1 Equilibrium and Efficiency

- We investigate the fundamental economic problem of allocation and price determination in a very simple economy. Our aim is to describe what outcomes might arise by giving individuals the opportunity to voluntarily exchange goods.

- Thus, we will follow two simple principles:
  (i) Optimization - individuals choose the best patterns of consumption that are affordable for them, and
  (ii) Equilibrium - prices adjust such that the amount that people demand of some good is equal to the amount that is supplied.

- We will determine equilibrium prices, equating demand and supply. We will show that these prices solve the allocation problem efficiently. Nonetheless the notion of equilibrium price is distinct from the notion of value, used in common parlance; for instance in, ”water is enormously valuable to each of us.” While this is not commonly made precise, we implicitly interpret the value of a good as a measure of how difficult it is to do without it. (Read your preferred rendition of King Midas’ fable. The king understands the value of simple things when his golden touch makes them completely unavailable to him. There is no reason (it is a logical fallacy) to conclude that values and not prices is what the noble man/woman or the social scientist should care about. There is a long list of social scientists/phylosophers which have fallen into a version of this fallacy.)

1.1 The Economy

- Pure exchange - no production; agents have fixed endowments of goods.

- Markets are competitive - individuals are price takers and optimize accordingly.
• 2×2 economy - two (types of) consumers and two goods economy.

1.1.1 Notation and Definitions

• \{1, 2\} - set of consumption goods.

• \{A, B\} - set of consumers.

• \( w_A = (w_A^1, w_A^2) \), \( w_B = (w_B^1, w_B^2) \) - consumer A’s and consumer B’s initial endowments respectively.

• \( x_A = (x_A^1, x_A^2) \), \( x_B = (x_B^1, x_B^2) \) - consumer A’s and consumer B’s consumption bundles respectively.

Definition 1 An allocation is a pair of consumption bundles, \( x_A \) and \( x_B \).

Definition 2 An allocation \( (x_A, x_B) \) is a feasible allocation if:

\[
x_A^1 + x_B^1 = w_A^1 + w_B^1
\]

and

\[
x_A^2 + x_B^2 = w_A^2 + w_B^2
\]

That is, if the total amount consumed of each of the goods is equal to the total amount available.

1.2 Pareto efficiency

Definition 3 A feasible allocation \( (x_A, x_B) \) is Pareto-efficient\(^1\) if there is no other feasible allocation \( (y_A, y_B) \) such that \( y_A \succ x_A \) and \( y_B \succ x_B \) with at least one \( \succ \).

• In words, an allocation is Pareto efficient if it is feasible and there is no other feasible allocation for which one consumer is at least as well off and the other consumer is strictly better off.

• This implies that at a Pareto efficient allocation (i) there is no way to make both consumers strictly better off, (ii) all of the gains from trade have been exhausted, that is, there are no mutually advantageous trades to be made.

• Is a Pareto efficient allocation fair? Define fair and think about this.

\(^1\)In honor of Vilfredo Pareto (1848-1923)
1.2.1 The Social Planner problem

• In this section will characterize the set of Pareto efficient allocations as
the set of solutions to a maximization problem called the Social Planner
Problem, (SP).

\[
\max_{x_A, x_B} u(x_A^1, x_A^2) \tag{1}
\]
subject to

\[
\begin{align*}
  x_A^1 + x_B^1 &= w_A^1 + w_B^1 = 2 \\
  x_A^2 + x_B^2 &= w_A^2 + w_B^2 = 2 \\
  u(x_A^1, x_B^2) &\geq \bar{u}
\end{align*}
\]

• Since both consumers’ utility is strictly increasing in both goods 1 and 2, it will never be optimal to give consumer B more utility than \( \bar{u} \). Thus the final constraint will be binding (i.e. an equality constraint).

• The set of all Pareto efficient allocations will be obtained as the solution to the Social Planner problem by varying \( \bar{u} \).

• The set of all Pareto efficient allocations can also be obtained as the solution to the following modified Social Planner problem:

\[
\max_{x_A, x_B} \theta u(x_A^1, x_A^2) + (1 - \theta) u(x_B^1, x_B^2) \tag{2}
\]
subject to

\[
\begin{align*}
  x_A^1 + x_B^1 &= w_A^1 + w_B^1 = 2 \\
  x_A^2 + x_B^2 &= w_A^2 + w_B^2 = 2
\end{align*}
\]

by varying the relative weight of agent A in the planner’s objective, that is by varying \( \theta \) between 0 and 1. [Can you prove this? Careful! It is not straightforward]

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\footnote{To be precise, we should also impose non-negativity constraints on allocations:}

\[
\begin{align*}
  x_A^1 &\geq 0; \\
  x_A^2 &\geq 0; \\
  x_B^1 &\geq 0; \\
  x_B^2 &\geq 0;
\end{align*}
\]
1.3 Competitive (Walrasian) equilibrium

Definition 4 A competitive or Walrasian\(^3\) equilibrium in a 2×2 economy is a pair of prices \((p_1^*, p_2^*)\) and allocations \((x_A^*, x_B^*)\) such that

- \((x_A^*, x_B^*)\) are demanded by agents A and B at prices \((p_1^*, p_2^*)\); and
- Markets clear:
  \[
  x_1^A + x_1^B = w_1^A + w_1^B \\
  x_2^A + x_2^B = w_2^A + w_2^B
  \]

- In words, in a competitive market / Walrasian equilibrium the total demand for each good should be equal to the total supply. Put differently, an equilibrium is a set of prices such that each consumer is choosing her most preferred (and affordable) bundle, and both consumers’ choices are compatible in the sense that the total demand equals the total supply for each of the goods.

- What happens to the budget set if both prices, \(p_1\) and \(p_2\) are proportionally increased to \(\lambda p_1\) and \(\lambda p_2\), for \(\lambda > 0\)?

  Note that your answer (well, the correct answer) implies that one price can always be normalized when looking at a competitive equilibrium, that is, \(p_1 = 1\) without loss of generality. Note that this is equivalent to saying that only relative prices, like \(\frac{p_2}{p_1}\), are determined at a competitive equilibrium.

- At a competitive equilibrium \((x_A, x_B, \frac{p_2}{p_1})\) prices satisfy (check this!):

  \[
  \frac{p_2}{p_1} = \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{\frac{\partial u(x_1^A, x_2^B)}{\partial x_1^A}}{\frac{\partial u(x_1^A, x_2^B)}{\partial x_2^B}}
  \]

  (3)

  That is: at a competitive equilibrium relative prices are equal to both agents’ marginal rates of substitution evaluated at the equilibrium allocation! The competitive equilibrium price of a good does not contain any information about the value of this good.

\(^3\)In honor of Leon Walras (1834-1910).
1.4 Welfare Economics

- Is the competitive market mechanism Pareto efficient? or in other words, Can it really extract all the possible gains from trade.

Theorem 1 (First Theorem of Welfare Economics) Suppose preferences are monotonic. Then, all competitive equilibria are Pareto efficient.

Proof. We proceed by contradiction. Suppose there exist a feasible allocation \((y_A, y_B)\) that Pareto dominates (is weakly preferred by both agents to, and is strictly preferred by at least one to) the competitive equilibrium \((x^*_A, x^*_B)\). Then, the allocation \((y_A, y_B)\) must be not budget feasible for at least one agent:

\[
p_1(y_1^1 - w_1^1) + p_2(y_2^1 - w_2^1) \geq 0
\]

\[
p_1(y_1^1 - w_1^1) + p_2(y_2^1 - w_2^1) \geq 0
\]

with at least one strict inequality sign. Summing up:

\[
p_1(y_1^A - w_1^A) + p_2(y_2^A - w_2^A) + p_1(y_1^B - w_1^B) + p_2(y_2^B - w_2^B) > 0 \quad (4)
\]

By monotonicity of preferences, prices are positive (convince yourself of this). Then equation (4) implies that either

\[
(y_1^A + y_1^B - w_1^A - w_1^B) > 0
\]

or

\[
(y_2^A + y_2^B - w_2^A - w_2^B) > 0
\]

which contradicts feasibility of \((y_A, y_B)\). ■

- The First Theorem of Welfare Economics says that all competitive equilibria are Pareto efficient. This is the formal argument behind Adam Smith invisible hand, in the 'Wealth of Nations,' (1776).

- Is the converse true? That is, Is any Pareto efficient allocation a competitive equilibrium for some endowments and prices?

Theorem 2 (Second Theorem of Welfare Economics) Suppose preferences are monotonic and convex. Then any Pareto efficient allocation is a competitive equilibrium for some prices and endowments.
1.4.1 The Edgeworth box

- The Edgeworth box\(^4\) is the graphical way to analyze the aspects of an economy with two consumers and two goods.

[Draw figure]

- What is the geometry of Pareto efficient allocation?
  The indifference curves of the two consumers must be tangent (in the interior of the box). If not, it must be that there exist some advantageous trade to explore.

**Definition 5** The set of all Pareto efficient allocations is called the contract curve.

- Typically, the contract curve stretch from consumer A’s origin to consumer B’s origin (why?).

[Draw Figure]

- The proof that any competitive equilibrium is Pareto efficient (First Welfare Theorem) has a graphical representation: a feasible allocation (in the Edgeworth box) allocation is Pareto efficient if the intersection of consumer A’s strictly preferred set and consumer B’s strictly preferred set is empty. However, in the competitive equilibrium the two sets of preferred allocation can not intersect since they lie on different sides of the prices’ ratio line.

1.5 Externalities

- Consider the case in which the utility of agent A depends also on the consumption choice of agent B:

\[ u_A = u_A(x^1_A, x^2_A, x^1_B) \]

We say that the consumption of good 1 by agent B has an externality on agent A (positive or negative depending on wether \(u_A(\cdot)\) is, respectively, increasing or decreasing on \(x^1_B\)).

\(^4\)In honor of Francis Edgeworth (1845-1926).
Consider the case in which the utility of both agents depend on a public good, which is bought adding the contributions of both agents:

\[ u_A = u(x_A^1, x_A^2 + x_B^2), \quad u_B = u(x_B^1, x_A^2 + x_B^2) \]

Examples of public good are bridges, parks, national defense, etc. They have the property of being non-exclusive, that is, an agent’s consumption of it does not preclude other agents’ consumption. You should immediately see that they introduce a (positive) externality of A on B and viceversa.

Show that in both cases, the First Welfare Theorem does not hold, and competitive equilibria are not Pareto Efficient:

Go back to the proof of the First Welfare Theorem, and identify which logical step of the proof breaks down.

Compare the first order conditions for the agents’ consumption problems (that must be satisfied at a competitive equilibrium) and the first order conditions for the Social Planner’s problem; show they are different.

Interpret the difference in first order conditions you have just identified.

1.6 References