1 The Financial Economy

Consider a two period economy, $t=0,1$, with uncertainty. At $t=1$, one of several possible states of the world will be realized, denoted by $s \in S =: \{1,\ldots,S\}$. The probability distribution over the states is supposed to be known to all agents.\footnote{Sometimes, the set of states will be enlarged to $\{0,1,\ldots,S\}$ to include the initial state.} This economy is populated by individuals $i \in I =: \{1,\ldots,I\}$ and firms $h \in : \{1,\ldots,H\}$. One single physical good is consumed and produced in each state.

**Endowments.** Each agent is endowed with a vector $\omega^i = (\omega^i_0, \omega^i_1, \ldots, \omega^i_S)$ of physical goods. We will assume

$\textbf{A1.} \; \omega^i \in \mathbb{R}^{S+1}_+$ for all $i \in I$.

**Preferences.** Let $u^i : \mathbb{R}^{S+1}_+ \rightarrow \mathbb{R}$ denote agent $i$’s utility function. We will assume

$\textbf{A2.} \; u^i$ is continuous, strongly monotonic, strictly quasi-concave and smooth, for all $i \in I$.

**Firms.** Each firm is endowed with a production set $Y^h \in \mathbb{R}^{S+1}_+$. We will assume

$\textbf{A.3.} \; Y^h \in \mathbb{R}^{S+1}_+$ is a convex cone, for all $h \in \{1,\ldots,H\}$.

**Asset structure.** Assets $j \in \{1,\ldots,J\}$ are traded in the economy. Let the payoff vector of asset $j$ be denoted by $a^j = (a^j_1,\ldots,a^j_S)$, where $a^j_s$ are the units of the single good that asset $j$ pays in state $s$. To summarize the payoffs of all the available assets, define the $S \times J$ asset payoff matrix

$$A = \begin{pmatrix}
a^j_1 & \cdots & a^j_1 \\
\vdots & \ddots & \vdots \\
a^j_S & \cdots & a^j_S
\end{pmatrix}.$$  

It will be convenient to define $A_s$ to be the $s$-th row of the matrix. Note that it contains the payoff of each of the assets in state $s$.

Whenever $\text{rank}(A) = S$ we shall say that the economy is one of complete (financial) markets. In this case, naturally, $J \geq S$. Whenever $J > \text{rank}(A)$ we shall say that the economy has redundant assets.
1.1 Equilibrium

Let \( x^i_s \) denote the amount of the physical good that agent \( i \) consumes in state \( s \). Let \( q = (q_1, \ldots, q_J) \in R^J \), denote the asset price vector. Quantities will be row vectors and prices will be column vectors.

**Agent \( i \)'s maximization problem:** Given prices \( q \) and the asset structure \( A \), agent \( i \) picks a consumption vector \( x^i \in R^{S+1} \) and a portfolio \( z^i \in R^J \) to

\[
\max u^i(x^i) \\
\text{s.t.} \\
x^i_0 - \omega^i_0 + qz^i = 0 \\
x^i_s - \omega^i_s = A_s z^i, \text{ for } s = 1, \ldots, S. 
\] (1)

**Firm \( h \)'s maximization problem:** Given prices \( q \), asset structure \( A \), and production set \( Y^h \), firm \( h \) picks a production vector \( y^h = (y^h_0, y^h_1) \in R^{S+1} \) and a financing portfolio \( \theta^h \in R^J \) to

\[
\max \max q\theta^h - y^h_0 \\
\text{s.t.} \\
(y^h_0, y^h_1) \in Y^h \\
y^h_1 = A\theta^h 
\] (2)

Note that in general \( y^h_0 < 0 \) represent firms \( h \)'s costs of production and hence \( \theta^h \) represents its financing of the production plan \( y^h \).

**Competitive equilibrium:**

**Definition 1** \( (x \in R^{(S+1)I}, z \in R^{JI}, y \in R^{(S+1)H}, \theta \in R^{I'H}, q \in R^J) \) is said to be a Financial Markets Equilibrium for the economy if

- For all \( i \in I \), \( (x^i, z^i) \) is an argmax of problem (1).
- For all \( h \in H \), \( (y^h, \theta^h) \) is an argmax of problem (2).

**Markets clear:**

\[
\sum_{i \in I} x^i_s - \omega^i_s - \sum_{h \in H} y^h_s = 0, \text{ for } s = 0, 1, \ldots, S \] (3)

\[
\sum_{i \in I} z^i - \sum_{h \in H} \theta^h = 0. \] (4)

1.2 No Arbitrage

Before introducing Modigliani-Miller, we shall invest some time in understanding the implications of no-arbitrage.

For convenience, define the \((S + 1) \times J\) matrix

\[
W = \begin{bmatrix} -q \\ A \end{bmatrix}.
\]
**Definition 2** (No-Arbitrage Condition) We will say that $W$ satisfies the No-Arbitrage Condition if there does not exist $z \in \mathbb{R}^J$ such that $Wz > 0$, i.e. all components of the $(S+1)$ vector $Wz$ are non-negative and at least one is positive.

The No-Arbitrage condition can be equivalently formulated in the following way. Define the span of $W$ to be

$$< W > = \{ \tau \in \mathbb{R}^{S+1} : \tau = Wz, \ z \in \mathbb{R}^J \}.$$ 

Note that this set contains all the feasible wealth transfers with the given asset structure $A$. Now, we can say that $W$ satisfies the No-Arbitrage condition iff

$$< W > \cap R^{S+1}_+ = \{0\}.$$ 

Clearly, requiring that $W = (-q, A)$ satisfies the No-Arbitrage condition is a milder requirement than equilibrium. Given that we assumed strong monotonicity in preferences, No-Arbitrage is equivalent to requiring the agent’s problem to be well defined. The next result is remarkable since it provides a foundation for asset pricing with milder requirements than imposing equilibrium in the economy.

**Theorem 3** [No-Arbitrage theorem]

$$< W > \cap R^{S+1}_+ = \{0\} \iff \exists \pi \in R^{S+1}_+ \text{ such that } \pi W = 0.$$ 

First, observe that there is no uniqueness claim on the $\pi$, just existence is claimed. Next, notice how $\pi W = 0$ provides a pricing formula for assets:

$$\pi W = \begin{pmatrix} ... \\ -\pi_0 q^j + \pi_1 a_1^j + ... + \pi_S a_S^j \end{pmatrix} = \begin{pmatrix} 0 \\ ... \end{pmatrix}$$

and, rearranging, we obtain for each asset $j$,

$$q^j = \frac{\pi_1}{\pi_0} a_1^j + ... + \frac{\pi_S}{\pi_0} a_S^j,$$

(5)

or$^3$ defining $p_s = \frac{\pi_s}{\pi_0}$, for $s=1,...S$, we obtain

$$q^j = p_1 a_1^j + ... + p_S a_S^j.$$ 

(6)

Two final remarks to this section. First, note that if we had an asset that paid one unit of physical good in state $s$ and nothing in all other states (an Arrow security), the price of such an asset, as dictated by (5), would be $p_s$.

$^3$Note how the positivity of all components of $\pi$ is necessary to obtain (5).
Secondly, should we expect to find a unique $p$ vector? Or, if there are many, how many? To address this question, notice how we can express the system of equations given by (6) in a simplified manner. Note that it is a system of $J$ equations with $S$ unknowns. Define the set of solutions to that system as

$$R(q) = \{ p \in R^S_+ : q = pA \}.$$ 

$R(q)$ will have generically dimension $S - \text{rank}(A)$. In particular, if markets are complete, $\text{rank}(A) = S$, the solution set has dimension zero, and there is a unique $p$ vector that solves (6).

## 2 Modigliani-Miller

The Modigliani-Miller Theorem is a couple of irrelevance propositions. In words,

**MM1** The value of the firm is independent of its capital structure (the proportion of debt and equity used to finance the firm’s operations).

**MM2** Provided the firm’s investments (projects) are unchanged, its value is independent of the firm’s dividend policy.

We shall study MM1.\(^4\) We will let MM2 to the reader. It is conceptually identical to MM1 but requires possibly a 3 period economy to be properly stated.

We start with a Lemma that is at the core of the result.

**Lemma 4** Assume $J = S$ and $\text{rank}(A) = S$ (complete markets). Consider an equilibrium allocation and prices $(x, y, z, \theta, q)$. Then $(x, y)$ constitutes an equilibrium allocation for any other asset payoff matrix $A'$ provided $\text{rank}(A') = S$.

**Proof.** Consider the consumer problem (1). Using $\text{rank}(A) = S$ the constraints can be written as

$$x^0_i - \omega^0_i + qA^{-1}(x^1_i - \omega^1_i) = 0 \quad (7)$$

$$x^1_i - \omega^0_i \in \text{span}(A) \quad (8)$$

Equation (8) is no restriction when $\text{rank}(A) = S$. Let $qA^{-1} =: p$, an $S$-dimensional vector.\(^5\) This reformulation of agent $i$’s problem is independent of $A$; also $z^i$ does not appear.

Consider now firm $h$’s problem, (2). In a similar manner, we can write firm $h$’s problem as

$$\max py^h_1 + y^h_0$$

s.t.

$$y^h \in Y^h \quad (9)$$

---

\(^4\)Actually we shall study a more general form of MM1 than usually stated. The added generality comes from free with our notation.

\(^5\)Note that, interpreting $p$ as state prices, equation (7) is a present discounted Arrow-Debreu budget constraint.
where we used \( p = qA^{-1} \) once more. This reformulation of firm \( h \)'s problem is independent of \( A \); also \( \theta^h \) does not appear.

Finally, notice that the market clearing condition (4) redundant as it is implied by (10).

We can now define an Arrow-Debreu equilibrium for the economy: allocation and prices \( (x \in \mathbb{R}^{I(S+1)}, y \in \mathbb{R}^{H(S+1)}, p \in \mathbb{R}^S) \) such that

For all \( i \in I \), \( x^i \) maximizes agent \( i \)'s utility subject to (7).

For all \( h \in H \), \( y^h \) is an argmax of problem (9).

Markets clear:

\[
\sum_{i \in I} x^i_s - \omega^i_s - \sum_{h \in H} y^h_s = 0, \text{ for } s=0,1,...,S \tag{10}
\]

It is immediate so see that equilibrium allocations, given \( A \), coincide with Arrow-Debreu equilibrium allocations. But since Arrow-Debreu equilibrium allocations are independent of \( A \), the statement follows.

We are now ready for MM1.

\textbf{Proposition 5 [Modigliani-Miller]} Assume \( J > S \) and \( \text{rank}(A) = S \) (complete markets with redundant assets). Each firm \( h \)'s value at equilibrium, \( q\theta^h - y^h_0 \), is independent of \( \theta^h \Theta^h = \left\{ \theta^h \in \mathbb{R}^J \mid y^h_1 = A\theta^h \right\} \), and hence the firm’s choice of \( \theta^h \in \Theta^h \) is indeterminate.

\textbf{Proof.} The statement follows directly from Lemma 4 , once it is realized that the Lemma trivially extends to the case \( J > S \) and \( \text{rank}(A) = S \).

One important remark is in order: Each firm \( h \) is price taker and it faces a linear financing technology, \( y^h_1 = A\theta^h \), for given production plan \( y^h \). As a consequence, given the prices \( q \) the firm’s financing decision are indeterminate simply by No-Arbitrage.\(^6\) If we properly interpret firm \( h \) as a mass 1 collection of infinitesimal firms, as we should to justify the price taking assumption, this is just saying that the financing decision of an infinitesimal firm is indeterminate. But the Modigliani-Miller results says more: it say that the indeterminacy extends to the financing plans of the whole firm \( h \), the whole mass! In other words, under our assumptions there are no implications regarding the observed debt-equity ratios (or any other statistics on the firms’ financing portfolio) except the trivial financing identity, \( y^h_1 = A\theta^h \) in our notation.

Let \( Y^h_1 = \{ y^h_1 \in \mathbb{R}^S \mid y^h \in Y^h \} \).

\textbf{Proposition 6 [Modigliani-Miller with incomplete markets]} Assume \( J > \text{rank}(A) \) and \( \text{rank}(A) < S \) (incomplete markets with redundant assets). If \( Y^h_1 \subseteq \text{span}(A) \), then the statement of Modigliani-Miller, Proposition 5, holds unchanged.

\(^6\)This is the analogous to saying that with a constant returns to scale technology the production plans of a firm are indeterminate.
Proof. We first show that Lemma 4 extends to the case \( J > \text{rank}(A) \) and \( \text{rank}(A) < S \). To this end we just need to use the No-Arbitrage theorem, theorem 3 to substitute \( q = Ap \) into agent \( i \)'s and firm \( h \)'s problems. While condition (8) cannot now be dispensed with, the Lemma holds. The Proposition follows then directly as previously. ■

The condition

\[ Y^h_1 \subseteq \text{span}(A) \]

implies that firm \( h \)'s cash flow is traded, independently of the firm’s activity.

Suppose this is not the case and firm \( h \)'s production set \( Y^h_1 \) contains production plans \( y^h_1 \) that she cannot finance by issuing existing securities, \( y^h_1 \notin \text{span}(A) \). Firm’s \( h \)'s problem then becomes:

\[
\begin{align*}
\max max & \quad q^h \theta + f_h(e^h) - y^h_0 \\
\text{s.t.} & \quad (y^h_0, y^h_1) \in Y^h \\
& \quad y^h_1 = A\theta^h + e^h
\end{align*}
\]

where \( e^h \) is the equity payoff of the firm and \( f_h : \mathbb{R}^S \to \mathbb{R} \) is a function mapping \( e^h \) into its value.\(^7\)

Agent \( i \)'s problem will include trading of equity \( E = [e_1, \ldots, e_H] \). Let \( \lambda^i \in \mathbb{R}^H \) denote agent \( i \)'s portfolio of equity and \( E_s \) the \( s \)-th row of the equity payoff matrix \( E \). Then agent \( i \)'s problem becomes:

\[
\begin{align*}
\max & \quad u^i(x^i) \\
\text{s.t.} & \quad x^i_0 - \omega^i_0 + qz^i + f \lambda^i = 0 \\
& \quad x^i_s - \omega^i_s = A_s z^i + E_s \lambda^i, \text{ for } s = 1, \ldots, S.
\end{align*}
\]

Two orders of problems arise:

The objective function of the firm: A firm \( h \) can not price univocally all elements of its production set. By no arbitrage at an equilibrium, \( \exists p \in \mathbb{R}^q_{++} \) that prices all existing assets, \( p \in R(q) \). But when markets are incomplete such \( p \) is not unique, and different \( p \in R(q) \) will price differently payoffs \( y^h_1 \notin \text{span}(A) \). Therefore, consider an equilibrium in which only assets \( j = 1, \ldots, J \) are traded and all firms’ production plans are in the span of \( A \). How can firm \( h \) evaluate the profits associated to a plan \( y^h \) which needs trading \( e^h \notin \text{span}(A) \)? (This is the right question, since each firm is to be interpreted as infinitesimal, and if a mass of firms trades shares containing a component that is not spanned by \( A \), then such component is in fact traded at equilibrium). Using \( pA\theta^h + pe^h - y^h_0 \), where \( p \in R(q) \), is a possibility, but there isn’t a unique such \( p \) (nor different vectors \( p \in R(q) \) produce the same value \( pe^h \) in general).

\(^7\)Note that we have to write the value of the component of cash flows which is not spanned by the existing securities as a function, because we do not in principle know that there is any linear representation of its value.
The price effects of (infinitesimal changes in) production plans: A firm $h$’s production choice $y^h$ may change the span of the securities traded by agents, and therefore has price effects that the firm cannot rationally anticipate in a competitive model. In fact, in our formulation, any infinitesimal amount of equity not in the span of $A$ opens almost-zero net supply trades in the equity (that is, consumers can go infinitely long and short in the equity). Therefore, even if at an equilibrium we could pick a specific vector $p \in R(q)$ to price equity, and hence the value of the firm was determined, it would change at equilibrium with infinitesimal changes of $y^h$.

These problems introduce inefficiencies in general (there is a large literature on this, with contributions of Dreze, Duffie-Shafer, Magill-Quinzii, Cass and many others). What is most important is that the price effects of (infinitesimal changes in) production plans are a sign of logical inconsistencies in the model. In such a context it is not even meaningful to ask Modigliani-Miller questions.

2.1 Stocks and Short Sales

The following modelling of stock issuances and short sales is instead logically consistent. We will introduce the model and then ask Modigliani-Miller questions to it.

Equity of each firm $h$ trade in the market at $t = 0$ at a price determined by the map $f_h : R^S \rightarrow R$, which will be determined at equilibrium. There is a given outstanding amount of equity of every firm, normalized to 1. Firm $h$ chooses the production plan which maximizes its value as in problem (11).

Firm $h$’s equity cannot be directly sold short. However, a financial intermediary can issue both short and long positions on (derivatives on) firm $h$’s equity. We assume here for simplicity that for each firm $h$ there is only one possible type of derivative claim, with the same unit return $e^h$ as the firm’s equity, and that there is an intermediary per firm.\(^8\)

Intermediaries bear no cost to issue claims, but face the possibility of default on the short positions they issue (i.e., on the loans granted via the sale of such positions).\(^9\) To protect themselves against the risk of default on the short positions issued, intermediaries have to hold an appropriate portfolio of claims (which acts then as a form of collateral). We consider the case in which the default rate on such positions is exogenously given and equal to $\delta$ in every state.\(^10\) In this case the best hedge against default risk is the equity of the same firm on which derivatives are written. In this set-up derivatives are thus ‘backed’ by equity in two ways: (i) the yield of each derivative of type $h$ is ‘pegged’ to the yield of equity of firm $h$;\(^11\) (ii) to issue any short position in the

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\(^8\)Both assumptions can be relaxed simply at notational cost.
\(^9\)Any other cost of intermediation, as long as proportional to the amount intermediated, would do.
\(^10\)A model with endogenous default is studied by Bisin-Gottardi (2001).
\(^11\)The role of equity as a benchmark to which the return on derivatives can be pegged can be justified on the basis of the fact that asset returns cannot be written as a direct function of future states of nature.
derivative, the intermediary has to hold - as a sort of collateral - an appropriate amount of equity of the same firm to whose return the derivative is pegged.

The self-financing constraint of the intermediary intermediating \( k_h \) units of firm \( h \)'s equity \( e^h \) is then:

\[
k_h e^h \leq k_h e^h (1 - \delta) + \gamma^h e^h
\]

where \( k_f \) is the number of long (and short) positions issued and \( \gamma^h \) the amount of equity of firm \( h \) acquired by the intermediary.

Let \( q^b_h \) (resp. \( q^s_h \)) be the price at which long (resp. short) positions in the derivative issued by the intermediary. The intermediary chooses then the amount of long and short positions in the derivative intermediated, \( k_h \in \mathbb{R}^+ \), and the amount held of equity as collateral, \( \gamma^h \in \mathbb{R}^+ \), so as to maximize their total revenue at date 0:

\[
\max (q^b_h - q^s_h) k_h - f_h(e^h) \gamma^h
\]

subject to the self-financing constraint (13).

The intermediation technology is characterized by constant returns to scale. If firm \( h \)'s production plan at equilibrium contains an equity component \( e^h > 0 \), the value of equity \( f_h(e^h) \) can be represented as a linear map, \( f_h e^h \). Moreover, a solution to the intermediary’s choice problem exists provided

\[
f_h \geq \frac{q^b_h - q^s_h}{\delta}
\]

and is characterized by \( k_h > 0 \) only if \( f_h = (q^b_h - q^s_h) / \delta \).

Let \( \lambda^+_i \in \mathbb{R}^H_+ \) denote agent \( i \)'s portfolio of long positions in the derivative, and \( \lambda^-_i \in \mathbb{R}^H_+ \) his portfolio of short positions in the derivative. The problem of every consumer \( i \) becomes:

\[
\begin{align*}
\text{max } u^i(x^i) \\
\text{s.t. } \\
x_0^i - \omega_0^i + q^i z^i + \sum_{h \in H} (f_h(e^h) \lambda^+_h + q^h \lambda^+_h - q^s_h \lambda^-_h) = 0 \\
x_s^i - \omega_s^i = A_s z^i + E_s (\lambda^+ - \lambda^-), \text{ for } s = 1, \ldots, S.
\end{align*}
\]

At an equilibrium of this economy, the whole price map \( f_h : \mathbb{R}^H \to \mathbb{R} \) must be determined for any firm \( h \). That is, to every possible vector \( e^h \in \mathbb{R}^H \) must be associated a price, including those vectors that do not represent the equity component of any firm’s production plan at equilibrium. There are two equivalent ways to do this: the first is to require market clearing for all possible vectors \( e^h \), the second is to specify a well chosen rational conjecture function that prices equity component not traded in the market. We follow the second route because simpler (it avoids market clearing in infinite dimension), but it is important to note they are equivalent.

We can then simply write market clearing as:

\[
\sum_{i \in I} x^i_s - \omega^i_s - \sum_{h \in H} y^h_s = 0, \text{ for } s = 0, 1, \ldots, S
\]
\[ \gamma^h + \sum_{i \in I} \lambda^i = 1, \text{ for } h = 1, \ldots, H \]

\[ \sum_{i \in I} \lambda^i_h = \sum_{i \in I} \lambda^i_{-h} = k_h, \text{ for } h = 1, \ldots, H \]

The value of equity \( f_h(e^h) \) is univocally determined at an equilibrium in which \( e^h \) is not a component of any firm’s production plan by rational conjectures:

\[ f_h(e^h) = \max \left\{ \max_i MRS^i(x^i)e^h, \frac{\max_i MRS^i(x^i)e^h - \min_i MRS^i(x^i)}{\delta} \right\} \]

In other words, conjectures are rational in that the value of equity \( e^h \), if traded in infinitesimal amount, would equal its maximal valuation at the margin among consumers and intermediaries.

For each firm \( h \), trading an equity component \( e^h \), two possible situations can arise at equilibrium:

1. \( q^h = (q^h_b - q^h_s)/\delta > q^h_b \), which in turn implies \( q^h_b > q^h_s/(1 - \delta) \). In this case equity sells at a premium over the long positions on the derivative claim issued by the intermediary (because of its additional value as input in the intermediation technology). Thus all the amount of equity outstanding is purchased by the intermediary, who can bear the additional cost of equity thanks to the presence of a sufficiently high spread \( q^h_b - q^h_s \) between the cost of long and short positions on the derivative.

2. \( q^h = q^h_b \). In this case there is a single price at which equity and long positions in the derivative with identical return \( e^h \) can be traded. Consumers are then indifferent between buying long positions in equity and the derivative and some if not all the outstanding amount of equity is held by consumers. When consumers hold all the outstanding amount of equity, intermediaries are non active at equilibrium and the bid ask spread \( q^h_b - q^h_s \) is sufficiently low (in particular, it is less or equal than \( \delta f_h \)).

We are now ready for our last Modigliani-Miller proposition, which follows in a straightforward manner from our equilibrium construction.

**Proposition 7** [Modigliani-Miller with incomplete markets, redundant assets, and no restrictions on the span of production plans.] Assume \( J > \text{rank}(A) \) and \( \text{rank}(A) < S \). The firm’s choice of \( \theta^h \in \Theta^h \) is indeterminate. But at an equilibrium in which firm \( h \)’s production plan \( y^h \) is such that \( y^h \notin \text{span}(A) \), for some \( h \in H \), the value of the firm depends on its financing decision, \( e^h \), which is therefore determined (jointly with the production choice \( y^h \)).

Needless to say, each infinitesimal firm’s choice of \( e^h \) is still indeterminate, but not the whole mass!
An attempt at an intuition of why with complete markets financing decisions are indeterminate, but they are determined with incomplete markets (if production plans are not in the span of the assets) follows. With complete markets, given $y$, the financing decision of the firm does not change the amount of risk in the economy: any extra equity issued by the firm must be held by consumers, but any extra equity supplied by a firm is matched by an exactly identical reduction at equilibrium in the supply of a security (or a portfolio of assets) held in financial markets. With incomplete markets, on the other hand, any extra equity supplied (or a different equity payoff) must be held by agents: there is no corresponding reduction in an identical security or portfolio because one such does not exist. Extra or new risk is held by consumers which substitute away other securities with different risks and require a compensation for the extra or new risk which depends on the amount to be held.

3 Appendix

Proof of Theorem 3. No-Arbitrage Theorem.

\[ \Rightarrow \]

Define the simplex in $\mathbb{R}^{S+1}$ as $\Delta = \{ \tau \in \mathbb{R}^{S+1} : \sum_{s=0}^{S} \tau_s = 1 \}$. Note that by No-Arbitrage condition, $\langle W \rangle$ intersection $\Delta$ is empty. The proof hinges crucially on the following separating result, which we shall state without proof.

**Lemma.** Let $X$ be a finite dimensional vector space. Let $K$ be a non-empty, compact and convex subset of $X$. Let $M$ be a non-empty, closed and convex subset of $X$. Furthermore, suppose $K$ and $M$ are disjoint. Then, there exists $\pi \in X \setminus \{0\}$ such that

\[ \sup_{\tau \in M} \pi \tau < \inf_{\tau \in K} \pi \tau. \]

Let $X = \mathbb{R}^{S+1}$, $K = \Delta$ and $M = \langle W \rangle$. Observe that all the required properties hold and so the lemma applies. As a result, there exists $\pi \neq 0$ such that

\[ \sup_{\tau \in \langle W \rangle} \pi \tau < \inf_{\tau \in \Delta} \pi \tau. \] (15)

Let us now show $\pi \in \mathbb{R}^{S+1}$. Suppose, on the contrary, that there is some $s$ for which $\pi_s \leq 0$. Then note that in (15), the RHS $\leq 0$. By (15), LHS $< 0$ but this contradicts the fact that $0 \in \langle W \rangle$ which implies that RHS is non-negative.

We still have to show that $\pi W = 0$, or in other words, that $\pi \tau = 0$ for all $\tau \in \langle W \rangle$. Suppose, on the contrary that there exists $\tau \in \langle W \rangle$ such that $\pi \tau \neq 0$. Since $\langle W \rangle$ is a subspace, there exists $\alpha \in \mathbb{R}$ such that $\alpha \tau \in \langle W \rangle$ and $\pi \alpha \tau$ is as large as we want. However, RHS is bounded above, which implies a contradiction.

\[ \Leftarrow \]
Suppose that there exists \( \pi \in R^{S+1}_{++} \) such that \( \pi W = 0 \), or \( q = pA \). From the agent’s problem, recall the budget constraint
\[
\begin{align*}
x^i_0 - \omega^i_0 + qz^i &= 0 \\
x^i_s - \omega^i_s &= A_s z^i, \text{ for } s = 1, ... S.
\end{align*}
\]
Plugging in \( q = pA \) and expanding the first equation yields
\[
\begin{align*}
x^i_0 - \omega^i_0 + \sum_{s=1}^{S} p_s (x^i_s - \omega^i_s) &= 0 \\
x^i_s - \omega^i_s &= A_s z^i, \text{ for } s = 1, ... S. \tag{16}
\end{align*}
\]
Observe now that the set defined in the first equation in (16) is a compact set (since it is a standard Arrow-Debreu budget set). Note further that the second equation defines a closed set. Both taken together are a closed subset of a compact set and therefore it itself is compact. The compactness of the budget set defined by (16) implies that the agent’s problem achieves a maximum level of utility. This, in turn, implies that no arbitrage opportunities were available to her.