

Appendix:
 Computations for
 ”Moral Hazard and Non-Exclusive Contracts”
 by A. Bisin and D. Guaitoli*

June 2002

The model is introduced in Section 2 of ”Moral Hazard and Non-Exclusive Contracts.”. Our strategy involves solving for d_0 as a function of (c_H, c_L) and then substituting the solution i) in the indifference curves and ii) in the incentive constraint.

By the first order condition of the agent’s choice problem, d_0 solves:

$$u'(w_0 + d_0) = \pi_a u'(w_H + d_H - d_0) + (1 - \pi_a) u'(w_L + d_L - d_0) \quad (1)$$

Under the assumption that $u(c) = \ln(c)$, the equation can be written as follows:

$$\frac{1}{w_0 + d_0} = \frac{\pi_a}{c_H - d_0} + \frac{1 - \pi_a}{c_L - d_0}, \quad (2)$$

where we have used $c_s = w_s + d_s$, $s = 0, H, L$.

Therefore d_0 solves $(c_H - d_0)(c_L - d_0) = (w_0 + d_0)(\pi_a c_L + (1 - \pi_a)c_H - d_0)$, that is the following quadratic equation:

$$2d_0^2 + \{w_0 - A - (c_H + c_L)\}d_0 + c_H c_L - w_0 A = 0 \quad (3)$$

where $A = \pi_a c_L + (1 - \pi_a)c_H$.

Let $d_{0i}(c_L, c_H)$ denote a distinct root of the quadratic equation (3) given $(c_L, c_H) \in \mathbb{R}_+^2$. It follows that:

$$d_{01}(c_L, c_H) = \frac{(c_H + c_L + A - w_0) + \sqrt{(c_H + c_L + A - w_0)^2 - 8(c_H c_L - w_0 A)}}{4},$$

$$d_{02}(c_L, c_H) = \frac{(c_H + c_L + A - w_0) - \sqrt{(c_H + c_L + A - w_0)^2 - 8(c_H c_L - w_0 A)}}{4}.$$

*Ryo Nakayama’s exceptionally skillful assistance in this Appendix is gratefully acknowledged.

We will show next that $d_{01}(c_L, c_H)$ is not in fact a solution of the agent's optimization problem. This is because either $c_H < d_{01}(c_L, c_H)$ or $c_L < d_{01}(c_L, c_H)$ for any $(c_L, c_H) \in \mathbb{R}^2$; and therefore $d_{01}(c_L, c_H)$ is necessarily associated to negative consumption allocations.

Consider $d_{01}(c_L, c_H)$. Figure 1 draws the contour plot of the function $f_H^1(c_L, c_H) := c_H - d_{01}(c_L, c_H)$ in the (c_H, c_L) space. Notice that $f_H^1(c_L, c_H) = 0$ holds for $c_H = c_L$. In fact, independently from the parametrization we use, it is easy to check that

$$c_s = d_{01}(c_s, c_s), \quad s = H, L \quad (4)$$

$$\partial d_{01}(c_L, c_H) / \partial c_H < 1 \quad (5)$$

$$\partial d_{01}(c_L, c_H) / \partial c_L > 1. \quad (6)$$

It follows that $f_H^1(c_L, c_H) < 0$ if $c_L > c_H$. Similarly, $f_L^1(c_L, c_H) < 0$ if $c_L < c_H$. As a result,

$$\begin{aligned} \text{If } c_L > c_H &\Rightarrow c_H - d_{01}(c_L, c_H) < 0 \\ \text{If } c_L < c_H &\Rightarrow c_L - d_{01}(c_L, c_H) < 0. \end{aligned}$$

We therefore study the indifference curves

$$u(w_0 + d_0) + \pi_a u(w_H + d_H - d_0) + (1 - \pi_a) u(w_L + d_L - d_0) - (a - b) = k \quad (7)$$

in the space (c_L, c_H) , under the assumption that d_0 solves the quadratic equation (3), and is in fact given by $d_{02}(c_L, c_H)$. Let us denote the utility function by

$$f(c_L, c_H) = \ln(w_0 + d_{02}) + \pi_a \ln(c_H - d_{02}) + (1 - \pi_a) \ln(c_L - d_{02}) - (a - b) \quad (8)$$

where $d_{02} = d_{02}(c_L, c_H)$

Figure 4 shows the contour lines of the function $f(c_L, c_H)$ on the space (c_L, c_H) for the parameter values $(w_0, \pi_a, \pi_b, a) = (5.0, 0.5, 0.2, 0.1)$. These isoquants represent the indifference curves for equation (7).

We now study the incentive compatibility constraint. It can be written as

$$u(w_0 + d_0) + \pi_a u(w_H + d_H - d_0) + (1 - \pi_a) u(w_L + d_L - d_0) - a \geq \quad (9)$$

$$2u\left(\frac{1}{2}w_0 + \frac{1}{2}(\pi_b(w_H + d_H) + (1 - \pi_b)(w_L + d_L))\right) \quad (10)$$

where d_0 in general solves:

$$u'(w_0 + d_0) = \pi_a u'(w_H + d_H - d_0) + (1 - \pi_a) u'(w_L + d_L - d_0) \quad (11)$$

The set of incentive constrained allocations will be obtained by substituting $d_0 = d_{02}(c_L, c_H)$ into inequality (9). It is characterized by the set of $(c_L, c_H) \in \mathbb{R}^2$ that satisfy:

$$g(c_L, c_H) = \ln(w_0 + d_{02}) - \pi_a \ln(c_H - d_{02}) + (1 - \pi_a) \ln(c_L - d_{02}) - (a - b) \quad (12)$$

$$-2 \ln \left[\frac{1}{2} w_0 + \frac{1}{2} (\pi_b c_H + (1 - \pi_b) c_L) \right] \geq 0 \quad (13)$$

$$\text{where } d_{02} = d_{02}(c_L, c_H) \quad (14)$$

Figure 2 draws a surface graph of $z = g(c_L, c_H)$ on the domain:

$$\left\{ (c_L, c_H) \in \mathbb{R}^2 \mid c_H - d_{02}(c_L, c_H) > 0, \quad \text{and} \quad c_L - d_{02}(c_L, c_H) > 0 \right\}, \quad (15)$$

given the parameter value $(w_0, \pi_a, \pi_b, a) = (5.0, 0.5, 0.2, 0.1)$. For this parameter values, the graph intersects the plane $z = 0$. Thus, the IC region exists. It is shown by Figure 3.

We now construct an equilibrium with number of active intermediaries $n \rightarrow \infty$. We follow the construction in the text, Section 3, as from Hellwig's example. Fix $w := (w_H, w_L) = (12, 2)$. Let $\theta(c_H, c_L)$ be defined as follows:

$$\theta(c_H, c_L) := \frac{c_H - w_H + d_0(c_H, c_L)}{c_L - w_L + d_0(c_H, c_L)} \quad (16)$$

An equilibrium (c_L^*, c_H^*) satisfies the following system equation:

$$\begin{aligned} g(c_H, c_L) &= 0 \\ \theta(c_H, c_L) &= MRS(c_H, c_L). \end{aligned}$$

where $g(c_L, c_H) \geq 0$ is the incentive compatibility constraint derived above, and $MRS(c_H, c_L)$ denotes the marginal rate of the substitution of an indifference curve $f(c_L, c_H) = k$ at the point of (c_H, c_L) .

Let us denote $f_s(c_L, c_H) = \partial f / \partial c_s$ for $s = H, L$. The MRS is thus given by

$$MRS(c_L, c_H) = -\frac{f_L(c_L, c_H)}{f_H(c_L, c_H)}.$$

Taking derivatives of Equation (7) yields

$$f_H(c_L, c_H) = \frac{d_{0H}(c_L, c_H)}{w_0 + d_0(c_L, c_H)} + \pi_a \frac{1 - d_{0H}(c_L, c_H)}{c_H - d_0(c_L, c_H)} - (1 - \pi_a) \frac{d_{0H}(c_L, c_H)}{c_L - d_0(c_L, c_H)} \quad (17)$$

$$f_L(c_L, c_H) = \frac{d_{0L}(c_L, c_H)}{w_0 + d_0(c_L, c_H)} - \pi_a \frac{d_{0L}(c_L, c_H)}{c_H - d_0(c_L, c_H)} + (1 - \pi_a) \frac{1 - d_{0L}(c_L, c_H)}{c_L - d_0(c_L, c_H)} \quad (18)$$

where $d_{0s}(c_L, c_H)$ denotes the partial derivative of $d_0(c_L, c_H)$ with respect to c_s for $s = L, H$. It is straightforward (though tedious¹) to obtain the partial derivatives d_{0L} and d_{0H} :

$$d_{0H}(c_L, c_H) = \frac{1 + (1 - \pi_a)}{4} - \frac{\{1 + (1 - \pi_a)\}B + 4\{w_0(1 - \pi_a) - c_L\}}{4\sqrt{B^2 - 8(c_L c_H - w_0 A)}} \quad (19)$$

$$d_{0L}(c_L, c_H) = \frac{1 + \pi_a}{4} - \frac{(1 + \pi_a)B + 4\{w_0 \pi_a - c_L\}}{4\sqrt{B^2 - 8(c_L c_H - w_0 A)}} \quad (20)$$

where A and B denote

$$\begin{aligned} A &= \pi_a c_L + (1 - \pi_a) c_H, \\ B &= c_H + c_L + A - w_0. \end{aligned}$$

By plugging these equations from (17) to (20) into the MRS equation above, we will have an implicit function, which represents the constraint $\theta(c_L, c_H) = MRS(c_L, c_H)$. Let us use $h(c_L, c_H)$ to denote the equation

$$h(c_L, c_H) = \theta(c_L, c_H) - MRS(c_L, c_H).$$

¹The computations have been done by *Mathematica*.

Now we have to solve the system equation as follows:

$$\begin{aligned} g(c_L, c_H) &= 0 \\ h(c_L, c_H) &= 0 \end{aligned}$$

Figure 5 illustrates the constraints of $g = 0$ and $h = 0$ on the $c_L - c_H$ space. These two curves have two crossing points, say P and Q. The coordinates of the points will solve the system equation above. Only Q is an equilibrium (the analysis in the text, Section 3, for insurance economies applies here; moreover, to point P is associated a negative value for $d_{02}(c_L, c_H)$; see figure 6 where we draw the region where $d_{02}(c_L^*, c_H^*) < 0$.) Thus, the coordinate Q is the unique equilibrium (c_L^*, c_H^*) of the system equation. It also provides a positive $d_{02}(c_L^*, c_H^*)$.

We also solved the system of equations

$$\begin{aligned} g(c_L, c_H) &= 0 \\ h(c_L, c_H) &= 0 \end{aligned}$$

by a numerical method, using the function `solve` provided by the *optimize package* in Matlab. Starting from an initial guess (5.0, 7.0), we obtained the equilibrium:

$$(c_L^*, c_H^*) = (4.8587, 6.9467)$$

Figure 7 explains the equilibrium construction graphically. The lower dotted line is the the tangent line of the indifferent curve $f(c_L, c_H) = 0$ at the equilibrium point $(c_L^*, c_H^*) = (4.8587, 6.9467)$. Its slope is given by $MRS(c_L^*, c_H^*) := -\frac{f_L(c_L^*, c_H^*)}{f_H(c_L^*, c_H^*)}$. It is parallel to the upper dotted line that goes through the point (w_L, w_H) and the point $(c_L^* + d_0(c_L^*, c_H^*), c_H^* + d_0(c_L^*, c_H^*))$. The slope is given by $\theta(c_L^*, c_H^*) := \frac{c_H^* - w_H + d_0(c_L^*, c_H^*)}{c_L^* - w_L + d_0(c_L^*, c_H^*)}$. These two slopes are same, and therefore $\theta(c_L^*, c_H^*) = MRS(c_L^*, c_H^*)$.

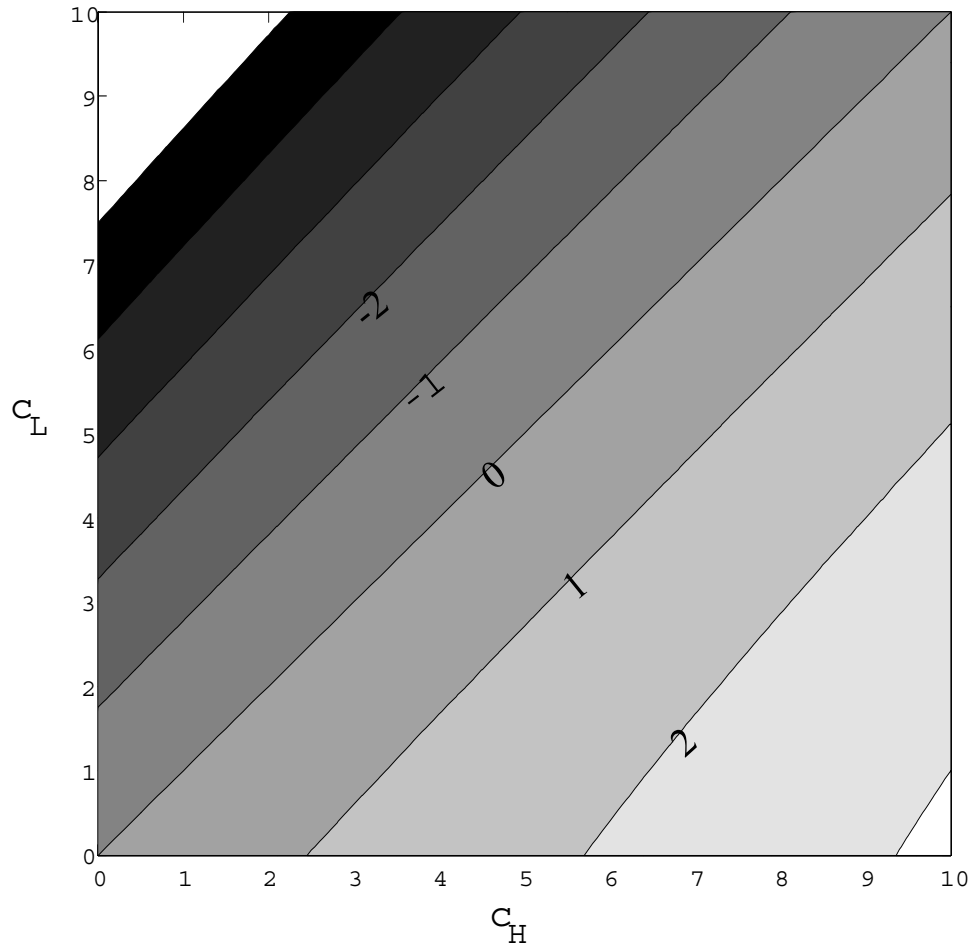


Figure 1: The contour graph of $f_H^1(c_L, c_H) = c_H - d_{01}(c_L, c_H)$

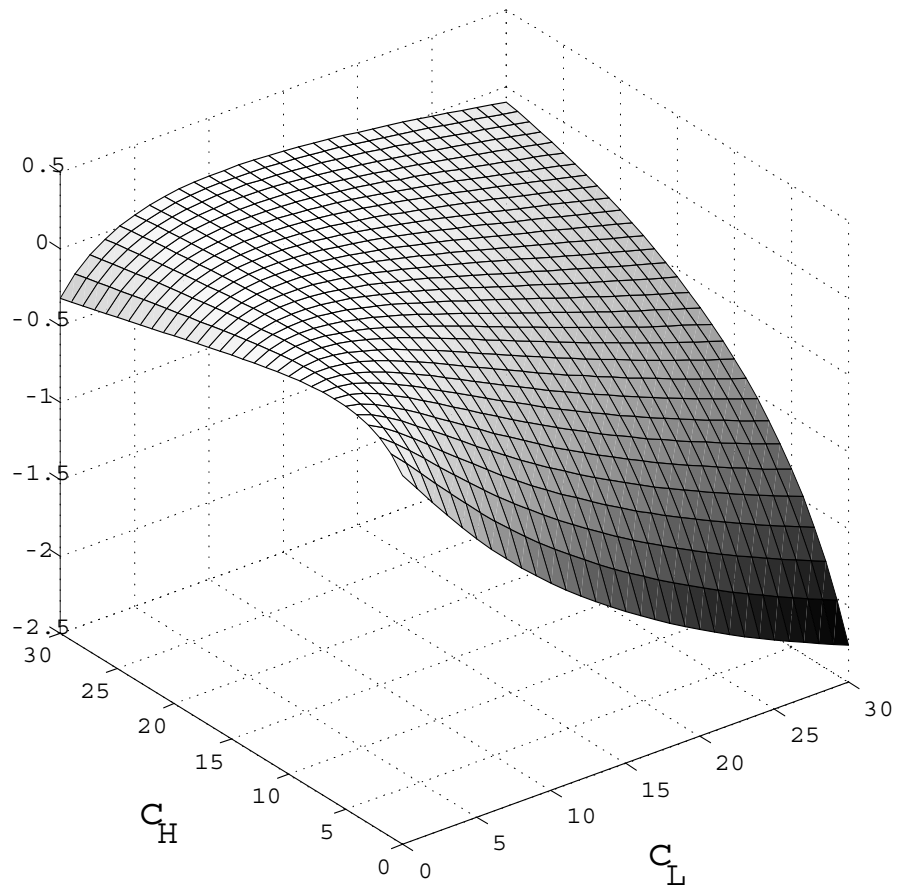


Figure 2: The graph of $z = g(c_L, c_H)$

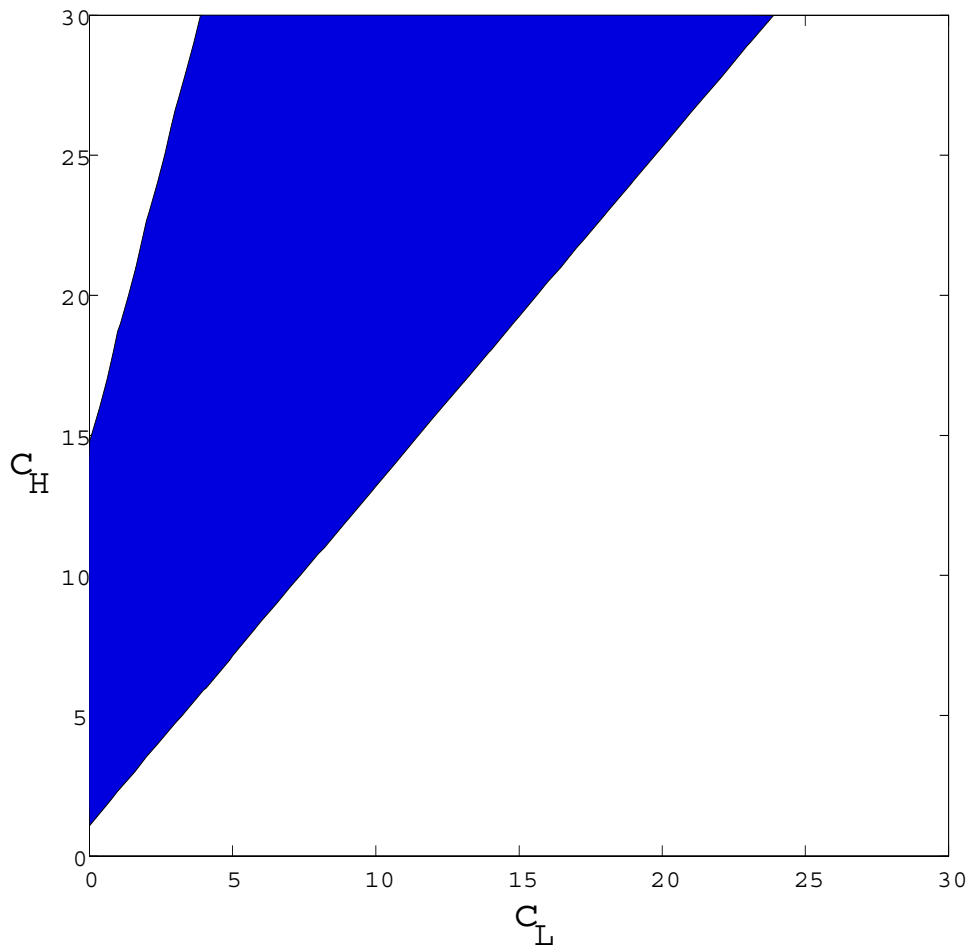


Figure 3: The Set of Incentive Compatible Allocations

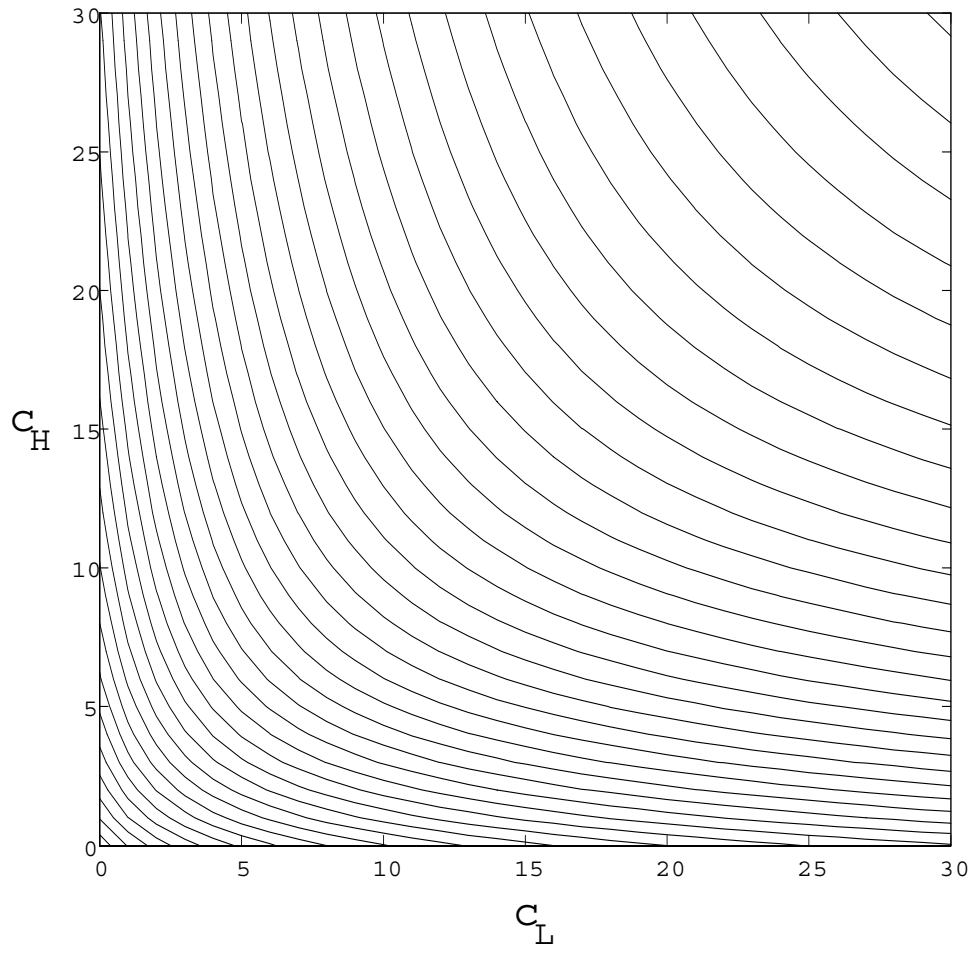


Figure 4: The Indifference Curves

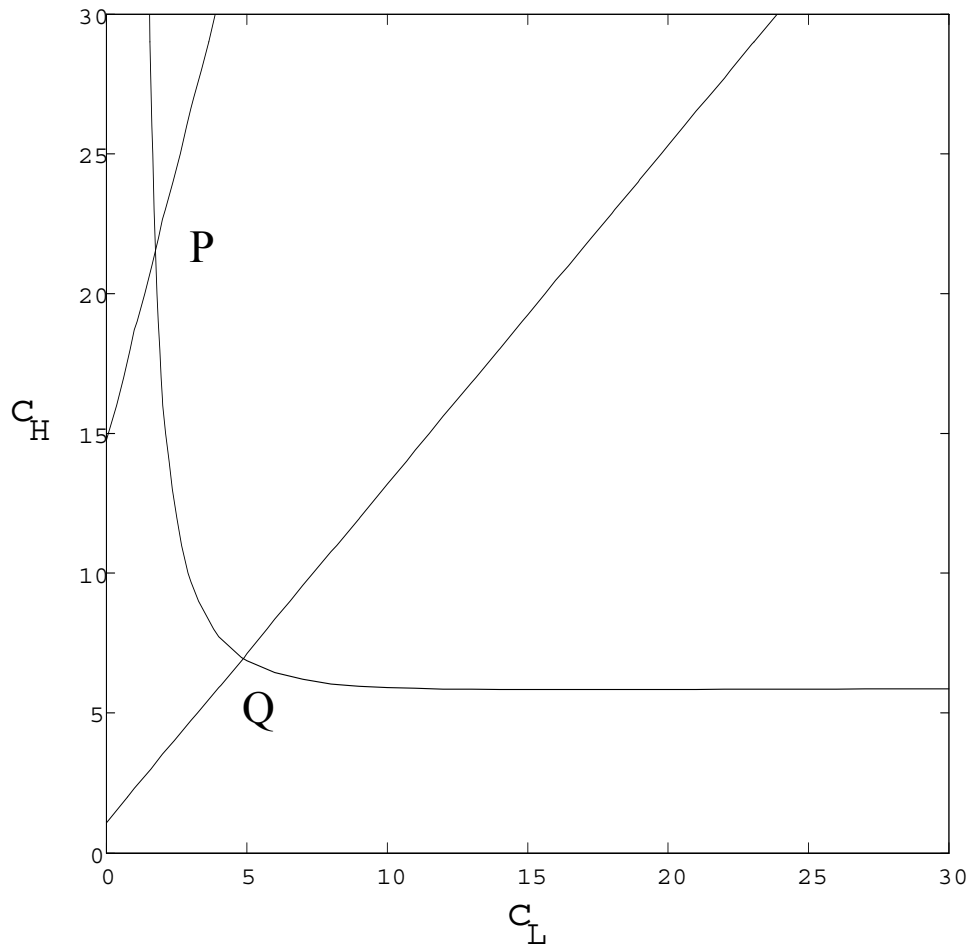


Figure 5: $g(c_L, c_H) = 0$ and $h(c_L, c_H) = 0$

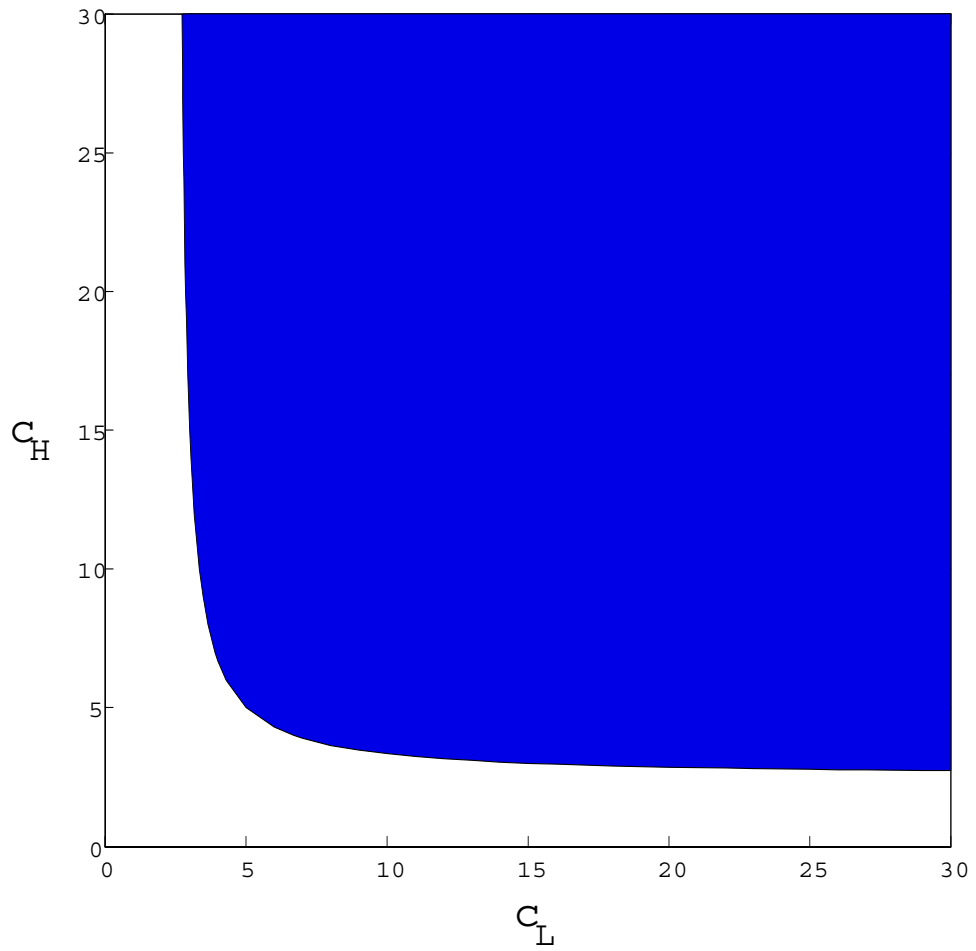


Figure 6: The Set of Allocations Satisfying $g(c_L, c_H) > 0$

Figure 7

