1 Introduction

Do competitive insurance markets function orderly in the presence of moral
hazard and adverse selection? What are the properties of allocations attainable
as competitive equilibria of such economies? And in particular, are competitive
equilibria incentive efficient?

The fundamental contribution on competitive markets for insurance con-
tracts is Prescott and Townsend (1984). They analyze Walrasian equilibria
of economies with moral hazard and with adverse selection when exclusive
contracts are enforceable. While for moral hazard economies they prove ex-
istence and constrained versions of the first and second theorems of welfare eco-
nomics, their method does not succeed in the case of adverse selection economies.
They conclude (p. 44) that “there do seem to be fundamental problems for the
operation of competitive markets for economies or situations which suffer from
adverse selection.1”

1 The standard strategic analysis of competition in insurance economies, due to Rothschild-
Stiglitz (1976), considers the Nash equilibria of a game in which insurance companies simulta-
neously choose the contracts they issue, and the competitive aspect of the market is captured
by allowing the free entry of insurance companies. Such equilibrium concept does not perform
too well: equilibria in pure strategies do not exist for robust examples (Rothschild-Stiglitz
(1976)), while equilibria in mixed strategies exist (Dasgupta-Maskin (1986)) but, in this set-
up, are of difficult interpretation. Even when equilibria in pure strategies do exist, it is not
clear that the way the game is modelled is appropriate for such markets, since it does not
More generally, the analysis of competitive equilibria of economies with asymmetric information has recently received renewed attention. For such economies the interaction between the private information dimension (e.g., the unobservable action in the moral hazard case, the unobservable type in the adverse selection case) and the observability of agents’ trades plays a crucial role, since trades have typically informational content over the agents’ private information. In particular, to decentralize incentive efficient Pareto optimal allocations the availability of fully exclusive contracts, i.e., of contracts whose terms (price and payoff) depend on the transactions in all other markets of the agent trading the contract, is generally required. The implementation of these contracts imposes typically the very strong informational requirement that all trades of an agent need to be observed. Full observability of trades is in fact the economic environment which Prescott and Townsend study. It is then of interest to analyze also situations where contracts traded are necessarily non-exclusive, because perfect monitoring of trades is not available. The case of complete anonymity of trades, where no transaction of the agents is observable, constitutes an important benchmark in this respect.

In summary, an important dichotomy arises in the study of economies with asymmetric information: economies in which each agent’s trades are observable behave very differently from economies in which trades are not observable (often these economies are referred to, respectively, as economies with exclusive and non-exclusive contractual relationships).

Many different approaches have been taken in the literature to analyze equilibria of economies with asymmetric information. Even restricting to Walrasian equilibria, many different definitions are available (a situation that Douglas Gale has referred to as “Balkanization”). In these notes we attempt an analysis of such concepts. To facilitate comparisons we apply all such concepts to the same simple moral hazard and adverse selection economy.

We will show that the equilibrium concepts developed for moral hazard economies have an analogous applications to adverse selection economies, and viceversa. We will also show that, in economies with observable trades, for which different concepts have been developed, all such concepts generate co-incident predictions in terms of equilibrium allocations and prices. In other words, different equilibrium concepts give rise to different equilibrium predictions only when they capture different assumptions about the observability of trades. All this for both moral hazard and adverse selection economies, in our simple example economies.

Our classification of equilibrium concepts follows. In the framework of a Walrasian competitive equilibrium model, alternative assumptions on the observability of agents’ trades may be captured in a reduced form by alternative assumptions on the possible non-linearities of equilibrium prices.

allow for dynamic reactions to new contract offers (Wilson (1977) and Riley (1979); see also Maskin-Tirole (1992)). Moreover, once sequences of moves are allowed, equilibria are not robust to ‘minor’ perturbations of the extensive form of the game (Hellwig (1987)).
Complete anonymity of trades (full non-exclusivity) corresponds to restricting price schedules to be a linear function of trades. The intermediate case in which only short and long trading positions can be distinguished, will turn out to be central in our analysis: a minimal form of non-linearity, e.g., the possibility of having a different price for buyers and sellers (a bid-ask spread), is in fact necessary and sufficient for competitive equilibria to exist; see Dubey and Geanakoplos (2004), Bisin and Gottardi (2000), Bisin, Geanakoplos, Gottardi, Minelli and Polemarchakis (2001).

At the other extreme, complete observability of trades (exclusivity) is captured by allowing price schedules to be general non-linear function of agents’ trades. We distinguish two main approaches to the analysis of such economies:

Prices are arbitrarily non-linear maps; as a consequence minimal restrictions are imposed by the equilibrium notion, and hence the plethora of resulting equilibria is refined by a formal concept in the spirit of sequential equilibria; see Gale (1993), Dubey and Geanakoplos (2004).

The specification of agents’ budget sets restricts admissible trades to lie in the set of incentive compatible trades; in other words, non-incentive compatible trades are just non available for trade, or, say, are traded at infinite price; Prescott and Townsend (1984); see also Bisin and Gottardi (2004) for the adverse selection case.

The literature on the strategic analysis of economies of asymmetric information presents us with many equilibrium concepts and strategic game forms, and few robust predictions about equilibrium allocations. We survey in this note instead Walrasian equilibrium concepts, with the objective of identifying robust predictions in terms of equilibrium allocations.

2 The Economy: Moral Hazard and Adverse Selection

We study two simple economies, workhorses of economics of uncertainty. The first economy is characterized by moral hazard in the form of hidden action; see Grossman and Hart (1983). The second economy is characterized by adverse selection in the form of unobservable risk types; see Rothschild and Stiglitz (1977), Wilson (19??). We will introduce such economies in a as much unified way as possible.

There is measure 1 of agents who live two periods, \( t = 0, 1 \), and consume, only in period 1, a single consumption good. Uncertainty is purely idiosyncratic, and is described by the collection of random variables \( \tilde{s}^\tau \), where \( \tau \) indexes the names of the agent and lies in a countable space. It is assumed that \( \tilde{s}^\tau \) are identically and independently distributed, with support \( S = \{H, L\} \); the realization of
all \(\tilde{\tau}\) variables is commonly observable.\(^2\) Uncertainty enters the economy via the agents’ endowments. The (date 1) endowment of an agent \(w(\tilde{s})\); let \(w^H \equiv w(H), w^L \equiv w(L)\) be the agent’s endowment in, respectively, the idiosyncratic state \(H\) and state \(L\).

The probability distribution of the period 1 endowment that each agent faces depends from the value taken by a variable \(e \in \{h, l\}\). The interpretation of \(e\) is what distinguishes moral hazard from adverse selection economies.

In moral hazard economies, \(e\) is an unobservable level of effort which is chosen by the agent. In adverse selection economies \(e\) describes the exogenously given risk type of an agent, and its realization is only privately observable. Let \(\xi^h\) (resp. \(\xi^l = 1 - \xi^h\)) be the probability that an agent is of type \(e = h\) (resp. \(e = l\)); by the Law of Large Numbers, \(\xi^h\) is then also the fraction of agents in the population which are of type \(h\). Importantly, in adverse selection economies, all markets open after agents observe the probability distribution of their endowments.

Let \(\pi^e_s\) be the probability of the realization \(s\) given \(e \in \{h, l\}\) (obviously \(\pi^e_H = 1 - \pi^e_L\), for any \(e\)). By the Law of Large Numbers, \(\pi^e_s\) is also the fraction of agents with \(e\) for which state \(s\) is realized. Agents’ preferences are represented by a (Von Neumann - Morgenstern) utility function of the following form:

\[
\pi^e_s u(c^H) + (1 - \pi^e_s)u(c^L) - v(e)
\]

where \((c^H, c^L)\) denotes consumption respectively in state \(H\) and \(L\); let \(c \equiv (c^H, c^L), w \equiv (w^H, w^L)\).

In moral hazard economies, \(v(e)\) denotes the disutility of effort \(e\), and we assume that

\[
\pi^h_H > \pi^l_H, \quad v(h) > v(l), \quad w^H > w^L > 0
\]

so that \(h\) is the ‘high’ effort and \(H\) is the ‘good’ state.

In adverse selection economies, \(v(e)\) is just a utility constant and hence is disregarded from the analysis without loss of generality.

**Assumption 1** Preferences are strictly monotonic, strictly concave, twice continuously differentiable, and \(\lim_{x \to 0} u'(x) = \infty\).

Let \(\Omega\) be the set of parameter values \((v(h), v(l), \pi^h_H, \pi^l_H, w^H, w^L)\) of the economy which satisfy the above assumptions.

\(^2\)Measurability issues arise in probability spaces with a continuum of independent random variables. We adopt the usual abuse of the Law of Large Numbers.
3 The Symmetric Information Benchmark

We consider now the benchmark case of symmetric information, in which \( e \), be it endogenous effort or risk type, is commonly observed. Even though, tautologically, no issues of moral hazard nor adverse selection arise with symmetric information, we nonetheless refer to the economy in which \( e \) is chosen by each agent (resp. \( e \) is exogenous) as a 'moral hazard' economy (resp. an 'adverse selection' economy).

**Definition 1** An allocation \((c, e) \in \mathbb{R}^2_+ \times \{h, l\}\) of consumption and effort is optimal in the moral hazard economy under symmetric information if it solves:

\[
\max_{c,e} \sum_s \pi_s^e u(c_s) - v(e) \tag{1}
\]

s.t.

\[
\sum_s \pi_s^e (c_s - w_s) = 0
\]

**Definition 2** An allocation \((c^h, c^l) \in \mathbb{R}^4_+\) of consumption is optimal in the adverse selection economy under symmetric information if it solves:

\[
\max_{c^h, c^l} \sum_e \kappa^e \sum_s \pi_s^e u(c_s^e) \tag{2}
\]

s.t.

\[
\sum_e \xi^e \sum_s \pi_s^e (c_s^e - w_s) = 0
\]

for some \((\kappa^h, \kappa^l) \gg 0\) such that \(\kappa^h = 1 - \kappa^l\).

Let \(q^e_s\) denote the (linear) price of consumption in state \(s\) for agents \(e\). By allowing the prices of the securities whose payoff is contingent on the idiosyncratic uncertainty to depend on \(e\), we effectively allow agents to trade in a complete set of markets.

In addition to consumers we introduce firms. Firms ‘pool’ payments in different states of the world. The Law of Large Numbers provides, in the economy under consideration, a mechanism - or a technology - for transforming aggregates of the commodity contingent on different individual states. Thus firms are characterized by the following constant returns to scale technology:

\[
Y = \{y \in \mathbb{R}^4 : \sum_e \sum_s \pi_s^e y_s^e \leq 0\}
\]

where \(y \equiv [y_s^e]_{s \in S}^{e \in E}\).

The firms’ problem is then the choice of a vector \(y\) of the commodity contingent on the agents’ individual states, lying in the set \(Y\) (i.e., a collection of
trades, or contracts to offer; contracts of the same type are then pooled and transformed according to the Law of Large Numbers) so as to maximize profits:

$$\max_{y \in Y} \sum_e \sum_s q_{es} y_{es} \quad (Pf)$$

taking prices $q$ as given.

**Definition 3** A Walrasian equilibrium with symmetric information in the moral hazard economy is given by prices $q^e \in \Delta^2$, for all $e$, a consumption allocation and effort choice $(c^e, e) \in \mathbb{R}_+^2 \times \{h, l\}$, and a production vector $y \in \mathbb{R}^4$, such that:

(i) $(c^e, e)$ solves the agent’s optimization problem

$$\max_{c^e, e} \sum_s \pi_s^e u(c^e_s) - v(e) \quad (3)$$

s.t.

$$\sum_s q_{es} (c^e_s - w_s) = 0$$

(ii) $y$ solves the firms’ profit maximization problem $(Pf)$, at prices $q$;

(iii) markets clear:

$$(c^e_s - w_s) \leq y_{es}, \forall s, e \quad (4)$$

**Definition 4** A Walrasian equilibrium with symmetric information in the adverse selection economy is given by prices $q^e \in \Delta^2$ and a consumption allocation $\tilde{c}^e \in \mathbb{R}_+^2$, for all $e$, such that:

(i) $\tilde{c}^e$ solves the optimization problem of agents of type $e$:

$$\max_{\tilde{c}^e} \sum_s \pi_s^e u(\tilde{c}^e_s) \quad (5)$$

s.t.

$$\sum_s q_{es} (\tilde{c}^e_s - w_s) = 0$$

(ii) $y$ solves the firms’ profit maximization problem $(Pf)$, at prices $q$;

(iii) markets clear:

$$(\tilde{c}^e_s - w_s) \leq y_{es}, \forall s, e \quad (6)$$

The First and Second Welfare theorems hold straightforwardly for both the moral hazard and the adverse selection economies under symmetric information.

**Proposition 1** Any Walrasian equilibrium with symmetric information in the moral hazard economy is optimal.

**Proposition 2** Any optimal allocation of a moral hazard economy with symmetric information can be decentralized as a Walrasian equilibrium with symmetric information with transfers.
Proposition 3 Any Walrasian equilibrium with symmetric information in the adverse selection economy is optimal.

Proposition 4 Any optimal allocation of an adverse selection economy with symmetric information can be decentralized as a Walrasian equilibrium with symmetric information with transfers.

4 Incentive Constrained Pareto Optimality

Definition 5 An allocation \((c, e) \in \mathbb{R}_+^2 \times \{h, l\}\) of consumption and effort is incentive constrained optimal in the moral hazard economy if it solves:

\[
\max_{c, e} \sum_s \pi^e_s u(c_s) - v(e)
\]

s.t.
\[
\sum_s \pi^e_s (c_s - w_s) = 0
\]
\[
\sum_s \pi^e_s u(c_s) - v(e) \geq \sum_s \pi^e_s u(c_s) - v(e'), \forall e, e'
\]

Definition 6 An allocation \((c^h, c^l) \in \mathbb{R}_+^4\) of consumption is incentive constrained optimal in the adverse selection economy if it solves:

\[
\max_{c^h, c^l} \sum_c \kappa^c \sum_s \pi^c_s u(c^c_s)
\]

s.t.
\[
\sum_c \xi^c \sum_s \pi^c_s (c^c_s - w_s) = 0
\]
\[
\sum_s \pi^h_s u(c^h_s) \geq \sum_s \pi^h_s u(c^l_s)
\]
\[
\sum_s \pi^l_s u(c^l_s) \geq \sum_s \pi^l_s u(c^h_s)
\]

for some \((\kappa^h, \kappa^l) \in \mathbb{R}_+^2\) such that \(\kappa^h = 1 - \kappa^l\).

5 Walrasian Equilibria: Fully Observable Trades

We consider first the case in which all the agents’ trades are observable, and hence exclusive contracts can be implemented.

For both moral hazard and adverse selection economies, in this case, two different equilibrium concepts have been proposed.

We first introduce Prescott-Townsend equilibria (introduced by Prescott-Townsend (1984) for moral hazard, and extended to encompass adverse selection by Bisin-Gottardi (2000). We then introduce refined non-linear prices equilibria.
5.1 Moral Hazard

We first consider Prescott-Townsend equilibria, as introduced by Prescott-Townsend for moral hazard economies. We then consider refined non-linear prices equilibria, by analogy with the adverse selection concept proposed by Gale (1993) and Dubey-Geanakoplos-Shubik (2004).

### 5.1.1 Prescott-Townsend Equilibria

The structure of existing markets is the same as under symmetric information: at the price \( q^e_s \) agents and firms can trade claims contingent on the agents’ individual state and effort level. To ensure the viability of markets for claims contingent on the agents’ effort even though this is now unobservable, the agents’ set of admissible trades is suitably restricted to the subset of trades and effort choices which are incentive compatible:

\[
Z_{MH} = \left\{(c, e) \in \mathbb{R}^2_+ \times \{h, l\} : \sum_s \pi^e_s c_s - v(e) \geq \sum_s \pi^{e'}_s c_s - v(e'), \ e' \neq e \right\}
\]

The incentive constraints require that the agents’ choice \((c, e)\) must be such that they prefer \((c, e)\) to any other allocation \((c, e')\), with \(e' \neq e\).

The agents’ optimization problem consists, as before, in the choice of a consumption bundle \(c \in \mathbb{R}^2_+\), specifying his level of consumption in the agent’s two possible individual states, and an effort level \(e \in \{h, l\}\) subject to the budget constraint and, now, to the additional restriction that admissible choices are restricted to lie in the set \(Z_{MH}\):

\[
\max_{c, e \in Z_{MH}} \sum_s \pi^e_s c_s - v(e) \quad (P_{PT})
\]

s.t.

\[
\sum_s q^e_s (c_s - w_s) \leq 0
\]

Firms’ technology \(Y\) is the same as in the symmetric information case, and so is the firms’ problem:

\[
\max_{y \in Y} \sum_e \sum_s q^e_s y^e_s \quad (P_f)
\]

taking prices \(q\) as given.

Thus the only difference, with respect to the symmetric information case, generated by the fact that effort is privately observed, is the fact that the consumer’s optimization problem is subject to the additional constraint that consumption and effort choices have to be incentive compatible (or that \((c, e)\) are restricted to lie in the set \(Z_{MH}\)).
Definition 7 A PT equilibrium is given by an allocation $c \in \mathbb{R}^2$, effort level $e \in \{h,l\}$, a production vector $y \in \mathbb{R}^4$, and a price vector $q \in \Delta^2 \times \Delta^2$ such that:

(i) $(c, e)$ solves the agent’s optimization problem $(P_{PT})$, at prices $q$
(ii) $y$ solves the firms’ profit maximization problem $(P_f)$, at prices $q$;
(iii) markets clear:

$$\forall s, e, (c^e_s - w_s) \leq y^e_s$$  \hspace{1cm} (9)

5.1.2 Refined Competitive Equilibria with Non-Linear Prices

A fully non-linear system of prices in principle is characterized by the fact that prices are an arbitrary - possibly non-linear function of the net trades of the consumption good in each state, $c - w = (c_H - w_H, c_L - w_L)$. Each agent solves the following problem:

$$\max_{c^H, c^L, e} \sum_s \pi^e_s u(c^e_s) - v(e) \hspace{1cm} (P_{NL,MH})$$

s.t.

$$q(c - w) \leq 0$$

No restriction is imposed here on the set of admissible trades and prices do not depend on the unobservable level of effort.

Let $C \equiv [0, \sum_s \pi^h_s w_s]^2$. With no loss of generality we can restrict consumers’ possible consumption choices to the compact set $C^2$. Given the non-convexity of the agents’ budget feasible choices of consumption and effort, we shall explicitly allow here for the possibility that they ‘randomize’ in their choices: for each $c \in C$, then $\lambda(c) \in [0, 1]$ be the fraction of agents choosing consumption $c$; similarly, let $h(c) \in [0, 1]$ be the proportion of the agents with consumption level $c$ who choose effort $h$, for all $c \in C$.

Unlike the previous case, firms are now unable to offer claims directly contingent on the agents’ effort level. Thus their problem consists in the choice of how much to offer of each contract. The set of all possible contracts is identified by the set of all possible specifications of net payments to the agents in the $H$ and $L$ states (geometrically, all points in the two-dimensional orthant); as for consumers, they can be restricted to the compact set $C - w$. The subset of contracts which is feasible is the subset of contracts which are self-financing, or require a net payment not exceeding zero. In the present situation, where contracts are not directly contingent on the agents’ effort level, the net payment depends on the level of effort which firms anticipate will be chosen by consumers trading the contract. For any $y \equiv (y_L, y_H) \in C - w$, let $h(y)$ denote the firms’

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[Strictly speaking, the agents’ consumption choice problem is non-convex also in PT-equilibria. However there, for the simplicity of the presentation, the definition of equilibrium was presented for the case in which consumers’choices are non-random (and symmetric); also there the price faced by agents choosing high and low effort was different, unlike here.]
anticipation over the proportion of agents trading this contract choosing effort $h$.

The set of feasible contracts is then:

$$Y(h(.)) = \{ y \in C - w : \sum_s (\pi_s^h h(y) + \pi_s^l (1 - h(y))) y_s \leq 0 \}$$

while the firms’ technology is the set of possible quantities which can be supplied of each feasible contracts:

$$Y^{NL}(h(.)) = \{ \mu : Y(h(.)) \to \mathbb{R}_+ \}$$

The technology is still characterized by constant returns to scale and is, now, dependent on the firms’ expectation over the agents’ effort choice $h(.)$.

The firms’ problem is then the choice of a vector $y$, specifying a contract, lying in the set $Y^{NL}(h(.))$ so as to maximize profits:

$$\max_{\mu \in Y^{NL}(h(.))} \int_{Y(h(.))} \mu(y) q(y) \quad (P_{NL})$$

Market clearing requires now that the market for each contract clears, i.e. that for each point in $C - w$ supply by firms equal demand by consumers:

$$\lambda(c) \leq \mu(c - w), \quad \forall c \in C$$

(10)

In addition, at equilibrium firms’ expectations over the agents’ effort choices have to be correct:

$$h^f(c - w) = h(c), \quad \forall c \in C$$

(11)

Note that, when $\lambda(c) = 0$, i.e. when consumers are not choosing the contract yielding consumption $c$, their effort choice for that level of $c$ is indeterminate: any $h(c) = [0,1]$ satisfies the agents’ optimization problem. In this case, the consistency condition (11) has then no bite; this condition in fact only restricts firms’ expectations for the contracts which are traded at equilibrium.

As a consequence, we can have a large variety of equilibria, sustained by different beliefs by firms over the effort levels chosen by agents for the non traded contracts. Some of these equilibria are however supported by beliefs which are clearly ‘unreasonable’. To rule them out, and restrict the possible equilibria, a refinement (in the spirit of ‘trembling hand’) will be introduced.

The refinement can be written naturally. Consider the economy perturbed as follows: for each $c \in C$, a fraction $\epsilon(c) > 0$ of agents, such that

$$\int_C \epsilon(c) = \epsilon,$$

is constrained to choose $c$ (and is then free to pick optimally $c$). No restriction on the perturbation is imposed other than $\epsilon(c) > 0$ for all $c \in C$; the refinement imposed will thus be very mild.
Indexing the perturbation by $\epsilon$ an equilibrium of the perturbed economy is obtained by requiring market clearing as in (11), where now, obviously,

$$\lambda(c) \geq \epsilon(c) > 0, \forall c \in C.$$ 

A refined non-linear prices equilibrium is then defined as a competitive equilibrium with non-linear prices which is a limit point of a sequence of equilibria of the perturbed economy, for $\epsilon \to 0$.

Given the constant returns to scale of the firms’ technology, the existence of a solution of their choice problem requires $q(y) \leq 0$ for all $y \in Y(h^f(\cdot))$. Thus, for all $y$ such that $\mu(y) > 0$ we have $q(y) = 0$. Note also that condition (10) imposes a separate market clearing condition for each quantity (contract) traded, i.e. embodies a no cross-subsidization condition across different contracts.

**Definition 8** A $NL, MH$-equilibrium is given by consumers’ and firms’ choices $\lambda(c), h(c)$ and $\mu(c)$, for all $c \in C$, and a price map $q(c)$ for all $c \in C$ such that:

(i) $\lambda(\cdot), h(\cdot)$ are a solution of the agent’s optimization problem ($P_{NL, MH}$), at prices $q(\cdot)$ (i.e., $\lambda(c) > 0, h(c) > 0$ $\Rightarrow \sum_s \pi^s u(c_s) - v(c) \geq \sum_s \pi^s u(c'_s) - v(c')$ for $c, c' \in C, c' \in \{h, l\}$; analogously if $\lambda(c) > 0, h(e) = 0$; $\int_c \lambda(c) = 1$;

(ii) $\mu(\cdot)$ is a solution of the firms’ optimization problem (i.e, $\mu(y) > 0$ $\Rightarrow \mu(y) \geq \mu(y')$ for all $y, y' \in Y^{NL}(h^f(\cdot))$, given $q(\cdot), h^f(\cdot)$)

(iii) markets clear, (10);

(iv) firms expectations are consistent with consumers’ choices (11).

A $NL, MH(\epsilon)$ is a $NL, MH$ of the perturbed economy in which $1 - \epsilon$ agents solve problem ($P_{NL, MH}$), while $\epsilon(c)$ agents solve problem ($P_{NL, MH}$), constrained by $c_L = c$, for all $c \in C$.

A $RNLMH$ is a $NL, MH$ such that $(c, e), q(c)$ are the limit points of the sequence of allocations associated to $NL, MH(\epsilon)$, for $\epsilon \to 0$.

Once again, (incentive constrained versions of) the Welfare Theorems are straightforward.

**Proposition 1** Any PT equilibrium in the moral hazard economy is incentive constrained optimal.

**Proposition 2** Any incentive constrained optimal allocation of a moral hazard economy with symmetric information can be decentralized as a PT equilibrium with transfers.

Moreover PT equilibrium and RNLMH are equivalent.

**Proposition 3** Any PT equilibrium in the moral hazard economy is a RNLMH; and viceversa.
5.2 Adverse Selection

We first consider Prescott-Townsend equilibria, as introduced by Bisin-Gottardi (2004) to encompass the adverse selection case. We then consider refined non-linear prices equilibria, as introduced by Gale (1993) and Dubey-Geanakoplos-Shubik (2000) (though in somewhat different environments).

5.2.1 Prescott-Townsend Equilibria

As in the moral hazard case, the structure of existing markets is the same as when information is symmetric. Therefore, we allow prices to depend on the agents’ type, even though this is only privately observed: \(q_e \in \mathbb{R}_+\) is the unit price at which any agent who claims to be of type \(e \in \{h, l\}\) can trade the commodity for delivery in his individual state \(s \in \{H, L\}\). To ensure the viability of these markets even though the agents’ type is now only privately observable, the set of admissible trades of agents will be restricted by imposing incentive compatibility constraints.

However, the incentive compatibility constraints faced by an agent who claims to be of type \(i\) depend on the level of trades by agents of type \(e \neq e'\) agents. Thus, unlike with moral hazard, the imposition of incentive compatibility constraints here will impose an externality in the specification of the agents’ set of admissible trades. More precisely, let \(z^e \equiv \{z^e_s\}_{s \in S}, e \in \{h, l\}\), the set of admissible net trades for every agent is then defined as follows:

\[
Z(\bar{z}^h, \bar{z}^l) = \begin{cases} 
  z^h, z^l \in \mathbb{R}^d : \\
  z^h - w \geq 0, z^l - w \geq 0 \\
  z^e \neq 0 \implies z^{e'} = 0 \quad \forall e, e' \neq e \in \{h, l\}, \quad \text{and} \\
  \sum_{s \in S} \pi^g_s (w_s + z^h_s) \geq \sum_{s \in S} \pi^l_s (w_s + \bar{z}^l_s) \\
  \sum_{s \in S} \pi^l_s (w_s + z^l_s) \geq \sum_{s \in S} \pi^h_s (w_s + \bar{z}^h_s) 
\end{cases}
\]

where \(z^l, z^h\) denote the net trades made in the market by agents (who claim to be) respectively of type \(l, h\), taken as exogenously given.

The above specification reflects the following facts. Every agent can claim to be of type \(h\) and trade in the market for the type \(h\); alternatively, he can claim to be of type \(l\) and trade in the market for \(l\). If he chooses to trade in the market for \(h\), i.e. \(z^h \neq 0\), then he cannot trade in the market for \(l\), \(z^l = 0\). Moreover, the level of his net trades in the market for \(h\) has to satisfy the incentive compatibility constraints with respect to the level of net trades made by agents who claim to be of type \(l\), \(z^l\), taken as exogenously given. Such constraints require that agents of type \(h\) prefer (at least weakly) \(z^h\) to \(z^l\), and
similarly that type \( l \) agents prefer \( z^l \) to \( z^h \). Symmetric restrictions hold if the agent chooses instead to trade in the market for the \( l \) types.

The choice problem of an agent of type \( e \in \{h, l\} \) has then the following form:

\[
\max_{z \in Z(z^h, z^l)} \sum_{s \in S, e' \in \{h, l\}} \pi^e_s u \left(w_s + z^e_s\right) \quad (P^EPT)
\]

s.t.

\[
\sum_{s \in S, e' \in \{h, l\}} q^e_s z^e_s \leq 0
\]

As with moral hazard, the consumers’ problem is the same as with symmetric information, except for the restriction imposed on agents’ trades. Furthermore, in this case the level of net trades chosen by the other agents (of both types) in the economy enters this problem as it restricts the set of admissible trades, via the incentive compatibility constraints. Thus we have, formally, an externality in the consumption space which is not internalized in the model.

The firms’ problem is again the same as when information is symmetric:

\[
\max_{y \in Y} \sum_{e} \sum_{s} q^e_s y^e_s \quad (P^f)
\]

taking prices \( q \equiv [q^e_s]_{s \in S} \) as given.

Following Bisin and Gottardi (2004), we denote the competitive equilibrium here as EPT, for PT equilibrium with an externality.

**Definition 9** An EPT is given by a collection of net trades for each consumers’ type \( \{z^l, z^h\} \), a production vector \( y \), a price vector \( q \) and a pair \( \{z^l, z^h\} \) such that:

(i) \( (z^e, 0) \) solves the optimization problem \( (P^EPT) \) of consumers of type \( e \), at \( (q, z^h, z^l) \), for \( e \in \{h, l\} \);

(ii) \( y \) solves the firms’ profit maximization problem \( (P^f) \), at the prices \( q \);

(iii) markets clear:

\[
z^e_s \leq y^e_s, \quad e \in \{h, l\}, s \in S
\]

(iv) agents’ choices are consistent with the level of trades in the market, taken as given by agents:

\[
z^e_s = z^e_s, \quad e \in \{h, l\}, s \in S
\]

Condition (i) requires that, faced with prices \( q^h \) and \( q^l \), agents of type \( h \) prefer to trade at prices \( q^h \) and type \( l \) prefers to trade at \( q^l \). This is ensured by the presence of the incentive compatibility constraints in the specification of the agents’ set of admissible trades together with the consistency condition (iv).
5.2.2 Refined Competitive Equilibria with Non-Linear Prices

As in the moral hazard case, let \( q(c - w) \) be price - possibly non-linear - of the net trades \((c - w)\) of the consumption good in each state. Each agent of type \( e \in \{h, l\} \) solves the following problem:

\[
\max_{c_{H}, c_{L}} \sum_{s} \pi_{s}^{e} u(c_{s}) \quad (P_{NL,AS})
\]

s.t.

\[
q(c - w) \leq 0
\]

With no loss of generality we can restrict again consumers’ possible consumption choices to the compact set \( C \), for \( C \equiv [0, \sum_{s} \pi_{s}^{h} w_{s}]^{2} \).

For each \( c \in C \), then \( h(c) \in [0, 1] \) (\( l(c) \in [0, 1] \)) be the fraction of agents of type \( h \) (respectively, \( l \)) choosing consumption level \( c \).

Firms’ problem consists in the choice of how much to offer of each contract, where the set of all possible contracts is the set of all possible specifications of net payments to the agents in the \( H \) and \( L \) states, which can be restricted to the compact set \( C - w \). To determine the subset of contracts which is feasible - or self-financing - we need to specify the proportions of the two types which firms anticipate will choose the contract, as this determine the expected net payment on the contract. For any \( y \equiv (y_{L}, y_{H}) \in C^{2} - w \), let \( h^{f}(y) \) denote the firms’ anticipation over the proportion of agents trading this contract who are of type \( h \).

The specification of the set of feasible contracts is then the same as with moral hazard:

\[
\mathcal{Y}(h^{f}(.)) = \{ y \in C - w : \sum_{s} \left( \pi_{s}^{h} h^{f}(y) + \pi_{s}^{l}(1 - h^{f}(y)) \right) y_{s} \leq 0 \}
\]

and so the firms’ technology:

\[
\mathcal{Y}^{NL}(h^{f}(.)) = \{ \mu : \mathcal{Y}(h^{f}(.)) \to \Re_{+} \}
\]

The firms’ problem is then the choice of a vector \( y \), specifying a contract, lying in the set \( \mathcal{Y}^{NL}(h(.)) \) so as to maximize profits:

\[
\max_{\mu \in \mathcal{Y}^{NL}(h^{f}(.))} \int\mathcal{Y}(h^{f}(.)) \mu(y)q(y) \quad (P_{NL}^{f})
\]

Market clearing requires now that the market for each contract clears, i.e. that for each point in \( C - w \), supply by firms equal demand by consumers:

\[
h(c) + l(c) \leq \mu(c - w), \quad \forall c \in C \quad (13)
\]

In addition, at equilibrium firms’ expectations over the agents’ effort choices have to be correct:
\[ h^f(c-w) = \begin{cases} \frac{h(c)}{\mu(c)}, & \text{if } h(c) + l(c) > 0 \\ \text{arbitrary}, & \text{if } h(c) + l(c) = 0 \end{cases} \quad \forall c \in C \tag{14} \]

Note that, when \( h(c) + l(c) = 0 \), i.e. when no consumer of any type chooses the contract yielding consumption \( c \), the firms’ expectation over the proportion of type \( h \) agents trading the contract yielding \( c \) is indeterminate: any \( h^f(c) = [0, 1] \) is consistent with consumers’ choices. In this case, the consistency condition (14) has then no bite; this condition in fact only restricts firms’ expectations for the contracts which are traded at equilibrium.

As a consequence, we can have a large variety of equilibria, sustained by different beliefs by firms over the proportion of \( h \) and \( l \) types which would choose any traded contract if this were issued. To restrict the set of possible equilibria, a refinement (in the spirit of ‘trembling hand’) will be introduced.

As in the case of moral hazard, the perturbed economy is characterized by the fact that a (small) fraction of agents \( \epsilon(c) > 0 \) is constrained to buy the contract yielding \( c \), for each \( c \in C \) such that

\[ \int_C \epsilon(c) = \epsilon, \]

However now we also have to specify the composition of this fraction given by \( h \) and by \( l \) types; different specifications of this composition lead to different equilibrium sets. We will consider here the case where the fraction \( \epsilon(c) > 0 \) of agents constrained to trade contract \( c \) is made entirely by agents of type \( h \), for every \( c \). The equilibria we obtain will clearly depend in this case from this particular specification of the refinement.

Indexing the perturbation by \( \epsilon \) an equilibrium of the perturbed economy is obtained by requiring market clearing as in (13), where now, obviously,

\[ \lambda(c) \geq \epsilon(c) > 0, \quad \forall c \in C. \]

A refined non-linear prices equilibrium is then defined as a competitive equilibrium with non-linear prices which is a limit point of a sequence of equilibria of the perturbed economy, for \( \epsilon \to 0 \).

Given the constant returns to scale of the firms’ technology, the existence of a solution of their choice problem requires \( q(y) \leq 0 \) for all \( y \in Y(h^f(\cdot)) \). Thus, for all \( y \) such that \( \mu(y) > 0 \) we have \( q(y) = 0 \). Note also that condition (13) imposes a separate market clearing condition for each quantity (contract) traded, i.e. embodies a no cross-subsidization condition across different contracts.

**Definition 10** A NL,AS-equilibrium is given by choices of the two consumers’ types and firms, \( l(c), h(c) \) and \( \mu(c) \), for all \( c \in C \), a price map \( q(c) \) and firms’ anticipations \( h^f(c) \), for all \( c \in C \), such that:

(i) \( h(\cdot) \) - resp. \( l(\cdot) \) - is a solution of the type \( h \) (l) agents optimization problem
\( (P_{NL,AS}^\epsilon) \), at prices \( q(.) \) (i.e., \( h(c) > 0 \Rightarrow \sum_s \pi_s^h u(c_s) \geq \sum_s \pi_s^h u(c'_s) \) for \( c, c' \in C \)); analogously for \( l(.) \);
(ii) \( \mu(.) \) is a solution of the firms’ optimization problem (i.e, \( \mu(y) > 0 \Rightarrow \mu(y') \geq \mu(y) \) for all \( y, y' \in Y_{NL}^{h,l} \)), given \( q(.), h(.), l(.) \)
(iii) markets clear, (13);
(iv) firms expectations are consistent with consumers’ choices (14).

A \( NL,AS(\epsilon) \) is a \( NL,AS \) of the perturbed economy in which \( 1 - \epsilon \) agents solve problem \( (P_{NL,MH}) \), while \( \epsilon \) agents solve problem \( (P_{NL,MH}) \), constrained by \( c_L = c \), for all \( c \in C \).

A \( RN,AS \) is a \( NL,AS \) such that \( (c, e), q(c) \) are the limit points of the sequence of allocations associated to \( NL,AS(\epsilon) \), for \( \epsilon \to 0 \).

We first prove existence and characterize EPT equilibria.

**Proposition 4** There exists a unique EPT competitive equilibrium, given by a price vector \( q \) satisfying, for all \( e, s \):

\[
q^e_s = \pi^e_s
\]

a production plan \( y \) satisfying (12), and a consumption allocation \( \{c^h, c^l\} \) such that:

i. \( c_L^l = c_H^l = \pi_L w_L + \pi_H w_H \);
ii. \( (c^h - w, 0) \) solves the optimization problem \( (P_{EPT}^h) \) of consumers of type \( h \), at \( q, z^h = c^h - w, \) and \( z^l = c^l - w \).

The reader will have recognized the equilibrium allocation as the Rothschild-Stiglitz ‘pooling’ allocation. It is well known that the pooling allocation might not be incentive constrained optimal. The First Welfare Theorem therefore does not hold for EPT. A weaker efficiency result can be proved, however.

**Proposition 5** All EPT equilibrium allocations are efficient within the restricted set of feasible allocations which are incentive compatible and, in addition, satisfy the condition

\[
\sum_s \pi^e_s (c^e_s - w_s) = 0, \text{ for } e \in \{h, l\}
\]

On the other hand, a second welfare theorem result holds for the present structure of markets: any incentive efficient consumption allocation can be decentralized as an EPT equilibrium with transfers (possibly dependent on the state but not the agents’ type).

**Proposition 6** For any incentive efficient consumption allocation \( (c^h, c^l) \) there exists a set of transfers \( (t_H, t_L) \) (common for all types) which are feasible, i.e.,

\[
\sum_s [\xi^h \pi^h s + \xi^l \pi^l t_s] \leq 0, \text{ and such that } (c^h, c^l) \text{ is an EPT equilibrium allocation for the economy under consideration when each agent receives a transfer } (t_H, t_L).
\]
Finally, once again, we prove the equivalence of EPT and RNL,AS.

**Proposition 7** Any EPT equilibrium in an adverse selection economy is a RNL,AS; and viceversa.

### 6 Conclusions

We have considered competitive economies for insurance contracts under the assumption that the agents’ trades are perfectly observable, and hence exclusive contracts can be implemented. We will consider in a subsequent installment of these notes competitive economies in which no information is available over the trades each agent makes. This is an extreme case of lack of exclusivity. In this situation prices cannot vary with the quantity traded by an agent, and quantities cannot be used to try to separate different types/effort choices; resulting in competitive equilibria with linear prices.

### 7 References


