1 Time and Uncertainty; or Savings and Portfolio Allocation

• In this section, by using the techniques that we have already learned, we examine i) the choice problem of a consumer saving over time, and ii) the choice problem of allocating wealth to a portfolio of assets with risky returns.

• In order to focus on saving and consumption over time and under uncertainty, we shall restrict our attention to examples in which there is a single consumption good.

1.1 Choice over time; or Saving

• We denote the income that the consumer will have in each period by \((m_1, m_2)\), and the amount of consumption in each period by \((c_1, c_2)\).

• Assume that the price of the consumption good in both periods is constant at 1. Hence, we assume that there is no inflation or deflation.

• Assume that the consumer can borrow and save money at some fixed interest rate \(r\).

1.1.1 The intertemporal budget constraint

• First, note that by our assumptions if the consumer saves in the first period so that \(m_1 > c_1\) then she will earn interest on the amount she saved, \(m_1 - c_1\). On the other hand, if the consumer borrows in the first period so that \(m_1 < c_1\) then she will have to pay interest on the amount she borrowed \(c_1 - m_1\).
Let $S$ denote the amount the consumer takes to or from period 1. If $S < 0$ (resp. $S > 0$) we say the consumer borrows (resp. saves). The budget constraint at time $t = 1$ is
\[ c_1 + S = m_1, \]
indicating that the consumer uses his income $m_1$ either to consume or to save.

The budget constraint in period $t = 2$ is then:
\[ c_2 = m_2 + (1 + r)S, \]
indicating that the consumer in the second period consumes his income plus the proceeds of his saving (minus the repayment of the borrowing, if $S < 0$).

We can now write the intertemporal budget constraint as follows:
\[ c_1 + \frac{1}{1 + r} c_2 = m_1 + \frac{1}{1 + r} m_2 \]
The right hand side of the budget constraint is called permanent income; it is the income of the agent evaluated as of time $t$.

Draw Figure

Consumption in period 1 and consumption in period 2 are two distinct goods. If you make the price of period 1 consumption as the numeraire (normalization), then the price of period 2 consumption is the amount of period 1 consumption that you have to give up to get an extra unit of consumption in period 2: this price is $\frac{1}{1+r}$. Thus, the maximum amount of period 1 consumption that the consumer can buy, if he borrows as much money as she can possibly repay in period 2 is $m_1 + m_2/(1 + r)$.

1.1.2 Intertemporal preferences for consumption

We write the consumer preferences for consumption at time $t = 1$ and $t = 2$ as follows:
\[ u(c_1) + \beta u(c_2), \quad 0 < \beta < 1 \]
• We have assumed that i) preferences for consumption are the same in each period, and ii) future utilities are discounted at rate $\beta < 1$.

• The function $u : \mathbb{R} \to \mathbb{R}$ is assumed continuous and concave (more on this later). We shall also assume:

$$\lim_{c \to 0} u(c) = \infty$$

that is, the marginal utility of consumption for an agent not consuming anything is arbitrarily high (infinity).

1.1.3 Choice over time, for given $r$

• The agent’s problem will be

$$\max_{c_1, c_2} u(c_1) + \beta u(c_2)$$

subject to

$$c_1 + \frac{1}{1+r} c_2 = m_1 + \frac{1}{1+r} m_2$$

• The first order conditions for the maximization problem (derive them by the Lagrangian method) include:

$$\frac{u'(c_1)}{\beta u'(c_2)} = 1 + r$$

(1)

Develop your intuition about what the condition means by drawing the corresponding indifference curve diagram.

Draw Figure

• Assume for simplicity that $\beta = \frac{1}{1+r}$. Then the first order condition (1) implies

$$c_1 = c_2;$$

and hence, using the budget constraint:

$$c_1 = c_2 = \frac{1}{2} \left( m_1 + \frac{1}{1+r} m_2 \right)$$

(2)
that is, consumption at time \( t = 1 \) (and \( t = 2 \)) depends on *permanent income*, not just today’s income. This is the fundamental insight in the modern (after the 60’s) theory of consumption, due to Milton Friedman and Franco Modigliani.

- Convince yourselves that the dependence of consumption on *permanent income* does not depend on our assumption that \( \beta = \frac{1}{1+r} \) (made just for illustration purposes).

- *Class example* - consider a consumer with an intertemporal utility function \( u(c_1, c_2) = c_1 c_2 \) (this formulation is equivalent to the one before in this notes, but with \( \beta = 1; \) can you see this?). The interest rate \( r = 10\% \) and the consumer’s income in period 1, \( m_1 = 100 \), and in period 2, \( m_2 = 121 \). Then, the consumer’s budget constraint is

\[
c_1 + \frac{c_2}{(1 + r)} = m_1 + \frac{m_2}{(1 + r)}
\]

and the ratio of the price of period 1 consumption to the price of period 2 consumption is

\[
\frac{1}{1+r} = 1 + r
\]

The consumer choice is a consumption bundle \((c_1, c_2)\) such that

\[
MRS = -\frac{M_u_1}{M_u_2} = -\frac{c_2}{c_1} = -(1 + r)
\]

That is,

\[
\frac{c_2}{c_1} = 1.1
\]

By using this condition and the budget constraint above we find that \( c_1 = 105 \) and \( c_2 = 115.5 \).

1.1.4 **Savings, for given \( r \)**

- Recall that we say that a consumer is a *saver* if \( m_1 > c_1 \) and is a *borrower* if \( m_1 < c_1 \).
Suppose the interest rate in the economy $1 + r$ increases. What happens to the savings of a saver? And, to the borrowings of a borrower? A figure helps: consider first the case in which $m_1 = 0$, so that the agent is certainly (under our assumptions on preferences) a borrower; and consider than the case in which $m_2 = 0$ and the agent is a saver.

Draw Figure

1.1.5 The interest rate $r$

The interest rate is just any other price. It is determined at a competitive equilibrium by the market clearing (demand-equal-supply) condition. In this case what is clearing is the demand and the supply of loans, that is, borrowing must equal lending. Suppose there are many agents in our economy, indexed by $i \in I$. Let the income of an arbitrary agent $i$ be denoted by $m_i$, and his consumption, respectively at time 1 and 2, by $c_i^1$ and $c_i^2$. The market clearing conditions in this economy are then written formally,

$$\sum_{i \in I} c_i^1 - m_i^1 = 0, \sum_{i \in I} c_i^2 - m_i^2 = 0$$

Note that the first condition captures borrowing equal lending. What is the role of the second condition then? In fact, none: if $\sum_{i \in I} c_i^1 - m_i^1 = 0$, then, by summing all budget constraints, it is necessarily the case that $\sum_{i \in I} c_i^2 - m_i^2 = 0$).

Consider as an example the case in which there is only one agent (or a lot of identical agents). Then the market clearing conditions are (no index $i$ is necessary, we drop it):

$$c_1 = m_1$$

$$c_2 = m_2$$

In this case an equilibrium requires that $1 + r$ is such that the agent does not save nor borrows (who would he be borrowing from? who would consume his savings and pay him back next period?)
Consider instead the case in which the economy is populated by a fraction \( \mu \) of agents 1 with income \( m_1 = 0, m_2 > 0 \), and a fraction \( 1 - \mu \) of agents 2 with income \( m_1^2 > 0, m_2^2 = 0 \). In this case, \( 1 + r \) will be determined so that

\[
-\mu c_1^1 + (1 - \mu) (m_1^2 - c_2^2) = 0 \\
\mu (m_2^1 - c_1^1) - (1 - \mu) c_2^2 = 0
\]

that is, the aggregate borrowing of one group of agents must equal the savings of the other group.

A classical model of equilibrium interest rate is one where in each time period there is a group of *young* agents (with no income) and a group of middle-aged (with positive income); or else one where in each time period there is a group of *middle-aged* agents (with positive income) and a group of retired agents (with no income).

### 1.2 Inflation and Monetary Neutrality

- It turns out that it is easy to modify our analysis to deal with the case of inflation. To do this, let us suppose that the consumption good has different prices in each period, \( p_1 \) and \( p_2 \). Given the prices of consumption in the two periods, \( p_1 \) and \( p_2 \), the rate of inflation, \( \pi \), is given by

\[
\pi = \frac{\Delta p}{p_1} = \frac{p_2 - p_1}{p_1}
\]

- Suppose the income of any arbitrary agent \( i \) in our economy, \( m_i^1 \) in the first period and \( m_i^2 \) in the second, are *real*, that is in units of goods. Let \( i \) denote the interest rate (we called it \( r \) before, but bear with me for a couple of lines). Then the his/her budget constraint is:

\[
p_1 c_i^1 + \frac{1}{1 + i} p_2 c_i^2 = p_1 m_i^1 + \frac{1}{1 + i} p_2 m_i^2
\]

\( ^1 \)These models, called *overlapping generation*, were developed by Maurice Allais and Robert Samuelson in the 50’s.
• It is convenient to normalize the price of period 1 consumption to be 1 and let the price of period 2 consumption be $p$ (we can do this, without loss of generality, remember?). Then,

$$\pi = \frac{p_2 - p_1}{p_1} = p_2 - 1$$

and the budget constraint becomes:

$$c_1^i + \frac{1 + \pi}{1 + i} c_2^i = m_1^i + \frac{1 + \pi}{1 + i} m_2^i$$

• Now, we are ready distinguish between the nominal interest rate, $i$, and the real interest rate, $r$, which we define by

$$1 + r = \frac{1 + i}{1 + \pi}$$

By rearranging we get

$$r = \frac{1 + i}{1 + \pi} - 1 = \frac{i - \pi}{1 + \pi}$$

which is the exact expression for the real interest rate. However, note that when there is no inflation the real and the nominal interest rates are equal.\(^2\)

• **Class example** - consider the consumer in the above example just that now there happened to be an inflation rate of 6% (that is $p_2 = 1.06 p_1$ and if $p_1$ is normalized to 1 then $p_2 = 1.06$). Suppose the nominal interest rate is 10%, then the real interest rate is given by

$$r = \frac{i - \pi}{1 + \pi} = \frac{1 - 0.06}{1 + 0.1} = 0.036$$

The consumer choice is a consumption bundle $(c_1, c_2)$ such that

$$MRS = \frac{M u_1}{M u_2} = -\frac{c_2}{c_1} = -(1 + \rho)$$

\(^2\)Note that for low inflation rates the real interest rate is approximately

$$r \approx i - \pi$$
That is,
\[ \frac{c_2}{c_1} = 1.036 \]

By using this condition and the budget constraint above we find that 
\[ c_1 = 108.4 \text{ and } c_2 = 112.3. \]

- Does inflation have any effect on the economy? Are equilibrium quantities changed because of inflation? Are prices? The answer to these questions is straightforward once you noticed that inflation only affects the units of the budget constraint. The equilibrium condition determines the real interest rate; the nominal interest rate is determined by adding inflation. If the preferences and the incomes of the economy are not changed, the equilibrium real interest rate will not change because of inflation, nor will equilibrium allocations. **Make sure you really understand this!**

- The situation is different if the agents’ income are nominal rather than real, that is, if \( m_1^i \) and \( m_2^i \) are in monetary units. In this case, the budget constraint of agent \( i \) becomes:

\[
 c_1^i + \frac{1 + \pi}{1 + i} c_2^i = m_1^i + \frac{1}{1 + i} m_2^i 
\]

and inflation acts as a tax, diminishing the purchasing power of each agent income at time 2, and hence diminishing his/her permanent income.

- We would expect that in the short run unexpected inflation has an effect on wages, and hence on agents’ income, but not in the long run. See Bob Lucas’ paper in my webpage on Monetary Neutrality. Monetary neutrality is nothing else that the equilibrium property we have just discussed, that is, inflation has no effects if incomes are real (in the long run). We call it Monetary Neutrality because we usually think of the inflation rate as essentially the same as the growth rate of the quantity of money. Can you see why is this the case?

1.3 Choice under Uncertainty; or Portfolio Allocation

We first introduce random variables and expected values.
1.3.1 Probability

- Let $\Omega = \{\omega_1, \omega_2\}$ be the set of all possible events; for instance, $\omega_1 = \text{it rains}$, $\omega_2 = \text{it does not rain}$.

- Consider the following probability distribution (on events):
  - event $\omega_1$ with probability $p$, $0 < p < 1$, and event $\omega_2$ with probability $1 - p$

- Consider the random variable $y$ taking values
  $$y_1 = y(\omega_1), \ y_2 = y(\omega_2)$$

- The expected value of $y$ is:
  $$py_1 + (1 - p)y_2,$$
denoted $Ey$.

- The variance of $y$ is:
  $$p(y_1 - Ey)^2 + (1 - p)(y_2 - Ey)^2,$$
denoted $\sigma^2(y)$; $\sigma(y) = \sqrt{\sigma^2(y)}$ is the standard deviation of $y$.

  Both the variance and the standard deviation are measures of the variability of $y$. Note for instance that if $y$ is constant, then $y = Ey$ and $\sigma^2(y) = \sigma(y) = 0$.

- Consider two random variables $y$ and $z$ taking values
  $$y_1 = y(\omega_1), \ y_2 = y(\omega_2), \ z_1 = z(\omega_1), \ z_2 = z(\omega_2)$$

  The covariance of $y$ and $z$ is:
  $$p(y_1 - Ey) \cdot (z_1 - Ez) + (1 - p)(y_2 - Ey) \cdot (z_2 - Ez),$$
denoted $\text{cov}(y, z)$; $\frac{\text{cov}(y, z)}{\sigma(y) \cdot \sigma(z)}$ is the correlation coefficient of $y$ and $z$.

  The covariance and the correlation coefficient of $y$ and $z$ are a measure of how the variables $y$ and $z$ move together, that is, in our example the covariance and the correlation coefficient are maximal when either $y_1$ and $z_1$ are both greater than their respective means $Ey$ and $Ez$; or else $y_1$ and $z_1$ are both smaller than their respective means $Ey$ and $Ez$. 
1.3.2 Expected Utility

- Consider an agent trying to evaluate the utility associated to consuming the random variable \( y \) (defined above).
- Let the utility function of the agent be \( u(c) \). His utility when consuming \( y \) is:
  \[
  u(y_1) \text{ with probability } p, \text{ and } u(y_2) \text{ with probability } 1 - p
  \]
- We assume an agent evaluates the utility of a random variable \( y \) by expected utility; that is,
  \[
  pu(y_1) + (1 - p)u(y_2)
  \]
- In other words, we assume that, when facing uncertainty, agents maximize expected utility. A lot of experiments document failures of this assumption in various circumstances. A lot of theoretical work addresses the failure, postulating different (still optimizing) behavior on the part of agents. Most economic theory still uses the assumption in first approximation.

1.3.3 Risk Aversion

- Does an agent with utility function \( u(c) \) prefer to consume the random variable \( y \) or its expected value \( Ey \)?
- **Theorem 1** Consider an agent with preferences represented by a continuous utility function \( u : \mathbb{R} \to \mathbb{R} \). Assume the agent is an expected utility maximizer. Then the agent will prefer the expected value \( Ey \) to the (non-degenerate) random variable \( y \) iff \( u(c) \) is strictly concave.
  
  **Proof.** Draw a strictly concave utility function, and represent in the figure \( u(Ey) \) and \( pu(y_1) + (1 - p)u(y_2) \). It will be apparent that
  \[
  u(Ey) > pu(y_1) + (1 - p)u(y_2)
  \]
  iff \( u \) is strictly concave. □
- If an agent has strictly concave utility function, we say he is risk averse.
1.3.4 Portfolio Selection

- Consider an economy with two financial assets:
  - A bond, which pays a riskless return $R$;
  - A stock, which pays a return $S$, a random variable taking values $S_1$ with probability $p$ and $S_2$ with probability $1 - p$.

- Consider an agent with an arbitrary sum of money, $w$, choosing the share $\alpha$ of bond and $(1 - \alpha)$ of stock in his portfolio to maximize the expected utility of his consumption after returns are payed.

- The agent will maximize, by choice of $\alpha$:

$$pu(\alpha Rw + (1 - \alpha) S_1 w) + (1 - p) u(\alpha Rw + (1 - \alpha) S_2 w)$$

- The first order condition for the maximization problem is:

$$pu'(c_1) (R - S_1) + (1 - p) u'(c_2) (R - S_2) = 0$$

(careful: it is only satisfied for an interior solution; by construction we must require $0 \leq \alpha \leq 1$, since $\alpha$ is a share !)

- Examine the condition (being careful about corners):
  - if $R > ES$, the condition is never satisfied, and $\alpha = 1$
  - if $R = ES$, the condition is satisfied at $\alpha = 1$
  - if $R < ES$, the condition is satisfied for some $\alpha < 1$
  - these answers are independent of $w$

- We conclude that, in this economy, if the stock has an expected return greater than the bond, the agent will always have some stocks in his portfolio.