Appendix:
Computations for
”Moral Hazard and Non-Exclusive Contracts”
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The model is introduced in Section 2 of ”Moral Hazard and Non-Exclusive Contracts.”. Our strategy involves solving for \( d_0 \) as a function of \((c_H, c_L)\) and then substituting the solution i) in the indifference curves and ii) in the incentive constraint.

By the first order condition of the agent’s choice problem, \( d_0 \) solves:

\[
u'(w_0 + d_0) = \pi_a u'(w_H + d_H - d_0) + (1 - \pi_a) u'(w_L + d_L - d_0)\tag{1}\]

Under the assumption that \( u(c) = \ln(c) \), the equation can be written as follows:

\[
\frac{1}{w_0 + d_0} = \frac{\pi_a}{c_H - d_0} + \frac{1 - \pi_a}{c_L - d_0},\tag{2}
\]

where we have used \( c_s = w_s + d_s, s = 0, H, L \).

Therefore \( d_0 \) solves \((c_H - d_0)(c_L - d_0) = (w_0 + d_0)(\pi_a c_L + (1 - \pi_a) c_H - d_0)\), that is the following quadratic equation:

\[
2d_0^2 + \{w_0 - A - (c_H + c_L)\} d_0 + c_H c_L - w_0 A = 0\tag{3}
\]

where \( A = \pi_a c_L + (1 - \pi_a) c_H \).

Let \( d_{01}(c_L, c_H) \) denote a distinct root of the quadratic equation (3) given \((c_L, c_H) \in \mathbb{R}_+^2\). It follows that:

\[
d_{01}(c_L, c_H) = \frac{(c_H + c_L + A - w_0) + \sqrt{(c_H + c_L + A - w_0)^2 - 8(c_H c_L - w_0 A)}}{4},
\]

\[
d_{02}(c_L, c_H) = \frac{(c_H + c_L + A - w_0) - \sqrt{(c_H + c_L + A - w_0)^2 - 8(c_H c_L - w_0 A)}}{4}.
\]

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We will show next that $d_{01}(c_L, c_H)$ is not in fact a solution of the agent’s optimization problem. This is because either $c_H < d_{01}(c_L, c_H)$ or $c_L < d_{01}(c_L, c_H)$ for any $(c_L, c_H) \in \mathbb{R}^2$; and therefore $d_{01}(c_L, c_H)$ is necessarily associated to negative consumption allocations.

Consider $d_{01}(c_L, c_H)$. Figure 1 draws the contour plot of the function $f_H^1(c_L, c_H) := c_H - d_{01}(c_L, c_H)$ in the $(c_H, c_L)$ space. Notice that $f_H^1(c_L, c_H) = 0$ holds for $c_H = c_L$. In fact, independently from the parametrization we use, it is easy to check that

\begin{align*}
c_s &= d_{01}(c_s, c_s), \quad s = H, L \\
\partial d_{01}(c_L, c_H)/\partial c_H &< 1 \\
\partial d_{01}(c_L, c_H)/\partial c_L &> 1.
\end{align*}

It follows that $f_H^1(c_L, c_H) < 0$ if $c_L > c_H$. Similarly, $f_H^1(c_L, c_H) < 0$ if $c_L < c_H$. As a result,

\begin{align*}
\text{If } c_L > c_H &\Rightarrow c_H - d_{01}(c_L, c_H) < 0 \\
\text{If } c_L < c_H &\Rightarrow c_L - d_{01}(c_L, c_H) < 0.
\end{align*}

We therefore study the indifference curves

\begin{equation}
u(w_0 + d_0) + \pi_a u(w_H + d_H - d_0) + (1 - \pi_a) u(w_L + d_L - d_0) - (a - b) = k
\end{equation}

in the space $(c_L, c_H)$, under the assumption that $d_0$ solves the quadratic equation (3), and is in fact given by $d_{02}(c_L, c_H)$. Let us denote the utility function by

\begin{equation}
f(c_L, c_H) = \ln(w_0 + d_0) + \pi_a \ln(c_H - d_0) + (1 - \pi_a) \ln(c_L - d_0) - (a - b)
\end{equation}

where $d_{02} = d_{02}(c_L, c_H)$.

Figure 4 shows the contour lines of the function $f(c_L, c_H)$ on the space $(c_L, c_H)$ for the parameter values $(w_0, \pi_a, \pi_b, a) = (5.0, 0.5, 0.2, 0.1)$. These isolines represent the indifference curves for equation (7).

We now study the incentive compatibility constraint. It can be written as

\begin{equation}
u(w_0 + d_0) + \pi_a u(w_H + d_H - d_0) + (1 - \pi_a) u(w_L + d_L - d_0) - a \geq 0
\end{equation}

\begin{equation}2u\left(\frac{1}{2}w_0 + \frac{1}{2}w_H + d_H \right) + (1 - \pi_b)(w_L + d_L))\right)
\end{equation}

where $d_0$ in general solves:

\begin{equation}u'(w_0 + d_0) = \pi_a u'(w_H + d_H - d_0) + (1 - \pi_a) u'(w_L + d_L - d_0)
\end{equation}
The set of incentive constrained allocations will be obtained by substituting $d_0 = d_{02}(c_L, c_H)$ into inequality (9). It is characterized by the set of $(c_L, c_H) \in \mathbb{R}^2$ that satisfy:

$$g(c_L, c_H) = \ln(w_0 + d_{02}) - \pi_a \ln(c_H - d_{02}) + (1 - \pi_a) \ln(c_L - d_{02}) - (a - b)$$

$$-2 \ln \left[ \frac{1}{2} w_0 + \frac{1}{2} (\pi_b c_H + (1 - \pi_b) c_L) \right] \geq 0$$

where $d_0 = d_{02}(c_L, c_H)$

Figure 2 draws a surface graph of $z = g(c_L, c_H)$ on the domain:

$$\left\{ (c_L, c_H) \in \mathbb{R}^2 \middle| c_H - d_{02}(c_L, c_H) > 0, \text{ and } c_L - d_{02}(c_L, c_H) > 0 \right\}$$

given the parameter value $(w_0, \pi_a, \pi_b, a) = (5.0, 0.5, 0.2, 0.1)$. For this parameter values, the graph intersects the plane $z = 0$. Thus, the IC region exists. It is shown by Figure 3.
We now construct an equilibrium with number of active intermediaries \( n \to \infty \). We follow the construction in the text, Section 3, as from Hellwig’s example. Fix \( w := (w_H, w_L) = (12, 2) \). Let \( \theta(c_H, c_L) \) be defined as follows:

\[
\theta(c_H, c_L) := \frac{c_H - w_H + d_0(c_H, c_L)}{c_L - w_L + d_0(c_H, c_L)}
\]

An equilibrium \((c^*_L, c^*_H)\) satisfies the following system equation:

\[
\begin{align*}
g(c_H, c_L) &= 0 \\
\theta(c_H, c_L) &= MRS(c_H, c_L).
\end{align*}
\]

where \( g(c_L, c_H) \geq 0 \) is the incentive compatibility constraint derived above, and \( MRS(c_H, c_L) \) denotes the marginal rate of the substitution of an indifference curve \( f(c_L, c_H) = k \) at the point of \((c_H, c_L)\).

Let us denote \( f_s(c_L, c_H) = \partial f/\partial c_s \) for \( s = H, L \). The MRS is thus given by

\[
MRS(c_L, c_H) = -\frac{f_L(c_L, c_H)}{f_H(c_L, c_H)}.
\]

Taking derivatives of Equation (7) yields

\[
\begin{align*}
f_H(c_L, c_H) &= \frac{d_0H(c_L, c_H)}{w_0 + d_0(c_L, c_H)} + \pi_a \frac{1 - d_0H(c_L, c_H)}{c_H - d_0(c_L, c_H)} \quad (17) \\
f_L(c_L, c_H) &= -\frac{d_0L(c_L, c_H)}{w_0 + d_0(c_L, c_H)} - \pi_a \frac{d_0L(c_L, c_H)}{c_H - d_0(c_L, c_H)} + (1 - \pi_a) \frac{1 - d_0L(c_L, c_H)}{c_L - d_0(c_L, c_H)} \quad (18)
\end{align*}
\]

where \( d_0_s(c_L, c_H) \) denotes the partial derivative of \( d_0(c_L, c_H) \) with respect to \( c_s \) for \( s = L, H \). It is straightforward (though tedious\(^3\)) to obtain the partial derivatives \( d_0_L \) and \( d_0_H \):

\[
\begin{align*}
d_0H(c_L, c_H) &= \frac{1 + (1 - \pi_a)}{4} \cdot \frac{[1 + (1 - \pi_a)]B + 4\{w_0(1 - \pi_a) - c_L\}}{4\sqrt{B^2 - 8(c_Lc_H - w_0A)}} \quad (19) \\
d_0L(c_L, c_H) &= \frac{1 + \pi_a}{4} \cdot \frac{(1 + \pi_a)B + 4\{w_0\pi_a - c_L\}}{4\sqrt{B^2 - 8(c_Lc_H - w_0A)}} \quad (20)
\end{align*}
\]

where \( A \) and \( B \) denote

\[
\begin{align*}
A &= \pi_a c_L + (1 - \pi_a)c_H, \\
B &= c_H + c_L + A - w_0.
\end{align*}
\]

By plugging these equations from (17) to (20) into the MRS equation above, we will have an implicit function, which represents the constraint \( \theta(c_L, c_H) = MRS(c_L, c_H) \). Let us use \( h(c_L, c_H) \) to denote the equation

\[
h(c_L, c_H) = \theta(c_L, c_H) - MRS(c_L, c_H).
\]

\(^3\)The computations have been done by Mathematica.
Now we have to solve the system equation as follows:

\[
\begin{align*}
g(c_L, c_H) &= 0 \\
h(c_L, c_H) &= 0
\end{align*}
\]

Figure 5 illustrates the constraints of \( g = 0 \) and \( h = 0 \) on the \( c_L - c_H \) space. These two curves have two crossing points, say P and Q. The coordinates of the points will solve the system equation above. Only Q is an equilibrium (the analysis in the text, Section 3, for insurance economies applies here; moreover, to point P is associated a negative value for \( d_{02}(c_L^*, c_H^*) \); see Figure 6 where we draw the region where \( d_{02}(c_L^*, c_H^*) < 0 \).) Thus, the coordinate Q is the unique equilibrium \((c_L^*, c_H^*)\) of the system equation. It also provides a positive \( d_{02}(c_L^*, c_H^*) \).

We also solved the system of equations

\[
\begin{align*}
g(c_L, c_H) &= 0 \\
h(c_L, c_H) &= 0
\end{align*}
\]

by a numerical method, using the function \texttt{solve} provided by the \textit{optimize package} in Matlab. Starting from an initial guess \((5.0, 7.0)\), we obtained the equilibrium:

\((c_L^*, c_H^*) = (4.8587, 6.9467)\)

Figure 7 explains the equilibrium construction graphically. The lower dotted line is the tangent line of the indifferent curve \( f(c_L, c_H) = 0 \) at the equilibrium point \((c_L^*, c_H^*) = (4.8587, 6.9467)\). Its slope is given by \( MRS(c_L^*, c_H^*) := -\frac{f_L(c_L^*, c_H^*)}{f_H(c_L^*, c_H^*)} \). It is parallel to the upper dotted line that goes through the point \((w_L, w_H)\) and the point \((c_L^* + d_0(c_L^*, c_H^*), c_H^* + d_0(c_L^*, c_H^*))\). The slope is given by \( \theta(c_L^*, c_H^*) := \frac{c_H^* - w_H + d_0(c_L^*, c_H^*)}{c_L^* - w_L + d_0(c_L^*, c_H^*)} \). These two slopes are same, and therefore \( \theta(c_L^*, c_H^*) = MRS(c_L^*, c_H^*) \).
Figure 1: The contour graph of $f_H(c_L,c_H) = c_H - d_{01}(c_L,c_H)$
Figure 2: The graph of $z = g(c_L, c_H)$
Figure 3: The Set of Incentive Compatible Allocations
Figure 4: The Indifference Curves
Figure 5: $g(c_L, c_H) = 0$ and $h(c_L, c_H) = 0$
Figure 6: The Set of Allocations Satisfying $g(c_L, c_H) > 0$
Figure 7

(cL, cH)

(wH, wL)

(wH - d0, wL - d0)

(cL, cH)

(cL*, cH*)