1 Asset Pricing in Infinite Horizon models

Assume a representative-agent economy with one good. Let time be indexed by \( t = 0, 1, 2, \ldots \). Uncertainty is captured by a probability space represented by a tree. Suppose that there is no uncertainty at time 0 and call \( s^0 \) the root of the tree. Without much loss of generality, we assume that each node has a constant number of successors, \( S \). At generic node at time \( t \) is called \( s^t \in S^t \). Note that the dimensionality of \( S^t \) increases exponentially with time \( t \) (abusing notation it is in fact \( S^t \)).

When a careful specification of the underlying state space process is not needed, we will revert to the usual notation in terms of stochastic processes. Let \( x := \{x_t\}_{t=0}^\infty \) denote a stochastic process for an agent’s consumption, where \( x_t : S^t \rightarrow \mathbb{R}_+ \) is a random variable on the underlying probability space, for each \( t \). Similarly, let \( \omega := \{\omega_t\}_{t=0}^\infty \) be stochastic processes describing an agent’s endowments. Let \( 0 < \beta < 1 \) denote the discount factor.

1.1 Contingent Markets Economy

Suppose that at time zero, the agent can trade in contingent commodities. Let \( p := \{p_t\}_{t=0}^\infty \) denote the stochastic process for prices, where \( p_t : S^t \rightarrow \mathbb{R}_+ \), for each \( t \).

**Definition 1** \( (\{x^{s^i}\}_i, p^*) \) is an Arrow-Debreu Equilibrium if

i. given \( p^* \), s.t.
\[
\sum_{t=0}^\infty p_t^* (x_t - \omega_t) = 0
\]

ii. and \( \sum_i x^{s^i} - \omega^i = 0 \).

The notation does not make explicit that the agent chooses at time 0 a whole sequence of consumption allocations, that is, the whole sequence of \( x(s^t) \) for any \( s^t \in S^t \) and any \( t \geq 0 \).

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1 Thanks to Francesc Ortega for research assistance.
1.2 Financial Markets Economy

Suppose that throughout the uncertainty tree, there are $J$ assets. We shall allow assets to be long-lived. In fact we shall assume they are and let the reader take care of the straightforward extension in which some of the assets pay off only in a finite set of future times. Let $z := \{z_t\}_{t=1}^\infty$ denote the sequence of portfolios of the representative agent, where $z_t : S_t \to R^J$. Assets’ payoffs are captured at each time $t$ by the $S \times J$ matrix $A_t$. Furthermore, capital gains are $q_t$, and returns are $R_t = \frac{A_t + q_t}{q_{t-1}}$.

In a financial market economy agents do not trade at time 0 only. They in fact, at each node $s_t$ receive endowments and payoffs from the portfolios they carry from the previous node, they re-balance their portfolios and choose state contingent consumption allocations for any of the successor nodes of $s_t$, which we denote $s_{t+1} | s_t$.

**Definition 2** $\{ (x^{*i}, z^{*i}) \}_i, q^*$ is a Financial Markets Equilibrium if

i. given $q^*$, at each time $t \geq 0$

\[
(x^{*i}, z^{*i}) \in \arg \max \{ u(x_t) + E_t[\sum_{\tau=1}^\infty \beta^\tau u(x_{t+\tau}|s^i)] \}
\]

s.t.

\[
x_{t+\tau} + q_{t+\tau} z_{t+\tau} = \omega_{t+\tau} + A_{t+\tau} z_{t+\tau-1},
\]

for $\tau = 0, 1, 2, \ldots$, with $z_{-1} = 0$

some no-Ponzi scheme condition

**Definition 3** ii. $\sum_i x^{*i} - \omega^i = 0$ and $\sum_i z^{*i} = 0$.

1.3 Conditional Asset Pricing

From the FOC of the agent’s problem, we obtain

\[
q_t = E_t \left( \frac{\beta u'(x_{t+1})}{u'(x_t)} A_{t+1} \right) = E_t (m_{t+1} A_{t+1}) \tag{1}
\]

or,

\[
1 = E_t \left( \frac{\beta u'(x_{t+1})}{u'(x_t)} R_{t+1} \right). \tag{2}
\]

**Example 1.** Consider a stock. Its payoff at any node can be seen as the dividend plus the capital gain, that is,

\[
R_{t+1} = \frac{q_{t+1} + d_{t+1}}{q_t},
\]

for some exogenously given dividend stream $d$. By plugging this payoff into equation (1), we obtain the price of the stock at $t$.

**Example 2.** For a call option on the stock, with strike price $k$ at some future period $T > t$, we can define

\[
A_t = 0, \ t < T, \ \text{and} \ A_T = \max\{q_T - k, 0\}.
\]
Define now
\[ m_{t,T} = \frac{\beta^{T-t} u(x_T)}{u(x_t)} \]
and observe that the price of the option is given by
\[ q_t = E_t (m_{t,T} \max\{q_T - k, 0\}) \]
Note how the conditioning information drives the price of the option: the price changes with time, as information is revealed by approaching the execution period \( T \).

**Example 3.** The risk-free rate is known at time \( t \) and therefore, equation (2) applied to a 1-period bond yields
\[ \frac{1}{R_{t+1}^f} = E_t \left( \frac{\beta u'(x_{t+1})}{u'(x_t)} \right). \]
Once again, note that the formula involves the conditional expectation at time \( t \). Therefore, while the return of a risk free 1-period bond paying at \( t + 1 \) is known at time \( t \), the return of a risk free 1-period bond paying at \( t + 2 \) is not known at time \( t \). [... relationship between 1 and \( \tau \) period bonds.... from the red Sargent book]

Conditional versions of the beta representation hold in this economy:
\[ E_t(R_{t+1}) - R_{t+1}^f = -\frac{Cov_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})} = \frac{Cov_t(m_{t+1}, R_{t+1})}{Var_t(m_{t+1})} \left( -\frac{Var_t(m_{t+1})}{E_t(m_{t+1})} \right) =: \beta_t \lambda_t. \]

### 1.3.1 Unconditional moment restrictions
Recall that our basic pricing equation is a conditional expectation:
\[ q_t = E_t(m_{t+1} A_{t+1}), \] (4)
In empirical work, it is convenient to test for unconditional moment restrictions. However, taking unconditional expectations of the previous equation implies in principle a much weaker statement about asset prices than equation (4):
\[ E(q_t) = E(m_{t+1} A_{t+1}), \] (5)
where we have invoked the law of iterated expectations. It should be clear that equation (4) implies but it is not implied by (5).

The theorem in this section will tell us that actually there is a theoretical way to test for our conditional moment condition by making a series of tests of unconditional moment conditions.
Define a stochastic process \( i_t \) to be \textit{conformable} if for each \( t \), \( i_t \) belongs to the time-\( t \) information set of the agent. It then follows that for any such process, we can write

\[ i_t q_t = E_t(m_{t+1} i_{t+1} A_{t+1}) \]

and, by taking unconditional expectations,

\[ E(i_t q_t) = E(m_{t+1} i_{t+1} A_{t+1}). \]

This fact is important because for each conformable process, we obtain an additional testable implication that only involves unconditional moments. Obviously, all these implications are necessary conditions for our basic pricing equation to hold. The following result states that if we could test these unconditional restrictions for all possible conformable processes then it would also be sufficient. We state it without proof.

\textbf{Theorem 4} If \( E(x_{t+1} i_t) = 0 \) for all \( i_t \) conformable then \( E_t(x_{t+1}) = 0 \).

By defining \( x_{t+1} = m_{t+1} A_{t+1} - q_t \), the theorem yields the desired result.

\subsection{1.4 Predictability or returns}

Recall the asset-pricing equation for stocks:

\[ q_t = E_t(m_{t+1}(q_{t+1} + d_{t+1})). \]

It is sometimes argued that returns are predictable unless stock prices to follow a random walk. (Where in turn predictability is interpreted as a property of \textit{efficient market hypothesis}, a fancy name for the asset pricing theory exposed in these notes). Is it so? No, unless strong extra assumptions are imposed. Assume that no dividends are paid and agents are risk neutral; then, for values of \( \beta \) close to 1 (realistic for short time periods), we have

\[ q_t = E_t(q_{t+1}). \]

That is, the stochastic process for stock prices is in fact a martingale. Next, for any \( \{\varepsilon_t\} \) such that \( E_t(\varepsilon_{t+1}) = 0 \) at all \( t \), we can rewrite the previous equation as

\[ q_{t+1} = q_{t+1} + \varepsilon_{t+1}. \]

This process is a random walk when \( \text{var}_t(\varepsilon_{t+1}) = \sigma \) is constant over time.

A more important observation is the fact that marginal utilities times asset prices (a risk adjusted measure of asset prices) follow approximately a martingale (a weaker notion of lack of predictability). Again under no dividends,

\[ u'(c_t)q_t = \beta E_t(u'(c_{t+1})(q_{t+1} + d_{t+1})), \]

which is a supermartingale and approximately a martingale for \( \beta \) close to 1.
1.5 Fundamentals-driven asset prices

For assets whose payoff is made of a dividend and a capital gain, FOC dictate

\[ q_t = E_t (m_{t+1}(q_{t+1} + d_{t+1})) , \]

where

\[ m_{t+1} = \frac{\beta u'(c_{t+1})}{w(c_t)} . \]

By iterating forward and making use of the Law of Iterated Expectations,

\[ q_t = \lim_{T \to \infty} E_t \left( \sum_{j=1}^{T} m_{t,t+j}d_{t+j} \right) + \lim_{T \to \infty} E_t \left( \sum_{j=1}^{T} m_{t,t+j}q_{t+j} \right) . \]

As we shall see, infinite horizon models (with infinitely lived agents) usually satisfy the no-bubbles condition, or

\[ \lim_{T \to \infty} E_t \left( \sum_{j=1}^{T} m_{t,t+j}q_{t+j} \right) = 0 . \]

In that case, we say that asset prices are fully pinned down by fundamentals since

\[ q_t = E_t \left( \sum_{j=1}^{\infty} m_{t,t+j}d_{t+j} \right) . \]

1.5.1 Conditional factor models and the conditional CAPM

[...from Cochrane...]

1.5.2 Frictions

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2 Bubbles: Santos and Woodford, Ecta 1997

Let \( N = X_{t=0}^{\infty}S^t \) be the set of nodes of the tree. Recall we denoted with \( s^0 \) denote the root of the tree and with \( s^t \) an arbitrary node of the tree at time \( t \). Denote by \( s^t - 1 \) the single (immediate) predecessor node to \( s^t \). Use \( s^{t+\tau}|s^t \) to indicate that \( s^{t+\tau} \) is some successor of \( s^t \), for \( \tau > 0 \).

At each node, there are \( J \) securities traded.

It is important that the notation includes Overlapping Generation economies. We need therefore to account for finitely lived agents. Let \( I(s^t) \) be the set of agents which are active at node \( s^t \). Let \( N^t \) be the subset of nodes of the tree
at which agent $i$ is allowed to trade. Also, denote by $\overline{N}^i$ the terminal nodes for agent $i$.

The following assumptions will not be relaxed.

1. If an agent $i$ is alive at some non-terminal node $s^t$, she is also alive at all the immediate successor nodes. That is, $s^t \in N^i \setminus \overline{N}^i \implies \{ s^{t+1} e N : s^{t+1} \mid s^t \} \subset N^i$.

2. The economy is connected across time and states: at any state there is some agent alive and non-terminal. Formally,

\[ \forall s^t, \exists i : s^t \in N^i \setminus \overline{N}^i. \]

Assets are long lived. Let $q : N \rightarrow F^I$ be the mapping defining the vector of security prices at each node $s^t$. Similarly, let $d : N \rightarrow F^J$ denote the vector-valued mapping that defines the dividends (in units of numeraire) that are paid by the assets at node $s^t$. We assume that $d(s^t) \geq 0$ for any $s^t$.

Each of the households alive at $s^0$ enters the markets with an initial endowment of securities $z^i(0)$. Therefore, the initial net supply of assets is given by

\[ z^i = \sum_{i \in I(s^0)} z^i. \]

As assets are long lived, a supply $z^i$ of assets is available at any $s^t$.

At each node $s^t$, each households in $I(s^t)$ has an endowment of numeraire good of $\omega^j(s^t) \geq 0$. We shall assume that the economy has a well-defined aggregate endowment

\[ \omega(s^t) = \sum_{i \in I(s^t)} \omega^i(s^t) \geq 0 \]

at each node $s^t$. This is the case, e.g., if $I(s^t)$ is finite, for any $s^t$. Taking into account the dividends paid by securities in units of good, the aggregate good supply in the economy is given by

\[ \bar{\omega}(s^t) = \omega(s^t) + d(s^t)z^i \geq 0. \]

The utility function of any agent $i$ is written

\[ U(x) = \sum_{i=0}^{\infty} \beta^t \sum_{s^t} \text{prob}_{s^t} u^i(x(s^t)). \]

Define the 1-period payoff vector (in units of numeraire) at node $s^t$ by

\[ A(s^t) = d(s^t) + q(s^t). \]

Agent $i$ chooses, at each node $s^t \in N^i$ a level of consumption $x^i(s^t)$ and a $J$ vector of securities $z^i(s^t)$ to hold at the end of trading, subject to the budget constraints:

\[ x^i(s^0) + q(s^0)z^i(s^0) \leq \omega^i(s^0) + q(s^0)z^i_{\omega}. \]
and at each node \(s^t \neq s^0\),
\[
x^i(s^t) + q(s^t)z^i(s^t) \leq \omega^i(s^t) + A(s^t)z^i(s^t - 1),
\]
with
\[
x^i(s^t) \geq 0 \quad q(s^t)z^i(s^t) \geq -B^i(s^t),
\]
where \(B^i : \mathbb{N} \rightarrow \mathbb{R}_+\) indicates an exogenous and non-negative household specific borrowing limit at each node. We assume households take the borrowing limits as given, just as they take security prices as given.

At equilibrium, markets clear: that is, at each \(s^t\),
\[
\sum_{i \in I(s^t)} x^i(s^t) = \tilde{\omega}(s^t) \\
\sum_{i \in I(s^t)} z^i(s^t) = z_\omega
\]

Given the price process \(q\), we say that no arbitrage opportunities exist at \(s^t\) if there is no \(z \in \mathbb{R}^J\) such that
\[
A(s^{t+1})z \geq 0, \text{ for all } s^{t+1}|s^t, \\
q(s^t)z \leq 0,
\]
with at least one strict inequality.

**Lemma 5** When \(q\) satisfies the no-arbitrage condition at \(s^t\), there exists a set of state prices \(\{\pi(s^{t+1})\}\) with \(\pi(s^{t+1}) > 0\) for all \(s^{t+1}|s^t\), such that the vector of asset prices at \(s^t\) can be written as
\[
q(s^t) = \sum_{s^{t+1}|s^t} \pi(s^{t+1})A(s^{t+1}). \quad (6)
\]

**Proof.** As usual, proof follows from applying an appropriate separation theorem. \(\blacksquare\)

Applying the Lemma at any \(s^t\) we can construct a stochastic process \(\pi : \mathbb{N} \rightarrow \mathbb{R}^J_+\). Let \(\Pi(s^t)\) denote the set of such processes for the subtree with root \(s^t\). Only under complete markets is the set \(\Pi(s^t)\) a singleton.

As a remark, note that completeness is an endogenous property since one-period payoffs \(A\) contain asset prices. Therefore, the rank property which defines completeness can only be assessed at each given equilibrium.

**Definition 6** For any state-price process \(\pi \in \Pi(s^t)\), define the \(J\) vector of fundamental values for the securities traded at node \(s^t\) by
\[
f(s^t, \pi) = \sum_{T=t+1}^{\infty} \sum_{s^T | s^t} \pi(s^T) d(s^T|s^t). \quad (7)
\]
Observe that the fundamental value of a security is defined with reference to a particular state-price process, however the following properties it displays are true regardless of the state prices chosen.

**Proposition 7** At each \(s^t \in N\), \(f(s^t, \pi)\) is well-defined for any \(\pi \in \Pi(s^t)\) and satisfies
\[
0 \leq f(s^t, \pi) \leq q(s^t).
\]

**Proof.** First of all, \(0 \leq f(s^t, \pi)\) follows directly from non-negativity of \(\pi\), the dividend, and the price process. We therefore turn to \(f(s^t, \pi) \leq q(s^t)\). From equation (6), we have
\[
q(s^t) = \sum_{s^{t+1}|s^t} \pi(s^{t+1})d(s^{t+1}|s^t) + \sum_{s^{t+1}|s^t} \pi(s^{t+1})q(s^{t+1})
\]
and, iterating on this equation we obtain
\[
q(s^t) = \sum_{T=t+1}^{\hat{T}} \sum_{s^T|s^t} \pi(s^T)d(s^T|s^t) + \sum_{s^T|s^t} m(s^T)q(s^T)
\]
for any \(\hat{T} > t\). Since by construction, \(q(s^T)\) is non-negative and \(\pi \in \Pi(s^t)\) is a positive state-price vector, the second term on the right-hand-side is non-negative. So,
\[
q(s^t) \geq \sum_{T=t+1}^{\hat{T}} \sum_{s^T|s^t} \pi(s^T)d(s^T|s^t), \text{ for any } \hat{T} > t.
\]
and
\[
q(s^t) \geq \sum_{T=t+1}^{\infty} \sum_{s^T|s^t} \pi(s^T)d(s^T|s^t) = \pi(s^t)f(s^t, \pi)
\]

We can correspondingly define the vector of asset pricing bubbles as
\[
\sigma(s^t, \pi) = q(s^t) - f(s^t, \pi), \quad (8)
\]
for any \(\pi \in \Pi(s^t)\) for the \(J\) securities. It follows from the proposition that
\[
0 \leq \sigma(s^t, \pi) \leq q(s^t),
\]
for any \(\pi \in \Pi(s^t)\). This corollary is known as the *impossibility of negative bubbles* result. Substituting (8) and (7) into (6) yields
\[
\sigma(s^t) = \sum_{s^{t+1}|s^t} \pi(s^{t+1})\sigma(s^{t+1}).
\]
This is known as the *martingale property* of bubbles: if there exists a (nonzero) price bubble on any security at date \(t\), there must exist a bubble as well on the
security at date $T$, with positive probability, at every date $T > t$. Furthermore, if there exists a bubble on any security at node $s^t$, then there must have existed a bubble as well on some security at every predecessor of the node $s^t$.

What does this imply for securities with finite maturity? Your answer must depend on how you define securities with finite maturity in this context.

In an economy with incomplete markets, the fundamental value need not be the same for all state-price processes consistent with no arbitrage. But even in this case, we can define the range of variation in the fundamental value, given the restrictions imposed by no-arbitrage.

Let $x : N \rightarrow R_+$ denote a stream of non-negative payoff. For any $s^t$, pick any $\pi \in \Pi(s^t)$ and define the present value at $s^t$ of $x$ with respect to $\pi$ by

$$V_x(s^t, \pi) = \sum_{T=t+1}^{\infty} \sum_{s^T|s^t} \pi(s^T)x(s^T).$$

Since this present value depends on the stochastic discount factor $\pi$, let us now define the bounds for the present value at $s^t$ of dividends $x$.

**Definition 8** For any $s^t$, define

$$\underline{V}_x(s^t) = \inf_{\pi \in \Pi(s^t)} \{V_x(s^t, \pi)\},$$

$$\overline{V}_x(s^t) = \sup_{\pi \in \Pi(s^t)} \{V_x(s^t, \pi)\}.$$ 

A few remarks follow from these definitions. First note that these definitions are conditional on a given price process $q$ since the set of no-arbitrage stochastic discount factors are defined with respect to $q$. Next observe that, for any security with payoff process $x^j$,

$$\underline{V}_{x^j}(s^t) \leq f^j(s^t, \pi) \leq \overline{V}_{x^j}(s^t),$$

for all $\pi \in \Pi(s^t)$, and

$$\underline{V}_{x^j}(s^t) < q^j$$ implies that there is a pricing bubble for the security with payoff process $x^j$.

Recall that to rule out Ponzi schemes when agents are infinitely lived, a lower bound on individual wealth is needed. Let us define a particular type of borrowing limit.

**Definition 9** An agent’s borrowing ability is only limited by her ability to repay out of her own future endowment if

$$B^i(s^t) = \underline{V}_{x^i}(s^t),$$

for each $s^t \in N^i \setminus \{s^t\}$.

It can be shown that these borrowing limits never bind at any finite date (see Magill-Quinzii, *Econometrica*, 94), but rather only constrain the asymptotic
behavior of a household’s debt. An important consequence of this specification is the following.

**Proposition 10** Suppose that agent $i$ has borrowing limits of the form (9). Then the existence of a solution to the agent’s problem for given prices $q$ implies that $\pi_{x^i}(s^i) < \infty$, at each $s^i \in N^i$, so that the borrowing limit is finite at each node.

This is because, if agent $i$ can borrow off of the value of $\tilde{\omega}^i$, this value must be finite at equilibrium prices for the agent’s problem to be well-defined. This must be the case, in fact, for the equilibrium price of $\tilde{\omega}^i$ (recall $\tilde{\omega}^i$ is traded) and hence it must be that $\pi(s^i) < \infty$ (recall that for any traded process $x^j$, $\pi_{x^j}(s^j) \leq \pi_{s^j}(s^j) \leq q^j$). We can now prove the following fundamental lemma.

**Lemma 11** Consider an equilibrium $\{x^{*i}, z^{*i}, q\}$. Suppose that the (supremum of the) value of aggregate wealth is finite, i.e., $\pi_\infty(s^i) < \infty$. Suppose also that there exists a bubble on some security in positive net supply at $s^i$ so that $\sigma(s^i)z(s^i) > 0$. Then, $\forall K > 0$, there exists a time $T$ and $s^T|s^i$ such that

$$\sigma(s^T)z(s^T) > K \sum_{s^T|s^i} \pi(s^T)\tilde{\omega}(s^T).$$

**Proof.** The martingale property of pricing bubbles implies,

$$\sigma(s^i)z(s^i) = \sum_{x^j|s^i} m(s^T)\sigma(s^T)z(s^T)$$

and hence $\sigma(s^i)z(s^i)$ explodes with positive probability. On the other hand, $\sum_{s^T|s^i} \pi(s^T)\tilde{\omega}(s^T)$ must converge to 0 in $T \to \infty$ to guarantee that $\pi_\infty(s^i) < \infty$.

That is, there is a positive probability that the total size of the bubble on the securities becomes an arbitrarily large multiple of the value of the aggregate supply of goods in the economy. The proof exploits crucially the martingale property of bubbles. It follows from this result that some agent must accumulate vast wealth.

We already learned that no bubbles can arise in securities with finite maturity. The next theorem shows that no bubbles can arise to securities in positive net supply as long as we are at equilibria with finite aggregate wealth. The proof uses the nonoptimality of the behavior implied by the previous lemma.

**Theorem 12** Consider an equilibrium $\{x^{*i}, z^{*i}, q\}$. Suppose that at each node $s^i \in N$, there exists $\pi \in \Pi(s^i)$ such that $V_\pi(s^i, \pi) < \infty$. Then

$$q^j(s^T) = f^j(s^T, \pi),$$

for all $s^T|s^i$ and $\pi \in \Pi(s^i)$, for each security $j$ traded at $s^T$ that has positive net supply, $z^j_\pi > 0$.

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2They are equivalent to requiring that the consumption process lies in the space of measurable bounded sequences. In the case of finitely lived agents, these borrowing limits are equivalent to imposing no-borrowing at all nonterminal nodes.
This is a crucial: if an agent’s endowment can be traded, then its value is on the right hand side of the present value budget constraint of the agent, and hence it must be finite. Note that if we have that at equilibrium $\pi_\omega(s^t) < \infty$, the condition of the theorem is satisfied.

The next two corollaries to the theorem provide conditions on the primitives of the model that guarantee that the value of aggregate wealth is finite at any equilibrium.

**Corollary 13** Suppose that there exists a portfolio $\tilde{z} \in \mathbb{R}^J_+$ such that
\[ d(s^t | s^0) \tilde{z} \geq \tilde{\omega}(s^t), \forall s^t \in N. \]

Then the theorem holds at any equilibrium.

Intuitively, if the existing securities allow such a portfolio $\tilde{z}$ to be formed, it must have a finite price at any equilibrium. But since the dividends paid by this portfolio are higher at every state than the aggregate endowment, the equilibrium value of the aggregate endowment is bounded by a finite number.

**Corollary 14** Suppose that there exists an (infinitely lived) agent $i$ and an $\varepsilon > 0$ such that i) $\omega^i(s^t) \geq \varepsilon \omega(s^t), \forall s^t \in N$ and ii) $B^i(s^t) = \pi_\omega(s^t), \forall s^t \in N$ and for all $i$. Then the theorem holds at any equilibrium.

Again, the result follows because in equilibrium, $\varepsilon \omega(s^t)$ must have a finite value as it appears on the right hand side of agent $i$’s budget set. If a positive fraction of aggregate wealth has a finite value in equilibrium, then aggregate wealth has finite value.

As a remark, note that these two corollaries share the same spirit and somewhat imply that bubbles are not a robust equilibrium phenomenon (for securities in positive net supply).

### 2.1 (Famous) Theoretical Examples of Bubbles

Recall that fiat money is a security that pays no dividends. Its only return comes from paying one unit of itself in the next period. Therefore if fiat money is in net supply and has a positive price in equilibrium, that is a bubble. The following two models have equilibria with such a property.

#### 2.1.1 Samuelson (1958)’s OLG model

Consider an economy in which $\bigcup_{s^t \in N} I(s^t)$ is (countably) infinite, even though $I(s^t)$ is finite for any $s^t \in N$. In this case even if $\pi_{\omega^i}(s^t) < \infty$, it is still possible that $\pi_{\omega^i}(s^t) = \infty$ (and hence that $\pi_{\omega^i}(s^t) = \infty$). The theorem does not apply and bubbles are possible.

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2.1.2 Bewley (1980)’s turnpike model

Consider the case in which stringent borrowing limits are imposed on trading, i.e., $B^i(s^t) < \pi^i(s^t)$. In this case, nothing will exclude the possibility that $\pi^i(s^t) = \infty$. The theorem does not apply and bubbles are possible.

2.1.3 More recent examples

Here’s a list.


3 Idiosyncratic shocks economies

The class of economies we studied for bubbles has a finite number of (types of) agents at each node of the tree $N$: $I(s^t)$ is finite for any $s^t$. In these economies the stochastic structure represented by the tree $N$ refers to the whole economy: all agents $i \in I(s^t)$ at time $t$ face the realization $s^t$. However, we often need to explicit the composition of the types, that is, to explicit them notationally as an infinity of ex-ante identical agents, so as to make (ab)use of the Law of large numbers.

In this section we show how to extend the notation to the case of idiosyncratic shocks economies. For simplicity, but without loss of generality, we consider the case in which all agents in the economy are of the same type: that is, agents are ex-ante identical in terms of preferences and stochastic process for endowments. Ex-post, however, the realization of their endowments are different across agents. Essentially, we now think of the tree $N$ as representing the stochastic process faced by each individual agent. Let each node $s^t$ have $S$ successors, that is, $s^{t+1}|s^t = \{1, 2, ..., S\}$. Since macroeconomics is typically set in terms of Markov processes, we adopt the assumption:

$$\text{prob}(s^{t+1} = s' | s^t = s) = \text{prob}(s'|s)$$

Obviously $\text{prob}(s' | s^t)$ defines a transition probability of a Markov chain with state space $\{1, 2, ..., S\}$. We assume the Markov chain is recurrent (e.g., it is
sufficient that $\text{prob} \ (s'|s) > 0$, for any $s', s \in \{1, 2, \ldots, S\}$) so that it has a stationary distribution $\lambda(s)$:

$$\lambda(s') = \sum_{s \in S} \text{prob} \ (s'|s) \lambda(s)$$

The (ab)use of the Law of large number consists in assuming the following:

*The fraction of agents whose realization of the individual shock is $s$ at time $t$ and $s'$ at $t+1$ is $\text{prob} \ (s'|s)$, for any $t \geq 0$."

In these economies, therefore, we can conveniently define individual endowments as a map $\phi: \{1, 2, \ldots, S\} \rightarrow \mathbb{R}^+$, for each time $t \geq 0$. Similarly, assets’ dividends are $d: \{1, 2, \ldots, S\} \rightarrow \mathbb{R}^J$, a $J$-dimensional Markov process in our probability space. Prices are a vector sequence $q_t \in \mathbb{R}^J$, for any $t \geq 0$. Each individual agent problem is then more easily written recursively:

$$v_t(z, s) = \max_{z' \leq -B} u(d(s)z + \omega(s) - qt z') + \beta \sum_{s' \in S} \text{prob} \ (s'|s) v_{t+1}(z', s'),$$

where $-B$ is the natural borrowing limit, as defined, e.g., in the section on bubbles. The policy function is

$$z_{t+1} = g_t(z_t, s_t; q_t),$$

and

$$x_t(z_t, s_t) = d(s_t)z_t + \omega(s_t) - qt g_t(z_t, s_t; q_t)$$

Construct then the stationary distribution

$$\lambda_{t+1}(z_{t+1}, s_{t+1}) = \sum_{s_t \in S} \text{prob} \ (s_{t+1}|s_t) \int_{z:z_{t+1}=g_t(z_t,s_t; q_t)} \lambda_t(z_t, s_t) dz_t$$

Using the ab(use) of the Law of large numbers, therefore, goods and financial market clearing at each time $t$ take the form:

$$\sum_{s_t \in S} \lambda_t(z_t, s_t) (x_t(z_t, s_t) - \omega(s_t)) = 0$$

$$\sum_{s \in S} \int \lambda_t(z_t, s_t) g_t(z_t, s_t; q_t) dz_t = 0,$$

respectively.

Summarizing:

**Definition 15** A financial markets equilibrium for the economy with idiosyncratic risk is represented by sequences of maps $x_t : S \times \mathbb{R}^J \rightarrow \mathbb{R}^+$,
\[ g_t : S \times R^J \to R^J, \text{ and a sequence of vectors } q_t \in R^J_+ \text{, for any } t \geq 0, \text{ such that } \\
z_{t+1} = g_t(z_t, s_t; q_t) \text{ solves:} \\

v_t(z, s) = \max_{z' \in B} u(d(s)z + \omega(s) - q_tz') + \beta \sum_{s' \in S} \prob(s'|s)v_{t+1}(z', s'), \\

and \\
x_t(z_t, s_t) = d(s_t)z_t + \omega(s_t) - q_tg_t(z_t, s_t; q_t), \\

Furthermore, financial markets clear:³ \\

\[ \sum_{s \in S} \int \lambda_t(z_t, s_t)g_t(z_t, s_t; q_t)dz_t = 0. \]

Consider now an extension of the previous section’s economy, which allows for aggregate risk, that is, a stochastic process for shocks \( a_t \in A = \{1, ..., A\} \) which affect all the agents. Essentially, we keep having the tree \( N \) as the representation of the stochastic process faced by each individual agent. But now think of each node on the tree \( N \) as a couple \( a^t, s^t \in A \times S \) with have \( A \times S \) successors, that is, \( a^{t+1}, s^{t+1} | a^t, s^t \in A \times S \). The transition probability for the Markov processes is: \\

\[ \prob(a^{t+1} = a', s^{t+1} = s'| a^t = a, s^t = s) = \prob(a', s'| a, s) \]

with stationary distribution \( \lambda(a, s) \). Each individual agent problem is then more easily written recursively: \\

\[ v_t(z, a, s) = \max_{z' \in B} u(d(a, s)z + \omega(a, s) - q_t(a)z') + \beta \sum_{a', s' \in S \times A} \prob(a', s'| a, s)v_{t+1}(z', a', s'), \]

and the policy function is \\

\[ z_{t+1} = g_t(z_t, a_t, s_t) \]

Summarizing:

**Definition 16** A Financial markets equilibrium for the economy with idiosyncratic and aggregate risk is represented by sequences of maps \( x_t : A \times S \times R^J \to R^J_+ \), \( g_t : A \times S \times R^J \to R^J_+ \), and \( q_t : A \to R^J_+ \), for any \( t \geq 0 \), such that \( z_{t+1} = g_t(z_t, a_t, s_t) \) solves: \\

\[ v_t(z, a, s) = \max_{z' \in B} u(d(a, s)z + \omega(a, s) - q_t(a)z') + \beta \sum_{a', s' \in S \times A} \prob(a', s'| a, s)v_{t+1}(z', a', s'), \]

³Goods market clearing, \\

\[ \sum_{s_t \in S} \lambda_t(z_t, s_t)(x_t(z_t, s_t) - \omega(s_t)) = 0, \]

is redundant.
and
\[ x_t(z_t, a_t, s_t) = d(a_t, s_t)z_t + \omega(a_t, s_t) - q_t(a_t)g_t(z_t, a_t, s_t) \]

Furthermore, financial markets clear:
\[ \sum_{a, s \in S \times A} \int A \lambda_t(z_t, a_t, s_t)g_t(z_t, a_t, s_t)dz_t = 0. \]

Typically, these economies have well-defined stationary equilibria.

**Definition 17** A stationary Financial markets equilibrium for the economy with idiosyncratic and aggregate risk is represented by maps \( x : A \times S \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \), \( g : A \times S \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \), and \( q : A \rightarrow \mathbb{R}_+ \) such that \( z' = g(z, a, s) \) solves:

\[ v(z, a, s) = \max_{z' \geq -B} \left( u(d(a, s)z + \omega(a, s) - \beta q(a)z') + \sum_{s' \in S} \text{prob} (a', s'|a, s) v(z', a', s') \right), \]

and
\[ x(z, a, s) = d(a, s)z + \omega(a, s) - q(a)g(z, a, s) \]

Furthermore, financial markets clear:
\[ \sum_{a, s \in S \times A} \int g(z, a, s)\lambda(z, a, s)dz = 0, \]

for a stationary distribution \( \lambda(z, a, s) \) which satisfies:
\[ \lambda(z', a', s') = \sum_{a, s \in S \times A} \text{prob} (a', s'|a, s) \int_{z:z'=g(z,a,s)} \lambda(z, a, s)dz. \]

The following result is not out there in the literature (I think). But it should be a consequence of the constrained efficiency result for incomplete market economies with one good. Can you prove it?

**Proposition 18** A stationary Financial market equilibrium for an incomplete market economy with idiosyncratic and aggregate risk is constrained Pareto efficient. On the other hand, a Financial market equilibrium for an incomplete market economy with idiosyncratic and aggregate risk (along the transition path) is generically constrained Pareto inefficient.

The constraint inefficiency result for Bewley economies, due to Davila, Hong, Krusell, Rios-Rull (2005), applies to production economies; see later.

### 3.1 When do Incomplete Markets matter?

In a series of papers, Telmer (1993), Aiyagari (1994) and Krusell and Smith (1998) among others, different authors have found support for a puzzling result. Even though theoretically the completeness of financial markets affects equilibrium allocations and prices in a fundamental way (e.g., equilibria are typically
efficient if and only if financial markets are complete), they do not seem to matter significantly in calibrations.

In fact, we can show theoretically that, when agents are infinitely lived and their endowments are stationary and idiosyncratic, incomplete markets tend to matter little (where "little" is precisely defined). This is the result from Levine and Zame, *Econometrica*, 2002. The intuition for this result is straightforward: over long horizons market incompleteness may not matter if traders can self insure i.e., if they can borrow in bad times and save in good times.

Consider the economy with idiosyncratic (no aggregate) risk in the previous section, extended to allow for a finite set of types \( i \in I \) (each of measure 1). Let \( \underline{\omega}^i \) and \( \underline{\omega} \) denote the *long-run average endowment (permanent income)* for agent \( i \) and for the aggregate economy, respectively:

\[
\underline{\omega}^i = \sum_{s \in S} \lambda(s)\omega^i(s),
\]
\[
\underline{\omega} = \sum_{i \in I} \sum_{s \in S} \lambda(s)\omega^i(s).
\]

Assume that only a bond is traded in the economy (\( J = 1 \)), that is an asset with payoff \( d(s) = 1 \), for any \( s \). It is easy to show that, the Pareto efficient allocations of this economy are given by the \( I \)-tuples of fixed shares of the constant *long-run average endowment* \( \underline{\omega} = \sum_i \underline{\omega}^i \). In particular, the complete market equilibrium allocation (hence Pareto efficient) for this economy is characterized by each agent \( i \) consuming his/her *long-run average endowment* at each node in \( N \).

The next theorem shows that, for an appropriate definition of "closeness", when markets are incomplete, equilibrium allocations are "close" to ones where each agent \( i \) consumes his/her long-run average endowment (permanent income) at each node. To this end, let \( N_i^t \subseteq N \) denote the set \( \{ s^t \in S^t \mid x^t_i(s_t) - \underline{\omega} > \varepsilon \text{ for some } i \} \) for some equilibrium allocation \( x^t_i(s_t) \). We say that the equilibrium stochastic process \( x^t_i \) is "close" to \( \underline{\omega}^i \), denoted \( x^t_i \approx \underline{\omega}^i \) for any \( i \) if,

for any \( \varepsilon, \delta > 0 \), there exists a \( \beta \) sufficiently close to 1 such that

\[
\delta > (1 - \beta) \sum_{t=0}^{\infty} \beta^t \sum_{s' \in N_i^t} \text{prob}(s^t | s^0)
\]

In other words, we say that the allocation process \( x^t_i \) is "close" to agent \( i \)'s long-run average endowment if the time-discounted probability that consumption deviates from the perfect risk sharing allocation by more than a given amount is small.

**Theorem 19** Suppose \( Du^i(x) \) is (weakly) convex, for any \( i \in I \). Any Financial market equilibrium allocation process of the economy with idiosyncratic risk, \( x^*i \), is "close" to perfect risk sharing,

\[
x^*i \approx \underline{\omega}^i.
\]
The proof involves constructing a budget feasible plan, a stochastic process $x^i_t$ for any $i$, whose utility is almost that of constant average consumption:

$$
\lim_{\beta \to 1} \left| E \left( \sum_{t=0}^{\infty} u^i(x^i_t) \right) - \frac{1}{1-\beta} u(\bar{x}^i) \right| = 0
$$

A crucial step in the argument is establishing that the riskless interest rate, $\frac{1}{q_t}$, is bounded above, with a bound close to 1, if $Du^i(x)$ is (weakly) convex, for any $i \in I$ (that is, if all agents have preferences for precautionary savings). This is important because the budget feasible plan constructed in the proof is financed by borrowing, and a low interest rate makes borrowing easy. A simple continuity argument then implies the result that $x^* \approx \bar{x}^i$.

In general, in the presence of aggregate risk, market incompleteness matters even if endowment processes are stationary (i.e., shocks are transitory). The reason is the following. When there is aggregate risk, the upper bound on the interest rate need not obtain; when the aggregate endowment is low, many traders will want to borrow, and this demand for loans may drive up the riskless interest rate. A high interest rate interferes with risk sharing because it makes borrowing difficult. Summing up, aggregate risk matters because it affects asset prices.

When there is more than one consumption good, market incompleteness matters again, even without aggregate risk. The reason is that commodity prices provide another source of untraded risk.

We conclude that in a one-good economy populated by infinitely-lived, patient agents, market incompleteness will not matter if shocks are transitory and risk is purely idiosyncratic. When there is aggregate uncertainty or more than one consumption good, market incompleteness matters, in general.

It is clear that Levine and Zame’s argument requires stationary individual endowments processes (otherwise $\bar{x}^i$ is not defined). Constantinides and Duffie, *Journal of Political Economy* 1996, show a partial converse: a particular non-stationary individual endowment process, when markets are incomplete, essentially any stochastic discount factor in terms of aggregate consumption.

Consider the idiosyncratic and aggregate risk economy introduced above. Assume agents have identical CRRA instantaneous preferences with risk aversion parameter $\alpha$:

$$
u(x) = \frac{1}{1-\alpha} x^{1-\alpha}
$$

Assume agents can trade two assets, i) a long-lived equity in positive (= 1) net supply, paying dividend $d_t = d(a_t) > 0$, and ii) a bond paying 1 in every state $a, s \in A \times S$. Let $x_t$ denote aggregate consumption at time $t$. Define

$$
\omega^i_t = \delta^i_t x_t - d_t, \text{ for any } t
$$
where
\[ \delta_t^i = \exp \left[ \sum_{\tau=1}^{t} \left( \eta^i_{\tau} y_{\tau} - \frac{(y_{\tau})^2}{2} \right) \right], \]
\[ y_t = \sqrt{\frac{2}{\alpha(\alpha + 1)}} \left[ \log \frac{m_t}{m_{t-1}} + \beta + \alpha \log \frac{x_t}{x_{t-1}} \right]^{1/2}, \]
and \( \eta^i_{\tau} \) are normal i.i.d. shocks, \( m_t \) is a stochastic discount factor consistent with no-arbitrage. Note that the fact that the driving shocks \( \eta^i_{\tau} \) affect cumulatively the individual share of aggregate income, as opposed to individual income, implies that individual income is not stationary.

It follows that:
\[ \sum_i \delta_t^i = \int \delta(s) \text{prob}(s) ds = 1 \]
\[ MRS_i^t = \frac{m_t}{m_{t-1}}, \text{ for any } i \]
and hence at equilibrium each agent consumes his own endowment. Finally, note that

i) the stochastic discount factor \( \frac{m_{t+1}}{m_t} \) can be expressed as a function of (the ratio of) aggregate consumption \( \frac{x_{t+1}}{x_t} \) with a degree of freedom represented by \( (y_{t+1})^2 \):
\[ E \left( R_{t+1} \beta \left( \frac{x_{t+1}}{x_t} \right)^{-\alpha} \exp \left( \frac{\alpha(\alpha + 1)}{2} (y_{t+1})^2 \right) \right) = 1, \]
for any return \( R_{t+1} \) in the span of the asset space; and

ii) \( (y_{t+1})^2 \) is the variance of the cross-sectional distribution of \( \log \frac{x_{t+1}}{x_t} \):
\[ \log \frac{x_{t+1}}{x_t} = \log \frac{\delta_{t+1}^i}{\delta_t^i} \sim N \left( -\frac{(y_{t+1})^2}{2}, (y_{t+1})^2 \right). \]

4 Macro with incomplete markets

In this section we consider two economies with incomplete markets and idiosyncratic shocks which have been studied in macroeconomics. The first, referred to as Bewley economies, are economies characterized by:

agents face stationary idiosyncratic endowment shocks (with/out an aggregate component), but
can only trade a riskless bond, typically with a no-short-sales constraint.

The second class of economies, referred to as individual risk economies, are characterized by:

agents face stationary idiosyncratic shocks (with/out an aggregate component) on the rate of return of savings, but can only trade a riskless bond, typically with a no-short-sales constraint.

Both Bewley economies and individual risk economies can be somewhat extended to allow for production. We discuss these extension in a future chapter.

4.1 Bewley economies

The prototypical Bewley economy is an economy with idiosyncratic shocks in which asset trading is restricted to a bond, which trades at time \( t \) for a price \( q_t \) normalized 1 for any \( t \geq 0 \), and pays \( 1 + r_{t+1} \) at time \( t + 1 \). This economy, originally studied by Huggett (1993); see Ljungvist-Sargent (2004), ch. 17.

This economy is straightforwardly modified to the case in which agents face a no-borrowing constraint, \( z_t \geq 0 \), and the rate of return on savings is an exogenous sequence \( r_t \); see Aiyagari (1994) and Ljungvist and Sargent (2004), ch. 17. An equilibrium is still characterized by a policy function of the form

\[
z_{t+1} = g_t(z_t, s_t),
\]

and a distribution \( \lambda_t(z_t, s_t) \), defined recursively from an initial given distribution \( \lambda_0(z_0, s_0) \). The distribution of wealth at time \( t \) is:

\[
\lambda_t(z_t) = \sum_{s_t \in S} \lambda_t(z_t, s_t).
\]

The limit distribution of wealth, if it exists, satisfies:

\[
\lambda(z) = \lim_{t \to \infty} \lambda_t(z_t).
\]

Proposition 20 The limit distribution of wealth in Bewley economies, \( \lambda(z) \), has thin tails, that is, all its moments are well defined.

This is perhaps problematic in lieu of the evidence that the distribution of wealth in the U.S. (as well as in many other developed countries; see Benhabib-Bisin-Zhu (2009) for a summary of this evidence).

\(^4\)We drop the dependence of the policy function from the whole sequence \( r_t \), for notational simplicity.
4.2 Bewley economies with production

A straightforward modification/reinterpretation/extension of this economy, has production to endogeneize the rate of return (on capital; re-interpret wealth as capital) and a wage rate. Let $F(K_t, N_t)$ the aggregate production function at time $t$, in terms of aggregate capital $K_t$ and labor $N_t$. Assume e.g., that labor supply is fixed and $\omega_t(s) = w_t s$, where $w_t$ is the wage rate at time $t$, and the shock $s$ is interpreted as labor productivity. Then, defining

$$K_{t+1} = \int \lambda_{t+1}(z_{t+1})dz_{t+1}, \quad N_t = N = \sum_{s \in S} \lambda(s)s$$

we have

$$r_t = \frac{\partial F(K_t, N)}{\partial K_t}, \quad w_t = \frac{\partial F(K_t, N)}{\partial N_t}.$$  

Aggregate shocks to the production function can be easily added. Typically we write it as $a_t F(K_t, N_t)$.

**Proposition 21** A financial market equilibrium for a Bewley economy with production is generically constrained Pareto inefficient.

This is the result, mentioned above, and due to Davila, Hong, Krusell, Rios-Rull (2005). The proof is obtained essentially by comparing the first order conditions at equilibrium with those at a constrained Pareto optimum. We only need to prove the result at an (ergodic) stationary equilibrium. Let $\lambda(z)$ denote the stationary distribution of wealth at equilibrium. Let the stationary interest rate and wage rate be denoted, respectively, $r(K)$ and $w(K)$ to make their dependence from the aggregate wealth of the economy explicit: $r = \frac{\partial F(K,N)}{\partial K}$ and $w = \frac{\partial F(K,N)}{\partial N}$.

The first order conditions at equilibrium include

$$\frac{d}{dx} u(r(K)z + w(K)s - z') \geq \beta r(K) \sum_{s' \in S} \text{prob}(s'|s) \frac{d}{dx} u(r(K)z' + w(K)s' - g(z,s));$$

while the first order conditions at a Pareto efficient allocation include

$$\frac{d}{dx} u(r(K)z + w(K)s - z') \geq \beta r(K) \sum_{s' \in S} \text{prob}(s'|s) \frac{d}{dx} u(r(K)z' + w(K)s' - g(z,s)) +$$

$$+ \beta \sum_{s' \in S} \text{prob}(s'|s) \left( \frac{dr(K)}{dK} z' + s' \frac{dw(K)}{dK} \right) \lambda(z)$$

Formally, and in general, they depend on the stationary distribution $\lambda(z,s)$. But in this economy, where the production function aggregates all individual wealth, this reduces to a dependence on $K = \int \lambda(z)dz$. 

20
As we are used to do by now, we avoid the details of the argument showing that, generically, the first order conditions at equilibrium and at the Pareto efficient allocation are different; that is, that the extra term in the conditions for Pareto efficiency

\[ +\beta \sum_{s' \in S} \text{prob}(s'|s) \frac{d}{dx} \left[ u \left( r(K)z' + w(K)s' - g(z, s) \right) \right] \left( \frac{dr(K)}{dK} z' + s' \frac{dw(K)}{dK} \right) \lambda(z) \]

induces generically a different policy function.

Intuitively, constrained Pareto inefficiency follows from \( r \) and \( w \) being endogenous.

As for the limit distribution of wealth, the same result holds for Bewley economies with production as with standard Bewley economies, as the interest rate is constant (or only dependent on aggregate risk \( a_t \)).

**Proposition 22** The limit distribution of wealth in Bewley economies with production, \( \lambda(z) \), has thin tails, that is, all its moments are well defined.

### 4.3 Investment risk economies

Suppose instead each agent faces an idiosyncratic exogenous rate of return on savings, a mapping \( r : \{1, 2, ..., S\} \rightarrow \mathbb{R}_+ \). We maintain the assumption that each agent can only save (not borrow) at the rate \( r_z \geq 0 \) for any \( t \geq 0 \). Given the process \( r \), each agent solves:

\[ v_t(z, s) = \max_{z' \geq 0} u \left( (1 + r(s)) z + \omega(s) - z' \right) + \beta \sum_{s' \in s} \text{prob}(s'|s) v_t(z', s') \]

The solution of this problem is a policy function of the form:

\[ z_{t+1} = g(z_t, s_t). \]

Let

\[ \lambda_{t+1}(z_{t+1}, s_{t+1}) = \sum_{s_t \in S} \text{prob}(s'\|s) \int_{z_t:z_{t+1}=g(z,s)} (z_t, s_t) \lambda_t(z_t, s_t) dz_t, \text{ for any } t \geq 0. \]

Then the distribution of wealth at time \( t + 1 \) in this economy is defined recursively, from an initial given distribution \( \lambda_0(z_0, s_0) \):

\[ \lambda_{t+1}(z_{t+1}) = \sum_{s_{t+1} \in S} \lambda_{t+1}(z_{t+1}, s_{t+1}) \]

The limit distribution of wealth, if it exists, satisfies:

\[ \lambda(z) = \lim_{t \to \infty} \lambda_t(z). \]

---

6Formally, the realization for \( s^{-1} \) is also given. Assume also agents are endowed with no portfolio positions on bonds at time 0: \( z_i'(s^{-1}) = 0 \), for all \( i \in I \).
Conjecture 23  The limit distribution of wealth in Investment risk economies, $\lambda(z)$, is a power law in the tail:

$$\lim_{z \to \infty} \lambda(z) \propto z^{-\alpha}, \quad \alpha > 1.$$ 

As a consequence it displays thick tails (let $k$ be the smallest integer such that $\alpha < k$, then the Pareto distribution with tail $z^{-\alpha}$ has no moments of order $k$ and higher).

4.4 Other incomplete market economies

Kubler-Schmedders

Heaton-Lucas