-Bellman equation, we can determine the constant A

$$A = \left(\frac{\theta + p - (1-\gamma)\left(r + p + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)}{\gamma\left(1 + p\chi^{\frac{1}{\gamma}}\right)}\right)^{-\gamma}.$$

From the budget constraint we obtain the wealth accumulation equation

$$dW(s,t) = \left(\frac{r-\theta}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(\alpha-r)^2}{\gamma\sigma^2}\right)W(s,t)dt + \frac{\alpha-r}{\gamma\sigma}W(s,t)dB(s,t). \blacksquare$$

6.2 Derivation of the forward Kolmogorov equations

Following Ross (1983), we heuristically derive the forward Kolmogorov equations (8) and (9).

Let $f(x,t;y)$ be the probability density of $X(t)$, given $X(0)=y$. Note that

$$\Pr\{X(t)=x|X(0)=y, X(t-\Delta t)=a\} = \Pr\{X(\Delta t)=x|X(0)=a\}$$

Let

$$f_{DB}(x;a) = \Pr\{X(\Delta t)=x|X(0)=a\}.$$

Thus

$$f_{DB}(x;a) = \frac{1}{x\sqrt{2\pi\kappa^2\Delta t}}\exp\left(-\frac{1}{2\kappa^2\Delta t}\left(\log x - \left(\log a + (g-\tilde{g}-\frac{1}{2}\kappa^2)\Delta t\right)\right)^2\right)$$

since $X(\Delta t)|X(0)$ is a lognormal distribution. When $x > x^*$, we have

$$
\begin{aligned}
& f(x,t;y) \\
= \; & (1-p\Delta t)\int_0^{+\infty} f(a,t-\Delta t;y) f_{DB}(\frac{x}{a})da + p\Delta t \cdot f(\frac{x}{\rho}, t-\Delta t; y)\frac{1}{\rho} \\
= \; & (1-p\Delta t)\int_0^{+\infty} \left[\begin{array}{c} f(x,t;y) + (a-x)\frac{\partial}{\partial x} f(x,t;y) - \Delta t \frac{\partial}{\partial t} f(x,t;y) \\ + \frac{(a-x)^2}{2}\frac{\partial^2}{\partial x^2} f(x,t;y) \end{array} \right] f_{DB}(x;a)da \\
& + p\Delta t \cdot f(\frac{x}{\rho}, t-\Delta t; y)\frac{1}{\rho} \\
= \; & (1-p\Delta t) \left[\begin{array}{c} (1-(g-\tilde{g})\Delta t + \kappa^2 \Delta t)\, f(x,t;y) \\ + x\left(-(g-\tilde{g})\Delta t + 2\kappa^2\Delta t\right) \frac{\partial}{\partial x} f(x,t;y) - \Delta t \frac{\partial}{\partial t} f(x,t;y) + x^2 \frac{\kappa^2}{2}\Delta t \frac{\partial^2}{\partial x^2} f(x,t;y) \end{array} \right] \\
& + p\Delta t \cdot f(\frac{x}{\rho}, t-\Delta t; y)\frac{1}{\rho} + o(\Delta t)
\end{aligned}
$$

where we use the Taylor expansion in the second and third equalities. Dividing by Δt on both sides and letting $\Delta t \to 0$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} f(x,t;y) \; = \; & (\kappa^2 - p - (g-\tilde{g})) f(x,t;y) + (2\kappa^2 - (g-\tilde{g})) x \frac{\partial}{\partial x} f(x,t;y) \\
& + \frac{\kappa^2}{2} x^2 \frac{\partial^2}{\partial x^2} f(x,t;y) + p f(\frac{x}{\rho}, t; y)\frac{1}{\rho}, \quad x > x^*
\end{aligned}
$$

Thus

$$
\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g}) x f(x,t)) - p f(x,t) + p f(\frac{x}{\rho}, t)\frac{1}{\rho}, \; x > x^*.
$$

Similarly, we have

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\kappa^2 x^2 f(x,t)) - \frac{\partial}{\partial x}((g-\tilde{g})xf(x,t)) - pf(x,t),\ x < x^*. \ \blacksquare$$

6.3 Proof of Proposition 2

Proof: Plugging $f(x) = Cx^{-\beta}$ into equation (11), we have

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g-\tilde{g}))\beta + \kappa^2 - p - (g-\tilde{g}) = 0.$$

Therefore,

$$\beta_1 = \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2}$$

To show that $\beta_1 < 1$ note that

$$\begin{aligned}\beta_1 &< 1 \Leftrightarrow \frac{\frac{3}{2}\kappa^2 - (g-\tilde{g}) - \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}}{\kappa^2} < 1 \\ &\Leftrightarrow \frac{1}{2}\kappa^2 - (g-\tilde{g}) < \sqrt{(\frac{1}{2}\kappa^2 - (g-\tilde{g}))^2 + 2\kappa^2 p}\end{aligned}$$

The last inequality holds since $\kappa > 0$ and $p > 0$. Thus $\beta_1 < 1$. Plugging $f(x) = Cx^{-\beta}$ into equation (10), we have

$$\frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g-\tilde{g}))\beta + \kappa^2 - p - (g-\tilde{g}) + p\rho^{\beta-1} = 0.$$

We next show that $\beta_2 > 2$. Let $\Gamma(\beta) = \frac{\kappa^2}{2}\beta^2 - (\frac{3}{2}\kappa^2 - (g-\tilde{g}))\beta + \kappa^2 - p - (g-\tilde{g}) + p\rho^{\beta-1}$. Note that $\Gamma(1) = 0$. Since $\frac{\kappa^2}{2} > 0$, we have

$$\lim_{\beta \to +\infty} \Gamma(\beta) = +\infty.$$

Also $\Gamma(2) = g - \tilde{g} - p + p\rho = -\tau$ by equation (6). Thus $\Gamma(2) < 0$. By the continuity of $\Gamma(\beta)$, we know that there exists $\beta > 2$ such that $\Gamma(\beta) = 0$. Since the function $\Gamma(\beta)$ is strictly convex, it can have at most two roots. Then there exist a unique β_2 which is greater than 2. ■

6.4 Proof of Proposition 3

Proof: We use the Embedded Markov Chain method to establish the ergodicity of the wealth distribution of newborns, which then implies the ergodicity of the wealth distribution of the whole economy.

As in Karlin and Taylor (1981), we construct the embedded Markov chain from the continuous time process, $X(\cdot, t)$. Let t_1, t_2, t_3, $\cdots$, denote the birth time of the generation 1, generation 2, generation 3,$\cdots$. By our notation, their starting wealth is $X(t_1, t_1)$, $X(t_2, t_2)$, $X(t_3, t_3)$, $\cdots$. Let

$$\Phi_0 = X(\cdot, 0), \quad \Phi_n = X(t_n, t_n), \quad n = 1, 2, 3, \cdots$$

Thus Φ_n is the newborns's starting wealth. Note that the state space for Φ_n is $S = [x^*, +\infty)$ by the subsidy policy of the government. The stochastic process Φ_n is a Markov chain. Note that the duration of the life follows an exponential distribution with parameter p. When the agent is alive, her wealth follows a geometric Brownian motion as in equation (3) in the text. Given the government subsidy policy for the newborns, the transition probability of Φ_n is

$$\begin{aligned}
P(\Phi_{n+1} &= x^* \mid \Phi_n = x) \\
&= \int_0^{x^*} \int_0^{+\infty} pe^{-pt} \\
&\quad \times \frac{1}{y\sqrt{2\pi t\kappa^2}} \exp\left(-\frac{1}{2\kappa^2 t} \left(\log\left(\frac{y}{\rho}\right) - \left(\log x + (g - \tilde{g} - \frac{1}{2}\kappa^2)t \right) \right)^2 \right) dtdy
\end{aligned}$$

and for $y > x^*$

$$\begin{aligned}
P(\Phi_{n+1} &= y \mid \Phi_n = x) \\
&= \int_0^{+\infty} pe^{-pt} \\
&\quad \times \frac{1}{y\sqrt{2\pi t\kappa^2}} \exp\left(-\frac{1}{2\kappa^2 t} \left(\log\left(\frac{y}{\rho}\right) - \left(\log x + (g - \tilde{g} - \frac{1}{2}\kappa^2)t \right) \right)^2 \right) dt
\end{aligned}$$

By Theorem 13.3.3 of Meyn and Tweedie (1993), $\{\Phi_n\}_{n=0}^{\infty}$ will be ergodic whenever it is positive Harris and aperiodic.

We need to show the following conditions to draw the conclusion of Proposition 3:

(1) Φ_n is ψ-irreducible, (2) Φ_n admits an invariant probability measure. (3) Φ_n is

Harris recurrent, (4) Φ_n is aperiodic.

It is easy to prove (1), due to the special lower bound of x^*. (2) is also true due to Proposition 2 in the text. The existence of the stationary wealth distribution, $f(x)$, implies the existence of the invariant probability measure of Φ_n. (3) is true since the government subsidy policy guarantees that, starting from any place in S, Φ_n visits x^* almost surely. (4) is obviously true. (For these mathematical concepts, see Meyn and Tweedie (1993)). ■