1 Introduction to Economic Analysis, Lab Session 2 - Maths Prelim - Addition - Solving the Cobb-Douglas Consumer’s Problem

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The second problem of section 1.4 of the second maths refresher homework is a classic example of a consumer maximization problem. You will see this problem, or a version of it, time and time again. It is therefore worth getting familiar with it now, as it will save you lots of time later on. We will also use it here to demonstrate the 5 steps that, quite often, will lead you to the solution of a constrained optimization problem. While its not advisable to think of maths in a cookbook fashion, you can treat this as such as a last resort. First, to restate the problem:

Maximize $x^\alpha y^{(1-\alpha)}$ subject to $p_x x + p_y y = I$. (Note that $\alpha, p_x, p_y, I$ are parameters for this problem your answer should be a function of $x$ and $y$ in terms of these values)

The interpretation of this problem is as follows: A consumer is has to choose between bundles of good $x$ and good $y$ in order to maximize her utility function $x^\alpha y^{(1-\alpha)}$. However, she only has a certain amount of income to spend, $I$. The price of the two goods is denoted by $p_x$ and $p_y$. So the consumer is trying to choose the bundles to maximize her utility function subject to the bundles being those that she can afford. As stated above, $\alpha, p_x, p_y, I$ are what we call parameters of the problem. In other words, they are the ‘rules of the game’, or things that the consumer cannot alter. Your solution to the problem will be a function of these parameters.

The cookbook method of solving these problems has 5 steps, described below:

1.1 Form the Lagrangian function

This always takes the form

$$L(x_1, x_2, \mu) = f(x_1, x_2) - \mu h(x_1, x_2)$$

Where $\mu$ is the lagrangian multiplier, $f(x_1, x_2)$ is the objective function and $h(x_1, x_2)$ is the constraint. In this case, we get:

$$L(x, y, \mu) = x^\alpha y^{(1-\alpha)} - \mu(p_x x + p_y y - I)$$

1.2 Take partial derivatives and set them equal to zero

I’ve tried to convince you that solving the constrained optimization problem is the same as solving the unconstrained problem for the lagrangian. If you are solving an unconstrained optimization problem, then the first thing we do is
take the partial derivatives and set them equal to zero. In general:

\[
\begin{align*}
\frac{\partial L(x_1, x_2, \mu)}{\partial x_1} &= \frac{\partial f(x_1, x_2)}{\partial x_1} - \mu \frac{\partial h(x_1, x_2)}{\partial x_1} = 0 \\
\frac{\partial L(x_1, x_2, \mu)}{\partial x_2} &= \frac{\partial f(x_1, x_2)}{\partial x_2} - \mu \frac{\partial h(x_1, x_2)}{\partial x_2} = 0 \\
\frac{\partial L(x_1, x_2, \mu)}{\partial \mu} &= h(x_1, x_2) = 0
\end{align*}
\]

And in this specific case:

\[
\begin{align*}
\frac{\partial L(x, y, \mu)}{\partial x} &= \alpha x^{\alpha - 1} y^{1-\alpha} - \mu p_x = 0 \\
\frac{\partial L(x, y, \mu)}{\partial y} &= (1 - \alpha) x^{\alpha} y^{-\alpha} - \mu p_y = 0 \\
\frac{\partial L(x, y, \mu)}{\partial \mu} &= p_x x + p_y y - I = 0
\end{align*}
\]

1.3 Divide the FOC from the choice variables by each other to get rid of \(\mu\)

We can always rewrite our partial derivatives from the choice variables by taking one term over to the other side of the equality. In general:

\[
\begin{align*}
\frac{\partial f(x_1, x_2)}{\partial x_1} - \mu \frac{\partial h(x_1, x_2)}{\partial x_1} &= 0 \Rightarrow \frac{\partial f(x_1, x_2)}{\partial x_1} = \mu \frac{\partial h(x_1, x_2)}{\partial x_1} \\
\frac{\partial f(x_1, x_2)}{\partial x_2} - \mu \frac{\partial h(x_1, x_2)}{\partial x_2} &= 0 \Rightarrow \frac{\partial f(x_1, x_2)}{\partial x_2} = \mu \frac{\partial h(x_1, x_2)}{\partial x_2}
\end{align*}
\]

We can then divide these equations by each other, giving

\[
\begin{align*}
\frac{\partial f(x_1, x_2)}{\partial x_1} \cdot \frac{\partial h(x_1, x_2)}{\partial x_1} &= \mu \\
\frac{\partial f(x_1, x_2)}{\partial x_2} \cdot \frac{\partial h(x_1, x_2)}{\partial x_2} &= \mu
\end{align*}
\]

This gets rid of the \(\mu\) from the top and bottom of our equation.

In this specific case, we get

\[
\begin{align*}
\alpha x^{\alpha - 1} y^{1-\alpha} - \mu p_x &= 0 \Rightarrow \alpha x^{\alpha - 1} y^{1-\alpha} = \mu p_x \\
(1 - \alpha) x^{\alpha} y^{-\alpha} - \mu p_y &= 0 \Rightarrow (1 - \alpha) x^{\alpha} y^{-\alpha} = \mu p_y
\end{align*}
\]

And so

\[
\begin{align*}
\frac{\alpha x^{\alpha - 1} y^{1-\alpha}}{(1 - \alpha) x^{\alpha} y^{-\alpha}} &= \frac{\mu p_x}{p_y} \Rightarrow \frac{\alpha x^{\alpha - 1} y^{1-\alpha}}{(1 - \alpha) x^{\alpha} y^{-\alpha}} = \frac{p_x}{p_y}
\end{align*}
\]
1.4 Solve for $x$ in terms of $y$ (or visa versa)

This equation should now be in terms only of $x, y$ and the parameters of the model, as we have got rid of $\mu$. In the case of Cobb-Douglas Utilities, something magical happens. Note that, on the left hand side of the equation, the powers of $\alpha$ on $x$ and $y$ cancel, leaving a linear model!

\[
\frac{\alpha x^{\alpha - 1} y^{1-\alpha}}{(1-\alpha)x^\alpha y^{-\alpha}} = \frac{\alpha y}{(1-\alpha)x} \\
\Rightarrow \frac{\alpha y}{(1-\alpha)x} = \frac{p_x}{p_y} \\
\Rightarrow y = \frac{(1-\alpha)x p_x}{\alpha p_y}
\]

1.5 Substitute into the constraint and solve

At present we have two unknowns. That means that we have to find another equation so we can solve the system. Luckily, we have such and equation - the constraint. We can now substitute the above into this equation

\[
p_x x + p_y y = I \\
p_x x + p_y \left( \frac{(1-\alpha)x p_x}{\alpha p_y} \right) = I
\]

but here we can cancel the $p_y$ terms, to give

\[
p_x x + \frac{(1-\alpha)}{\alpha} p_x x = I
\]

We can neaten this up a bit by multiplying the left most term top and bottom by $\alpha$

\[
\frac{\alpha}{\alpha} p_x x + \frac{(1-\alpha)}{\alpha} p_x x = I \\
\frac{\alpha}{\alpha} p_x x + \frac{(1-\alpha)}{\alpha} p_x x = I \\
\frac{p_x x}{\alpha} = I \\
x = \frac{I \alpha}{p_x}
\]

Similar work will tell you that

\[
y = \frac{I (1-\alpha)}{p_y}
\]

We now have an expression for $x$ and $y$ in terms only of the parameters, so we have solved the model. Victory!. Note that this can be rewritten as $p_x x = \alpha I$
where the left hand side of the equation is just the amount spent on good $x$. This tells us that, with these preferences, the amount spent on good $x$ will be a constant fraction of income, whatever the price!