1 Maths Review Questions - Set 1 - Answer sheet

1.1 Sets and numbers

Let \( A = \{1, 2, 3, \ldots, 10\} \), \( B = \{0, 0.5, \sqrt{2}, 2, 10\} \), \( \mathbb{R} \) be the real numbers, \( \mathbb{N} \) be the natural numbers, \( \mathbb{Z} \) be the integers and \( \mathbb{Q} \) be the rationals. In each case below, describe the content of the set \( C \)

1. \( C = \{x \in A : x > 6\} \)
   
   This is all the numbers in set \( A \) that are strictly greater than 6. In other words \( C = \{7, 8, 9, 10\} \)

2. \( C = A \cup B \)
   
   All the elements that are either in set \( A \) or set \( B \) (or both). \( C = \{0, 0.5, 1, \sqrt{2}, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)

3. \( C = A \cap B \)
   
   All the elements that are in both set \( A \) and set \( B \). \( C = \{2, 10\} \)

4. \( C = A \setminus B \)
   
   All the elements that are in set \( A \) but not in set \( B \). \( C = \{1, 3, 4, 5, 6, 7, 8, 9\} \)

5. \( C = B \cap \mathbb{R} \)
   
   All the elements of set \( B \) that are also real numbers. As all the numbers that we will come across in this course are real numbers, everything in set \( B \) is also in \( \mathbb{R} \). So \( C = \{0, 0.5, \sqrt{2}, 2, 10\} = B \)

6. \( C = B \cap \mathbb{N} \)
   
   All the elements of set \( B \) that are also natural numbers. Remember that the natural numbers are \( \mathbb{N} = \{1, 2, 3, \ldots\} \) so \( C = \{2, 10\} \)

7. \( C = B \cap \mathbb{Z} \)
   
   All the elements of set \( B \) that are also integers. Remember that the natural numbers are \( \mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \) so \( C = \{0, 2, 10\} \)

8. \( C = B \cap \mathbb{Q} \)
   
   All the elements of set \( B \) that are also rational numbers. Remember that the rational numbers are all the numbers that can be expressed as a ratio of an integer and a natural number, and that \( \sqrt{2} \) is not a rational number but 0.5 is, as it is equal to \( \frac{1}{2} \). So \( C = \{0, 0.5, 2, 10\} \)

9. \( C = \mathbb{Z} \setminus \mathbb{N} \)
   
   All the numbers which are integers but not natural numbers. As the integers are defined as \( \mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \) and the natural numbers as \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( C = \{\ldots -3, -2, -1, 0\} \), or all the integers which are not strictly positive.
10. \( C = \mathbb{N}\backslash\mathbb{Z} \)  

All the natural numbers that are not integers. We can see from above that any natural must also be an integer. The set of natural numbers which are not integers is therefore empty. We write this as \( C = \emptyset \)

1.2 Derivatives

Take the first derivatives of the following functions

1. \( f(x) = (x^2 + x)(x^3 - 3x^2) \)
   
   Product rule. Let \( g(x) = (x^2 + x) \) and \( h(x) = (x^3 - 3x^2) \). Then \( g'(x) = (2x + 1) \) and \( h'(x) = (3x^2 - 6x) \). The product rule tells us that if \( f(x) = g(x)h(x) \) then \( f'(x) = g'(x)h(x) + h'(x)g(x) \). In this case,  
   \[ f'(x) = (2x + 1)(x^3 - 3x^2) + (3x^2 - 6x)(x^2 + x) \]

2. \( f(x) = \frac{(x^2 + x)}{(x^3 - 3x^2)} \)
   
   Quotient rule. Let \( g(x) = (x^2 + x) \) and \( h(x) = (x^3 - 3x^2) \). Then \( g'(x) = (2x + 1) \) and \( h'(x) = (3x^2 - 6x) \). The quotient rule tells us that if \( f(x) = \frac{g(x)}{h(x)} \) then \( f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2} \). In this case  
   \[ f'(x) = \frac{(2x + 1)(x^3 - 3x^2) - (x^2 + x)(3x^2 - 6x)}{(x^3 - 3x^2)^2} \]

3. \( f(x) = (2x^2 + 3x)^{3/2} \)
   
   Chain rule. Let \( g(x) = (2x^2 + 3x) \) and \( h(y) = y^{3/2} \) then \( f(x) = h(g(x)) \). Then \( g'(x) = (4x+3) \) and \( h'(y) = \frac{3}{2} y^{1/2} \). By chain rule, \( f'(x) = h'(g(x))g'(x) \) in this case  
   \[ f'(x) = \frac{3}{2}(2x^2 + 3x)^{1/2}(4x + 3) \]

4. \( f(x) = \log(x^2) \)
   
   Two different methods. First, by chain rule, let \( g(x) = x^2 \) and \( h(y) = \log y \) so that \( f(x) = h(g(x)) \). Then \( g'(x) = 2x \) and \( h'(y) = \frac{1}{y} \). By chain rule, \( f'(x) = h'(g(x))g'(x) \) in this case  
   \[ f'(x) = \frac{1}{x}2x = \frac{2}{x} \]
   
   Alternatively, we can always rewrite \( \log x^2 \) as \( 2 \log x \), the differential of which is \( \frac{2}{x} \).

5. \( f(x) = e^{x^2} \)
   
   Again, chain rule. Let \( g(x) = x^2 \) and \( h(y) = e^y \) then \( f(x) = h(g(x)) \). Then \( g'(x) = 2x \) and \( h'(y) = e^y \). By chain rule, \( f'(x) = h'(g(x))g'(x) \) in this case  
   \[ f'(x) = e^{x^2}2x \]

Take the first and second derivatives of these functions.
1. $f(x) = 6x^3 + 4x^2 + x$
   First derivative $f'(x) = 18x^2 + 8x + 1$
   Second derivative $f''(x) = 36x + 8$

2. $f(x) = \log(x)$
   First derivative $f'(x) = \frac{1}{x}$
   Second derivative $f''(x) = -\frac{1}{x^2}$

3. $f(x) = e^x x^2$
   First derivative $f'(x) = 2xe^x + x^2e^x$ (product rule)
   Second derivative $f''(x) = 2xe^x + 2e^x + 2xe^x + x^2e^x = (2 + 4x + x^2)e^x$

Take all the partial derivatives of the following functions

**Two things to remember:**
1: There are the same number of partial derivatives as there are arguments in a function. If we have $f(x, y, z)$, then we will have three partial derivatives
2: When we are taking the partial derivatives, we take all the other arguments as constant. So in the case of $f(x, y, z) = xyz + x^2y + z^3$, if we are taking the derivative with respect to $x$, we can write $f(x, y, z) = ax + bx^2 + c$, where $a = yz, b = y$ and $c = z^3$. The derivative of this function is $\frac{df}{dx} = a + 2bx = yz + 2yx$

1. $f(x, y, z) = xyz + x^2y + z^3$
   $\frac{df}{dx} = yz + 2xy$
   $\frac{df}{dy} = xz + x^2$
   $\frac{df}{dz} = xy + 3z^2$

2. $f(x, y) = xy(y^2 + x^2)$
   $= xy^3 + yx^3$
   $\frac{df}{dx} = y^3 + 3yx^2$
   $\frac{df}{dy} = 3xy^2 + x^3$

3. $f(x, y) = x \log y$
   $\frac{df}{dx} = \log y$
   $\frac{df}{dy} = \frac{x}{y}$

For the following two functions, sketch a graph of the isoquants for $f(x_1, x_2)$ equals 1, 2 and 5 in $x_1, x_2$ space. Write down an equation for the slope of the isoquants
1. $f(x_1, x_2) = x_1 + \frac{x_2}{2}$
   
   For the case of $f(x_1, x_2) = 1$, we are looking for the set of $x_1, x_2$ such that $x_1 + \frac{x_2}{2} = 2$. Rearranging this gives $x_2 = 4 - 2x_1$. This is clearly a linear function with a slope of $-2$. We can also calculate the slope of the isoquant using the result that $\frac{\partial f(x_1, x_2)}{\partial x_1} = 1$ and $\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{1}{2}$, this confirms our result. Note that the slope is constant along the graph, and doesn’t vary between the isoquants 1, 2 and 5.

2. $f(x_1, x_2) = x_1x_2$
   
   For the case of $f(x_1, x_2) = 1$, we are looking for the set of points such that $x_1x_2 = 1$. Rearranging, this gives $x_2 = \frac{1}{x_1}$. This shape is a hyperbola. While you may not know this, you should be able to work out that, as $x_1$ gets very large, $x_2$ gets very small (but not negative) and as $x_1$ gets close to zero, $x_2$ gets very large.

   We can work out the slope of the isoquant either directly, or using the ratio of the partial derivatives. As $\frac{\partial f(x_1, x_2)}{\partial x_1} = x_2$ and $\frac{\partial f(x_1, x_2)}{\partial x_2} = x_1$, the above formula tells us that $\frac{dx_2}{dx_1} = -\frac{x_2}{x_1}$. But we can do more. Note that $x_2 = \frac{1}{x_1}$ in this case, so we can substitute back in to give $\frac{dx_2}{dx_1} = -\frac{1}{x_1^2}$. Note that the slope of the isoquant changes both along the curve and between isoquants.