Competition in Financial Innovation*

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Abstract

This paper examines the incentives to innovate securities provided by frictionless competitive markets (with short sales) to entrepreneurs. In economies with symmetric investor utilities, we provide the conditions under which a firm’s market value is maximized by a complete financial structure – that is, when the mechanism that gives rise to market incompleteness is the one illustrated in the classical example of Allen and Gale (1991). The foundation for incompleteness in that case is derived from innovation costs and free riding among entrepreneurs. For the class of preferences where an incomplete financial structure maximizes the firms’ market values, markets are incomplete with a probability of one, even when innovation is cost free. In this sense, we provide an alternative to Allen and Gale’s foundation for market incompleteness.

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1 Introduction

Over the past three short decades, an unprecedented number of asset innovations have been introduced to financial markets. Straps, swaps, CMOs, and putable convertibles have all been developed in this period. The present paper examines incentives that financial markets offer to innovate assets. The choice as to which securities to offer is fundamental for central

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banks, treasury departments and other institutional investors. Understanding incentives to innovate assets is, perhaps, even more relevant from a macroeconomic perspective. Financial innovation determines whether markets are complete or incomplete and, hence, is critical for understanding both efficiency and how markets respond to various economic shocks.

Extensive literature identifies several potential reasons why financial innovation can occur in a market – the most primitive being a “spanning” motive: by introducing new securities innovators benefit from demand for risk-sharing. Other strands of literature attribute innovation incentives to frictions: asymmetric information, or transaction costs. This paper belongs to the literature that focuses on the span of transfers of revenue permitted by the set of securities available for trade in a market. We are interested in understanding entrepreneurs’ incentives to innovate in frictionless markets, in which investors are allowed short sales. The seminal paper of Allen and Gale (1991) introduces a framework in which entrepreneurs’ innovate assets and asset structures are determined endogenously in equilibrium. In addition, they established abstract results about equilibrium properties – such as existence – and provided an important numerical example. However, as emphasized by Duffie and Rahi (1995) in their survey, there are few concrete normative results in the spanning literature and they have been demonstrated in specific numerical examples. Overall, the general message from the spanning literature on frictionless markets inspired by Allen and Gale (1991) is that even though a complete financial structure maximizes a firm’s market value, markets can be incomplete due to innovation costs and free riding problems. Such predictions are specific to parametric assumptions. This paper contributes to the literature as follows. In a model with symmetric investor utilities, general conditions are provided with regard to investors preferences under which complete (or incomplete) financial structures maximize a firm value. For markets wherein the incomplete market foundations of Allen and Gale (1991)—based on innovation costs and free riding problems—does not apply, we characterize a mechanism that robustly gives rise to market incompleteness in large markets, even when innovation is without cost.

The conditions that determine the optimality of (in)complete financial structure are based on the (shape) of investors’ marginal utilities. In the model with two states, the financial structure that maximizes the market value of a firm is characterized completely: when investor marginal utility is concave, a complete financial structure is optimal. However, when investor marginal utility is convex, selling equity (i.e., selling shares of a real asset) maximizes revenue. For concave marginal utility, the optimal nature of complete financial structures extends to settings with arbitrary numbers of states. However, market value need not be monotone in the security span when there are more than two states of the world. It follows that in the case of convex marginal utility, although complete financial structures remain suboptimal, financial structures that maximize market value may be richer than equity. The
particular form of the optimal financial structure depends on the relative convexity of the marginal utility across the states, the firms’ returns in each state and the heterogeneity of investors endowments.

The characterization of optimal financial structure allows us to find clear results regarding market (in)completeness in economies with large numbers of entrepreneurs. In economies with concave marginal utilities, our predictions coincide with those of Allen and Gale (1991): when entrepreneurs simultaneously choose financial structures, large markets can be incomplete with positive probability, provided that innovation costs are positive. In a two-state economy with convex marginal utility, our predictions change dramatically: markets are incomplete with \textit{probability one} for an arbitrary number of entrepreneurs and regardless of innovation costs. Quite surprisingly, this result need not extend to settings with more than two states, where markets can be complete with positive probability even when each entrepreneur strictly prefers (any) incomplete financial structure and innovation costs are positive. Such an undesirable (from the entrepreneurs’ perspective) complete market outcome can arise in economies in which the value of a firm is not monotone in the security span. This results from the entrepreneurs’ inabilities to coordinate on one optimal financial structure. The outcome, that markets are complete with positive probability, results in \textit{ex post} regret on the part of each entrepreneur and should not be expected in long run interactions. To capture this intuition, we complement our static analysis by considering a financial innovation model in which entrepreneurs choose their financial structures sequentially. In a dynamic economy, the coordination problem does not arise, and under general conditions, large markets are complete with concave marginal utility (as long as innovation costs are not prohibitively high), while they are incomplete when marginal utilities are convex.

A central economic insight from our paper is that large market completeness critically depends on the shape of investors’ marginal utilities. Determining whether convex or concave marginal utility is more plausible requires a theory on the third derivative of the utility functions of financial investors. Empirical evidence supporting theories based on the third derivative, e.g., precautionary savings, provides some support in favor of convex marginal utility (“prudence”). Given such preferences (e.g., logarithmic, CARA, CRRA utility functions), firm’s market value is maximized by an incomplete financial structure. Thus, in frictionless markets with short sales, demand for risk-sharing does not provide sufficient incentive for entrepreneurs to complete financial markets. Then, the incentive to innovate must originate elsewhere, e.g. commissions from trading assets (Pesendorfer, 1995), short sale restrictions, asymmetric information, or innovation-subsidizing policies. Our results also provide strong welfare recommendations. An innovation-encouraging policy that reduces innovation costs might be effective if marginal utility is concave, but such policy is ineffective in the case of convex marginal utility.
2 Model

In the literature, the normative predictions regarding incentives to innovate in frictionless markets by entrepreneurs are based on (versions of) the classic example of Allen and Gale (1991).

Example 1. (Allen and Gale): Consider a two-period economy with \( N \) entrepreneurs and a continuum of investors. Each entrepreneur is endowed with a real asset (a firm) which in the second period gives return \( z = (0.5, 2.5) \) in terms of numéraire. In the first period, entrepreneurs, who derive utility from consumption in period one, sell their claims to the return to two types of competitive investors. Investor types differ in utilities \( U_1(c_1, c_2) = 5 + c_1 - \exp(-10c_2) \) and \( U_2(c_1, c_2) = 5 + c_1 + \ln(c_2) \) and mass of each type is normalized to \( 0.5N \). Entrepreneurs simultaneously choose among two financial structures: each can costlessly issue equity, in which case one market opens and shares of a firm are traded; alternatively, at a cost, the entrepreneur can innovate by issuing two contingent claims, in which case, two markets open.

There are no assets in the economy before entrepreneurs issue securities. Therefore, if all entrepreneurs choose to issue equity, financial markets are incomplete. If one or more entrepreneur innovates, financial markets become complete. A central question in the literature on asset innovation is whether competition among entrepreneurs provides sufficient incentives to innovate so that large markets are complete. The Allen and Gale example demonstrates that markets can be incomplete with positive probability even if they grow large: Arbitrage ensures that, in equilibrium, firms with identical returns have the same market values, which, in the numerical example of Allen and Gale, is greater under complete markets (i.e., \( MV_C = 0.58603 > 0.58583 = MV_I \)). Thus, completing the market is essentially a public good: all entrepreneurs are better off if one pays an innovation cost to introduce contingent claims. In equilibrium, each entrepreneur chooses to innovate with a positive probability, which decreases in market size. With a larger number of entrepreneurs, the free riding problem becomes more severe—ceteris paribus, for each entrepreneur, the probability that at least one other entrepreneur introduces contingent claims increases. This reduces individual incentives to innovate and the probability that one or more entrepreneurs innovate is bounded away from one. A general lesson from the Allen and Gale example is that large, frictionless markets may be incomplete, due to the free riding problem in the presence of innovation costs. Clearly, the fact that innovation is costly is necessary for the free riding mechanism to operate, for otherwise markets are complete.

Notice however that, as it turns out, if the utility of type-1 investors is \( U_1(c_1, c_2) = 5 + c_1 + \ln(c_2 + 2) \), the example’s predictions change dramatically. A firm’s market value is
maximized in *incomplete* markets ($MV_C = 2.0952 < 2.3228 = MV_I$). (An inequality reversal may occur if one changes asset payoffs or investor endowments as instead.) Innovation is then no longer a public good. Rather, it becomes a *public “bad”*: all entrepreneurs are worse off if at least one of them innovates. As a result, large markets are *incomplete with a probability of one, even if asset innovation is cost free*.

Both examples describe markets with plausible investor preferences. Yet, the corresponding predictions regarding endogenous market incompleteness differ markedly. Which offers more robust predictions? Which more aptly captures financial markets? Ideally, a theory of financial innovation identifies the economic mechanisms that underlie the distinct equilibrium predictions. The primary result of this paper is the determination of such a mechanism to offer sharp predictions in a general model with symmetric utilities. In particular, we provide the conditions that determine whether complete or incomplete asset structures maximizes market value, which allows us to characterize markets in which a public good mechanism, underlying the Allen and Gale example, operates. For the market environments where it does not, we also uncover an alternative rationale for equilibrium market incompleteness, which does not rely on innovation cost or free-riding, and which is stronger in the sense that market incompleteness occurs with a probability of one rather than a positive probability. We give conditions under which such a mechanism is present in financial markets.

### 2.1 Model with Symmetric Utilities

As in Allen and Gale (1991), we consider a two-period economy with $N$ entrepreneurs indexed by $n \in \{1, 2, \ldots, N\}$ and a continuum of competitive investors. Each entrepreneur $n$ owns a real asset (e.g., a firm) that pays in terms of numéraire at date 2 in $S$ states of the world, $z_n \in \mathbb{R}_+^S$. To sell claims to asset returns $z_n$, entrepreneurs (simultaneously) issue securities at date 1. Each entrepreneur can choose from a wide variety of alternative selling strategies. One possibility is to open an equity market and sell shares of the asset. Another alternative is to issue state-contingent claims that each pay one unit of the numéraire in a corresponding state and to sell the quantity given by the payoff of the real asset $z_n$ in a given state. More generally, an entrepreneur can issue a portfolio of $I_n$ securities. A *financial structure* specifies payments of issued securities $F_n = \{f_1, \ldots, f_{I_n}\}$, where $f_i \in \mathbb{R}^S$ is a promised payment of security $i$ in terms of numéraire, and the supply of each issued security is $t_n \in \mathbb{R}_{I_n}^S$. We treat $F_n$ as an $S \times I_n$ matrix. A financial structure $(F_n, t_n)$ is required to exhaust returns to the real asset, so that the entrepreneur is solvent in the second period; that is, $F_n t_n = z_n$. At date 1, $I = \sum_{n=1}^N I_n$ markets open for all securities: $F = \{F_1, \ldots, F_N\}$, which is an $S \times I$ matrix with supply $t = (t_1', \ldots, t_N')' \in \mathbb{R}^I$. The collection of all financial structures $(F, t)$ such that each $(F_n, t_n)$ satisfies the solvency condition for each $n$ is denoted
by $\mathcal{F}$. At date 2, payments against securities are made and consumption occurs.

The (column) span of $F$, which is the linear subspace of $\mathbb{R}^S$ defined as $\langle F \rangle \equiv \{ x^k \in \mathbb{R}^S | Ft^k = x^k \text{ for some } t^k \in \mathbb{R}^I \}$, gives the set of all numéraire transfers at date 2 that can result from trades of offered securities $F$. The financial structure $(F, t)$ is said to be complete if the rank of $F$ equals $S$ ($\langle F \rangle = \mathbb{R}^S$); otherwise, $(F, t)$ is incomplete.

Competitive investors derive utility from consuming numéraire in both periods. Investor preferences are quasilinear, Von Neumann-Morgenstern $U_1(c_1, c_2) = c_1 + E[u(c_2)]$, where $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies the standard assumptions of $C^2$, strict monotonicity, strict concavity and Inada condition, $\lim_{c_2 \to 0} u'(c_2) = \infty$. There are $K$ types of non-atomic investors indexed by $k \in \{1, \ldots, K\}$, who differ in initial endowments at date 2, $e^k \in \mathbb{R}_+^S$; the mass of each type is normalized to $N$. Thus, investors have a common utility function over consumption; heterogeneity in initial endowments gives rise to heterogeneity in preferences over securities trades. Entrepreneurs do not know the realizations of initial investor endowments; each holds probabilistic beliefs over profile $(e^1, \ldots, e^K)$, given by the joint distribution function $G$ defined over $\mathbb{R}_+^{S \times K}$. For some results, we assume that distribution $G$ is absolutely continuous with respect to the Lebesgue measure. No other restrictions are placed on $G$. In particular, marginal distributions need not be the same across investors and the joint distribution can feature the arbitrary interdependence of endowments, as long as correlations are not perfect, which is the case for absolute continuity.

3 Financial Structure and Market Value

This section determines how different financial structures $(F, t) \in \mathcal{F}$ affect the expected market value of a real asset $z_n$, given $G$. Section 3.1 presents two lemmas that shed light on the structure of the competitive equilibrium in financial markets for any given financial structure $(F, t) \in \mathcal{F}$ (Lemmas 1 and 2). Section 3.2 then characterizes a profit-maximizing financial structure in Propositions 3 and 1 and Corollary 1. Section 3.3 provides a geometric interpretation of our results.

3.1 Characterization of Competitive Equilibrium

Lemma 1 characterizes the allocation of numéraire among investors in the second period, resulting from security trading in competitive financial markets. Define the set of $F$-feasible allocations as the set of all feasible allocations for which the individual transfers of each investor are in the security span; in other words, allocations that can result from certain
security trades,

\[ X((F)) \equiv \left\{ x \in \mathbb{R}^{S \times K}_+ \mid \sum_{k=1}^{K} x^k = \sum_{k=1}^{K} e^k + \sum_{n=1}^{N} z_n \text{ and } (x^k - e^k) \in \langle F \rangle \text{ for all } k \right\}. \] (1)

Let \( V : \mathbb{R}^{S}_+ \to \mathbb{R}_+ \) be the expected utility of investor \( k \) in the second period, defined as 
\( V(x^k) = \sum_{s=1}^{S} P_s \times u(x^k_s) \), where \( P_s \) is the probability of state \( s \), and let \( V^P(x) \equiv \sum_{k=1}^{K} V(x^k) \) be the planner utility defined over all allocations. Given transferable utility, the following equivalent characterization of a competitive allocation of numéraire is obtained.

**Lemma 1.** (Allocative Equivalence) Security allocation \((\tilde{t}^1, \ldots, \tilde{t}^K)\), satisfying \( \sum_{k=1}^{K} \tilde{t}^k = t \), is an allocation in a competitive equilibrium if, and only if, the resulting allocation of numéraire at date 2, \((\tilde{x}^1, \ldots, \tilde{x}^K)\), given by \( \tilde{x}^k = e^k + F\tilde{t}^k \), solves the planner problem, 
\[ \max_{x \in X((F))} V^P(x). \]

Thus, competitive financial markets allocate numéraire at date 2 in the same way as a planner whose choice is restricted to \( F \)-feasible allocations. The equivalent numéraire allocation between the market and planner problems has important implications. For any financial structure \((F,t)\), numéraire allocation is unambiguously determined in competitive equilibria, even if securities trades are not (as is the case, for instance, of linearly dependent securities). Moreover, numéraire allocation at date 2 depends on financial structure \((F,t)\) only though \( \text{span } \langle F \rangle \); that is, for any two financial structures \((F,t)\) and \((F',t')\), such that \( \langle F \rangle = \langle F' \rangle \), the numéraire allocations coincide.\(^1\) Let \( \mathcal{L}_z \) be the collection of all linear subspaces of \( \mathbb{R}^S \) that contain real return \( \{z_n\}_{n=1}^{N} \) and allocation \( x : \mathcal{L}_z \to \mathbb{R}^{S \times K}_+ \) give a numéraire allocation \( x(L) \) observed for any choice of \((F,t)\) with \( \text{span } \langle F \rangle = L \in \mathcal{L}_z \). Finally, define \( \kappa : \mathcal{L}_z \to \mathbb{R}^S_+ \) as the average marginal utility evaluated at equilibrium allocation,
\[ \kappa(L) \equiv \frac{1}{K} \sum_{k=1}^{K} DV(x^k(L)). \] (2)

Function \( \kappa(\cdot) \) gives Arrow prices for any, complete or not, financial structure. Competitive securities prices are given by \( p^T = \kappa(\langle F \rangle) \cdot F \). The latter is straightforward for a complete financial structure—then, consumption and individual marginal utilities coincide for all investors. With an incomplete \( F \), consumption vectors and, hence, marginal utilities, differ

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\(^1\)The existence of a competitive equilibrium allocation in markets that open once the entrepreneurs choose \( F \) follows from the compactness of \( X((F)) \) and the continuity of \( V^P(x) \), while uniqueness holds by the convexity of \( X((F)) \) and the strict concavity of \( V^P(x) \). The dependence of numéraire allocation through \( \text{span } \langle F \rangle \) alone obtains because, in the planner problem, \((F,t)\) enters the planner constraint only through \( \text{span } \langle F \rangle \).
across investors; while Arrow prices are not unique, (2) remains one possibility.\footnote{In fact, any vector from the set \( \{\kappa(L)\} + L^+ \) constitutes Arrow prices. In particular, each vector whose average defines \( \kappa(L) \) does so. The marginal utilities at equilibrium consumption can differ only in the components that are orthogonal to security span and their differences are irrelevant for security pricing. Characterization of \( \kappa(L) \) as an average is useful in determining a financial structure that maximizes market value.} Lemma 2 characterizes the firm’s market value.

**Lemma 2. (Market Value)** Given \( G \) the expected market value of firm \( z_n \) for any \((F,t)\) is given by

\[
E[MV_n(F,t)] = E[\kappa(\langle F \rangle)] \cdot z_n.
\]  

(3)

Two implications are immediate. First, for any financial structure \((F,t)\), the expected market value is unambiguously defined—any two financial structures from \( F_z \) can be ranked in terms of profitability. In addition, just as with numèraire allocation, market value depends on financial structure \((F,t)\) only through span \( \langle F \rangle \). Thus, financial structures collections that induce the same span define equivalence classes for market value.

### 3.2 Market Value

We now characterize the relation between financial structures \((F,t)\) and firm’s market value.

**Existence.** We first show that within the menu of all financial structures \( F \), a financial structure exists that maximizes the expected market value of \( z_n \). There are two difficulties with demonstrating existence: First, even if one restricts attention to financial structures with a fixed number of securities \( I \), the domain over which the entrepreneur optimizes—given by the set of all financial structures \( \mathbb{R}^{(S+1)\times I} \) is non-compact. Additionally, market value is discontinuous in \((F,t)\).\footnote{Consider the following sequence of financial structures with two securities

\[
F^h = \begin{bmatrix} 1/h & 0 \\ 0 & 1/h \end{bmatrix}, \ h \in \mathbb{N}_+.\]

For any finite \( h \), markets are complete and the set of \( F^h \)–feasible allocations \( X(\langle F^h \rangle) \) comprises all feasible allocations. In the limit as \( h \to \infty \), security span collapses to a zero-dimensional subspace and \( X(\langle F^h \rangle) \) becomes the autarky point. Consequently, numèraire allocation and, hence, the average marginal utility are discontinuous.} Allen and Gale (1991) do not face these difficulties, since they consider entrepreneurs who choose from an exogenously pre-specified finite set. Here, the aim is to characterize the market value maximizing financial structure from an unconstrained set.

To deal with these two problems, we take the following approach. Since any two financial structures with the same span are equivalent in terms of market value (Lemma 2), the
problem of optimal financial structure can be recast as a choice of a span that maximizes market value—a linear subspace from the set of all linear subspaces of $\mathbb{R}^S$, rather than optimizing over financial structures $(F, t)$ directly. The optimization problem over linear subspaces is more tractable: for any dimension $D \leq S$, the set of all $D$-dimensional linear subspaces of $\mathbb{R}^S$ is a compact manifold (known as the Grassmannian) and market value $MV$ is continuous on it. This allows us to recover the compactness of the domain and continuity of the objective function while establishing that a financial structure that maximizes market value exists.

**Lemma 3.** *(Existence)* A financial structure $(F^*, t^*) \in \mathcal{F}$ exists, such that $E[MV(F^*, t^*)] \geq E[MV(F, t)]$, for all $(F, t) \in \mathcal{F}$.

We now characterize the financial structure that maximizes real asset value.

**Completeness of the Optimal Financial Structure.** Proposition 1 asserts that the financial structure that dominates in terms of market value depends on the shape of the marginal utility, $u'()$. Specifically, any degree of market incompleteness is superior, inferior or equivalent to completeness in terms of market value, depending on whether the marginal utility is convex, concave or linear on the relevant part of the domain. More formally, let $\mathcal{X}$ be a convex set that contains all the equilibrium consumption allocations (given by the image of $x(L)$ on the support of $G$).

**Proposition 1.** *(Optimal Financial Structure)* Consider any incomplete $(F, t) \in \mathcal{F}$ and any complete $(F', t') \in \mathcal{F}$. Then:

(i) If $u'''() > 0$ on $\mathcal{X}$, $(F, t)$ strictly dominates $(F', t')$ in terms of market value, $G$—almost surely (and is surely not dominated);

(ii) If $u''() < 0$ on $\mathcal{X}$, $(F, t)$ strictly dominates $(F', t')$ in terms of market value $G$—almost surely (and is surely not dominated);

(iii) If $u''() = 0$ on $\mathcal{X}$, then $(F, t)$ and $(F', t')$ give the same market value.

Since in a model with $S = 2$, as in the Allen and Gale example, the entrepreneur effectively chooses between a complete and incomplete (equity) financial structure, Proposition 1 fully characterizes the optimal financial structure, which is worth highlighting as a corollary.

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4Heuristically speaking, suppose $S = 2$ and entrepreneur chooses among all one-dimensional linear subspaces. Each subspace is represented by a line passing through the origin and is uniquely identified by a point on a semicircle with the radius one (See Figure 1A). A bijection that enlarges the distance of any point on the semicircle by a factor of two (around the circle) translates a semicircle into a full circle. Given such parameterizations of linear subspaces, the entrepreneur effectively chooses a point on a circle: a compact set. In addition, the dimensionality of any linear subspace in the domain of optimization—each represented by a point on the circle—is, by construction, the same and equal to one; $X(L)$ is a continuous correspondence defined on the circle. By the Maximum Theorem and Lemma 1, the equilibrium numéraire allocation $x(L)$ is continuous and so are Arrow prices given by the average marginal utility.
Corollary 1. (Two-State Model) Suppose $S = 2$. If $u'(\cdot)$ is strictly convex (concave) on $X$, the financial structure is market value maximizing if, and only if, it is complete (consists of equity only).

Observe that, in the Allen and Gale example, a complete financial structure maximizes market value. This prediction is consistent with our model only if marginal utility is concave. For standard utility functions used in macroeconomics and finance, such as CARA or CRRA, an incomplete financial structure dominates in terms of market value.

We provide a simple example that highlights a key economic intuition. Note that the result holds in the strong, ex-post sense. For transparency of the arguments, in all examples presented in the paper can, thus, be considered deterministic initial holdings.

Example 2. Consider two (types of) investors with utility $u(c_2) = 2 \ln(c_2)$, one entrepreneur with riskless asset $z_1 = (1,1)^T$. At date 2, there are two equally likely states and initial endowments given by $e^1 = (1,0)^T$ and $e^2 = (0,1)^T$. In an economy with two states, there are two choices of financial structures: a complete financial structure (e.g., equity and debt) and an incomplete financial structure (equity alone). With a complete financial structure, the equilibrium allocation of numéraire is Pareto efficient, $x^1 = x^2 = (1,1)$; the marginal utility of an investor in each state, given by $1/c_2$, is the same and equal to 1; and the market value with two securities is 2. If, instead, only equity is offered, each investor obtains half of the claims to $z_1$, which gives the equilibrium allocation $x^1 = \left(\frac{3}{2}, \frac{1}{2}\right)$ and $x^2 = \left(\frac{1}{2}, \frac{3}{2}\right)$. The average marginal utility for each good is $\frac{1}{2} \left(\frac{2}{3} + 2\right) = 1\frac{1}{3}$ while the market value of a real asset equals $2\frac{2}{3}$. Hence, an incomplete financial structure dominates a complete one in terms of market value. It is straightforward to show that when marginal utility is linear, both complete and incomplete financial structures yield the same market value, yet when strictly concave, only the complete financial structure maximizes market value.

In the example, with a complete financial structure, each investor purchases only consumption in a state for which his initial endowment is zero, and the marginal utilities of investors coincide in each state. When only equity is available, in order for an investor to obtain consumption in the desired state, he must purchase the security that pays (the same quantity of) numéraire in the other state. Thus, by introducing a wedge in consumption, an incomplete financial structure creates a wedge in marginal utility between the two investors in each state. With convex $u'(\cdot)$, the wedge increases the willingness of investors to pay with a lower equilibrium consumption above the Pareto efficient level by more than it reduces the willingness of the investor who consumes more to pay. Therefore, in each state, an incomplete financial structure induces a higher equilibrium average marginal utility compared to complete markets. Since the average willingness to pay remains high after trade in each state, the equilibrium value of equity remains high as well.
Notice that in Example 2, the market value of an asset increases only if the equilibrium allocation of numéraire is inefficient. In general, even with Pareto inefficient endowments (which occur \( G \) — almost surely given that \( G \) is absolutely continuous with respect to the Lebesgue measure), the final allocation for an incomplete financial structure may still be Pareto efficient. However, for any given incomplete financial structure, endowments realizations that give efficient outcomes are non-generic—the equilibrium allocation is \( G \) — almost surely Pareto inefficient.

**Security Span and Monotonicity of Market Value.** More generally, with \( S \geq 3 \), Proposition 1 asserts that a complete financial structure is almost surely dominated by (dominates) any incomplete financial structure when marginal utility is convex (concave). We are then led to ask: Does the market value monotonically increase when reducing investor hedging possibilities, as measured by a security span? That is, for any \( (F, t) \) and \( (F', t') \) such that \( \langle F \rangle \subseteq \langle F' \rangle \), does market value satisfy \( MV(F, t) \geq MV(F', t') \) so that opening a single equity market is always optimal? Example 3 demonstrates that, in general, this need not be the case, even for symmetric, quasilinear utilities.

**Example 3.** Consider \( S = 3 \), one entrepreneur with a riskless asset \( z_1 = (1, 1, 1)^T \) and two types of investors, whose utility function is given by

\[
u(c_2) = 3 \times \left\{ \begin{array}{ll}
2c_2 - \frac{1}{2} (c_2)^2 - \frac{3}{2} & \text{if } c_2 \leq 1 \\
\ln c_2 & \text{otherwise}
\end{array} \right. \quad (4)
\]

(Note that this function is \( C^2 \).) The initial holdings of goods are \( e^1 = (\frac{1}{2}, 0, 1)^T \) and \( e^2 = (0, \frac{1}{2}, 1)^T \). By symmetry, when only equity is offered, that is, \( F = \{(1, 1, 1)^T\} \), equilibrium allocation is given by \( x^1 = (1, \frac{1}{2}, \frac{3}{2})^T \) and \( x^2 = (\frac{1}{2}, 1, \frac{3}{2})^T \), Arrow prices are \( (\frac{5}{4}, \frac{5}{4}, \frac{2}{3}) \), and market value is \( 3 \frac{1}{6} \). Now, consider the following (not necessarily optimal) financial structure with the state-one Arrow security and a security that pays one in states two and three:

\[
F' = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\quad (5)
\]

Observe that \( \langle F \rangle \subseteq \langle F' \rangle \). Since security \( f_2 \) pays in the second state, it is more attractive to investor one and, in equilibrium, the allocation of securities is \( t^1 \simeq (\frac{1}{4}, \frac{2}{3}) \) and \( t^2 \simeq (\frac{3}{4}, \frac{1}{3}) \). The implied allocation of numéraire is \( x^1 \simeq (\frac{3}{4}, \frac{2}{3}, \frac{5}{3})^T \) and \( x^2 \simeq (\frac{3}{4}, \frac{5}{6}, \frac{4}{3})^T \), the Arrow prices are \( (\frac{5}{4}, \frac{5}{4}, \frac{27}{40}) \) and market value is \( 3 \frac{7}{40} > 3 \frac{1}{6} \). Financial structure \( F' \) strictly dominates \( F \) in terms of market value. Utility function (4) can be perturbed so that marginal utility is strictly convex on the whole domain while \( F' \) still yields a strictly higher market value than \( F \).
In Example 3, financial structure $F$ introduces a wedge in the numéraire consumption in the first two states, whereas the allocation is Pareto efficient in the third state. Given that in the first two states, consumption takes place in the domain of quadratic utility, distortion brings no increase in market value relative to complete markets—the average marginal utility remains intact. In contrast, while the two-security financial structure $F'$ improves the efficiency of the first two states’ allocation, it introduces a wedge in the allocation of the third state. Given that consumption in this state is in the domain of a logarithmic function with strictly convex marginal utility, the wedge in the third state increases the Arrow price for that state as well as the firm’s market value. The lack of monotonicity extends to strictly concave marginal utility environments.

As a more general insight, a financial structure that maximizes the market value of $z_n$ distorts allocation, relative to the Pareto-efficient allocation, in the states in which: (1) the convexity of marginal utility—and, hence, the potential increase in average willingness to pay—is greatest; (2) the asset return is the largest; and (3) the probability of the greatest heterogeneity in initial endowments is the highest, ceteris paribus.

Example 3 demonstrates that, in general, the market value of a real asset need not be monotone in a security span—with convex or concave investor marginal utility. As the analysis from Section 4.2 implies, non-monotonicity does not stem from non-monotonicity of welfare in asset span. In the important instance of the CARA utility and a riskless real asset, market value is indeed monotone in span and the optimal financial structure involves selling a riskless security (bond) alone. This is demonstrated in the following example.

**Example 4.** Consider an entrepreneur with riskless asset $z_n = (\lambda, ..., \lambda)^T$ for some $\lambda > 0$, investors with CARA utility $u(c_2) = -e^{-\alpha c_2}$ and $G$ is arbitrary. The expected market value of $z_n$ is monotone in a security span. In particular, opening a market for riskless asset maximizes market value.

In the next section we provide intuition behind monotonicity in markets with CARA utility and riskless asset.

**Discussion.** Concluding the characterization of a financial structure that maximizes market value requires some final remarks about Proposition 1. First, Proposition 1 extends to state-dependent investor (Bernoulli) utilities. Second, since the inequalities are strict with $G$—probability 1 in claims (i) and (ii) of the proposition, it follows that the result is robust to sufficiently small asymmetries in investor utility functions. However, Proposition 1 does not generalize to arbitrary asymmetries in utility functions across investors. In fact, in the Allen and Gale example, investor marginal utilities are strictly convex, yet it is a complete

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5 This is clearly the case for a given financial structure and, by the compactness argument used in the Proposition 3 proof, extends to all incomplete financial structures.
financial structure that maximizes the market value of an asset. Taken together, Proposition 1 and the Allen and Gale example suggest that in markets with asymmetric investor utilities, for convex or concave marginal utility, no general normative predictions based solely on investors preferences can be obtained in that the optimality of complete or incomplete financial structure then depends on the model’s details, such as endowment or asset return distributions.

The assumptions of the three claims in Proposition 1 hold over some convex subset of the respective domains, which is large enough to include all the relevant equilibrium allocations of numéraire. We introduce this qualification, because otherwise, the class of preferences under consideration with \( u'''(\cdot) \leq 0 \) on the whole domain is vacuous.\(^6\) If distribution \( G \) has a bounded support, we can always find a bounded set of outcomes \( \mathcal{X} \) to qualify the assumptions on the shape of marginal utilities. Finally, notice that claims (i) and (ii) in the proposition hold true if all marginal utilities weakly satisfy the assumptions of convexity or concavity; one of them does so strictly over the set \( \mathcal{X} \). However, we cannot draw any general conclusions for the cases in which there is no clear second-order behavior over the relevant consumption set.

In summary, using symmetric utility, one can obtain general predictions regarding the optimality of complete or incomplete financial structures based on the shape of marginal utility itself which holds for all distributions of endowments and all returns of real assets. Determining which of the three types of preferences is more plausible requires a theory of the third derivative of utility. Empirical evidence from the literature that recognizes the importance of the third derivative, such as precautionary savings, provides some support in favor of convex marginal utility (“prudence”).\(^7\)

### 3.3 Geometric Interpretation

We provide a geometric interpretation of our results to elucidate the role of span for equilibrium, as well as the impact of asymmetry in investor utility on predictions about incentives to innovate. In doing so, we exploit the equivalence of equilibrium numéraire allocation between the entrepreneur and the planner problem (Lemma 1). The set of all feasible allocations of

\(^6\)While a global assumption would not be problematic for the claim (j), given the Inada assumption about utility, a strictly concave marginal utility function does not exist wherein marginal utilities are always strictly positive and concave or linear.

\(^7\)Loosely speaking, the mechanism that underlies the theory of precautionary saving shares the implication of convex marginal utility that lowering consumption increases an agent’s marginal utility by more than increasing consumption improves it. However, the precautionary savings effect involves in a single-agent problem, whereas ours operates, crucially, as an equilibrium mechanism, through heterogeneity across agents. Further, while precautionary savings phenomenon concerns differences in marginal utilities (and transferring consumption) across states, the conditions for optimality of (in)complete financial structure involve differences in marginal utilities and consumption across agents within states.
numéraire in Example 2 are represented by the Edgeworth box in Figure 2. In markets with two states, a complete financial structure $F$, in which case $F$–feasible set $X((F))$ comprises all allocations in the box. With equity $F'$, $X((F'))$ is represented by a line segment that connects the endowment points. The social planner objective $V^P(x) = \sum_{k=1}^{K} V(x^k)$ attains the bliss point at the Pareto efficient allocation, where both investors consume the same quantities while planner utility decreases for allocations further away from the center (Figure 2.A). Thus, the following are possible equilibrium allocations: If financial markets are complete, the planner chooses his unconstrained maximum, whereas with equity, the equilibrium allocation coincides with the constrained planner maximum on the $F$–feasible set.

Figure 2.B depicts the entrepreneur preference map, each curve comprising all allocations that give rise to a given firm value. As a result of the symmetry of the investor marginal utility, the critical point of the market value function, $MV = \frac{1}{K} \sum_{k=1}^{K} DV(x^k) \cdot z$, is at the Pareto efficient allocation as well. Whether the Pareto efficient allocation yields a minimum or a maximum depends on whether the marginal utility and, hence, the market value function is convex or concave. With the logarithmic utility as in Example 2, market value increases for allocations located further away from the Pareto efficient allocation. Conversely, with a strictly concave marginal utility, the Pareto efficient allocation maximizes market value, and a complete financial structure is optimal. In the case of a quadratic utility, all allocations in the box are equivalent in terms of market value and entrepreneurs are indifferent to the planner’s allocation choice.

In general, the planner preference and market value maps need not overlap, which in settings with $S > 2$, may result in the non-monotonicity of profit in the security span. In Example 4, by offering two securities ($F'$) rather than equity ($F$), the entrepreneur enlarges the $F$-feasible set in the direction for which the planner can improve the overall welfare, which also gives rise to higher market value. For CARA utility with a riskless asset, the two maps coincide: the exponential utility function $u(c_2)$ satisfies $u'(c_2) = const \times u(c_2)$ and market value is, thus, proportional to the negative of the planner utility. Thus, smaller security span and hence choice set in the planner program will never reduce market value. Additionally, outside of CARA utility, it is apparent that one can specify endowments and an asset payoff such that increasing the span increases profit.

With asymmetric investor utilities, predictions regarding the optimality of an incomplete financial structure depend on the details of the environment for the following reason. The

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8In the two-investor economy, the concavity of market value function in allocation in the Edgeworth Box is defined as the concavity of $\frac{1}{2} (DV(x^1) + DV(e^1 + e^2 + z - x^1)) \cdot z$ in $x^1$. More generally, concavity is defined with respect to consumption of the first $K - 1$ investors and consumption of the $Kth$ investor is the residual of total resources $\sum_{k=1}^{K} e^k + z$. It is straightforward to show that if marginal utilities are convex, then market value is convex as well.
Pareto efficient and the allocations that minimize market value do not necessarily coincide, even with a convex marginal utility, as is the case in the Allen and Gale example depicted in Figure 3; with equity only, the equilibrium allocation is the point on the line segment that maximizes the planner’s utility, while with a complete financial structure, it is the unconstrained maximum, which gives a higher market value. Thus, with convex investor marginal utility, separation of the Pareto efficient and profit-minimizing allocations derived from the asymmetry of investor utilities—is necessary (but not sufficient) for market completion to be profitable for the entrepreneurs.

4 Competition in Security Innovation

A central question in the literature on financial innovation concerns whether large markets are incomplete, or whether competition provides sufficient incentives to complete the market. To study how endogenous financial structures are affected by competition among entrepreneurs, like Allen and Gale (1991), we study the interactions among entrepreneurs who choose which securities to issue. By the standard argument (e.g., Kreps (1979)), entrepreneurs can affect prices, even in large markets, so long as they can affect the joint span of $F$. We first consider a market in which, as in Allen and Gale (1991), entrepreneurs simultaneously choose portfolios of issued securities. We also examine sequential competition. Likewise, as Allen and Gale (1991), we introduce a per-security innovation cost $\gamma > 0$ that discourages entrepreneurs from excessive asset innovation.  

4.1 Equilibrium Financial Structure

Simultaneous Innovation. In markets with $u''(\cdot) < 0$ and $S = 2$, our model predictions regarding asset innovation are in line with those of the Allen and Gale example: entrepreneurs benefit from completing markets and—due to the public-good nature of innovation and individual incentives to free-ride—large markets are incomplete with positive probability as long as innovation is costly; the probability that equilibrium financial structure is incomplete tends towards zero as innovation costs vanish.

In markets with $u''(\cdot) > 0$ and $S = 2$, markets are incomplete in equilibrium with probability one, even if innovation is cost free,  

9 When entrepreneurs simultaneously choose financial structures, the model with costless innovation has a (trivial) multiplicity of Nash equilibria: If one entrepreneur chooses a complete financial structure, it is a weak best response for all other entrepreneurs to issue a complete financial structure as well, regardless of market primitives (by changing $F_n$, the entrepreneurs have no impact on the aggregate financial structure $F$).

10 While innovation cost overcomes the (trivial) multiplicity of equilibria, it is not essential for the free-riding mechanism, which gives rise to market incompleteness with convex marginal utility in large markets.
of endowments and for an arbitrary—potentially large—number of entrepreneurs.

**Proposition 2. (Financial Structure with Simultaneous Innovation.)** Suppose $u'' \geq 0$ and $S = 2$. For any $\gamma > 0$, $N < \infty$ and $\{z_n\}_{n \in N} \not\in \mathbb{R}^2$, in any mixed-strategy Nash equilibrium, $F$ is incomplete with probability one.

No entrepreneur has an incentive to innovate, even if innovation costs are negligible. Quite surprisingly, in a more general model with richer uncertainty, $S > 2$, markets may be complete with positive probability, even when marginal utility is strictly convex and innovation costs are non-negligible. The following example illustrates the key economic mechanism.

**Example 5.** Consider $S = 3$, two entrepreneurs with a riskless real asset $z = (1, 1, 1)^T$ and two types of investors whose utility function and endowments are as in Example 3. The mass of each investor type is normalized to 2. Let $F^\ast$ denote the financial structure that maximizes market value when innovation is cost free and let $MV^\ast$ denote the maximum market value. Lemma 4 in the Appendix shows that: (1) $F^\ast$ consists of two independent securities; (2) $F^\ast \cup F^\ast\ast$ is complete. With a sufficiently small innovation cost $\gamma$, when entrepreneur $n' \neq n$ issues equity $F_{n'} = \{(1, 1, 1)^T\}$, it is optimal for entrepreneur $n$ to choose either $F_n = F^\ast$ or $F_n = F^\ast\ast$. The market value then equals $MV^\ast$. However, if entrepreneur $n'$ chooses $F_{n'} = F^\ast$ or $F_{n'} = F^\ast\ast$, then, given costly innovation, issuing equity alone maximizes profit. This gives rise to a mixed-strategy Nash equilibrium in which entrepreneurs randomize over the three financial structures $\{(1, 1, 1)^T\}$, $F^\ast$ and $F^\ast\ast$. The probabilities with which $F^\ast$ and $F^\ast\ast$ are chosen are

$$\sigma = \frac{1}{3} \left(1 - \frac{\gamma}{MV^\ast - MV_C}\right),$$

where $MV_C$ is the market value in a complete market.\footnote{Suppose entrepreneur $n'$ follows the mixed strategy $(1 - 2\sigma, \sigma, \sigma)$ over $F, F^\ast$ and $F^\ast\ast$, with $\sigma$ from (6). For entrepreneur $n$, profit from having chosen $F^\ast$ can be found as follows: entrepreneur $n'$ chooses equity with probability $(1 - 2\sigma)$ and issues $F^\ast$ with probability $\sigma$. In either case, $\langle F \rangle = \langle F^\ast \rangle$ and market value is $MV^\ast$. With probability $\sigma$, entrepreneur $n'$ chooses $F^\ast\ast$. Then $\langle F \rangle = \mathbb{R}^3$ and profit equals $MV_C < MV^\ast$. Thus, the expected total profit of entrepreneur $n$ from financial structure $F^\ast$ is $(1 - \sigma) \pi^\ast + \sigma \pi^{CM} - 2\gamma$. Now take equity $F$. With probability $(1 - 2\sigma)$ entrepreneur $n'$ also offers equity and market value that coincides with $MV_C$ (there is no distortion in the third state, see Example 3). With probability $2\sigma$, either $\langle F \rangle = \langle F^\ast \rangle$ or $\langle F \rangle = \langle F^\ast\ast \rangle$, each of which gives the maximum market value $MV^\ast$. The expected profit of entrepreneur $n$ from issuing equity is $(1 - 2\sigma) \pi^{CM} + 2\sigma \pi^\ast - \gamma$. Equating the two expected profits gives (6).} Since $MV^\ast > MV_C$, for a sufficiently small innovation cost $\gamma$, probability $\sigma$ is strictly positive. In equilibrium, the probability that markets are complete equals $\sigma^2 > 0$.

For intuition, the market value in the example is not monotonically decreasing in security span; each entrepreneur is willing to pay innovation costs in order to partially complete the
market. Either of the two incomplete financial structures \( F^* \) and \( F^{**} \) maximizes market value and, in the described equilibrium, entrepreneurs are unable to always agree on one of them. Consequently, different profit-maximizing financial structures may be chosen—an undesirable outcome for both entrepreneurs, as the equilibrium financial structure \( F \) is then complete and \( MV_C < MV^* \). Example 5 generalizes to markets with \( N \) entrepreneurs in a straightforward way. In large markets, the probability of market completeness is bounded away from zero as \( N \to \infty \). An example of such an economy in which, in any equilibrium, only one market opens is an environment with CARA investor utility and riskless asset. In this case, profit is monotone in asset span and all entrepreneurs have incentives to preserve the minimal span.

In the analysis of competition in asset innovation so far, entrepreneurs choose which securities to issue without observing the financial structures issued by other entrepreneurs. As shown in Example 5, simultaneous modes of competition generate \textit{ex post} regret in the event of complete markets—\textit{ex post}, each entrepreneur prefers to shut down some securities markets. Thus, a complete market outcome is not likely to be stable in the long run. Therefore, next we consider a competition in which the entrepreneurs choose financial structures sequentially.

**Sequential Innovation.** Prior to choosing a financial structure, entrepreneur \( n \) can now observe the financial structures chosen by entrepreneurs \( 1, 2, \ldots, n-1 \).

**Proposition 3.** (Financial Structure with Sequential Innovation 1) Suppose \( u''(\cdot) < 0 \) and \( N < \infty \). There exists \( \tilde{\gamma} > 0 \) such that for any innovation cost \( \gamma \leq \tilde{\gamma} \), in any Subgame Perfect Nash equilibrium, \( F \) is complete. Absent uncertainty as to the securities issued by other entrepreneurs, markets are complete in equilibrium, so long as innovation costs are not prohibitively high. On the other hand, with \( u''(\cdot) \geq 0 \), markets are incomplete, even when innovation is cost free, for any \( N \).

**Proposition 4.** (Financial Structure with Sequential Innovation 2) Suppose \( u''(\cdot) \geq 0 \). For any \( \gamma > 0 \), \( N < \infty \) and \( \{z_n\}_{n \in \mathbb{N}} \neq \mathbb{R}^S \), in any Subgame Perfect Nash equilibrium, \( F \) is incomplete. With a riskless asset \( z = (\lambda, \ldots, \lambda)^T \), for some \( \lambda > 0 \) and CARA investor utility function, equilibrium \( F \) comprises only the riskless assets.

Thus, with a weakly convex marginal utility, for any number of entrepreneurs, markets are incomplete. Entrepreneurs weakly prefer (any) incomplete financial structure and, hence, none has any incentive to innovate. Since each entrepreneur can observe the financial structure choices of his predecessors, a lack of coordination from Example 5 does not arise. Therefore, with convex marginal utility, frictionless markets in which investors can sell short
provide few incentives to innovate. Markets are incomplete in equilibrium and there is tension between profit maximization and efficiency. Accordingly, incentives to innovate must then result from the presence of asymmetric information, short selling restrictions or other frictions.

4.2 Asset Innovation and Welfare

The ability to alter the security span and, hence allocation among investors, by issuing securities allows entrepreneurs to affect prices even in markets with large numbers of entrepreneurs. A natural question arises as to how the power of entrepreneurs to create markets affects welfare.

Our model has the following implications for the welfare appraisal of asset innovation. Clearly, to achieve Pareto efficiency of market outcomes, a policy should induce a full-span portfolio of securities. As suggested in Section 3.3, this recommendation can be strengthened: introduction of an additional security is never detrimental to welfare, even if asset innovation does not fully complete the financial structure. Formally, denote by $DWL(F)$ a deadweight loss that results from financial structure $F$. As monetary transfers sum to zero across investors and the entrepreneur, for any pair $(F, t), (F', t') \in \mathcal{F}_z$ such that $\langle F \rangle \subseteq \langle F' \rangle$, by Lemma 1,

$$DWL(F) - DWL(F') = \max_{x \in X(\langle F' \rangle)} \sum_{k=1}^{K} u(x^k) - \max_{x \in X(\langle F \rangle)} \sum_{k} u(x^k).$$  \hspace{1cm} (7)

Since $X(\langle F' \rangle) \subseteq X(\langle F \rangle)$, it follows that a deadweight loss is (weakly) decreasing in the span of a financial structure, $DWL(F) \geq DWL(F')$.

A financial structure $(F, t)$ that maximizes market value necessarily distorts allocation in markets where investor marginal utility is convex: maximization of the market value of an asset requires market incompleteness, which $(G-$ almost surely) introduces a wedge in investor marginal utility in equilibrium. Indeed, the very mechanism through which market incompleteness provides an effective means to increasing the entrepreneur’s profit is the introduction of inefficiency in the allocation of numéraire. Nevertheless, as the analysis from Section 3 implies, while the exercise of market power through reducing the span of $F$ introduces a profit-efficiency trade-off, market value is not necessarily monotone in a deadweight loss and the entrepreneur’s benefit need not be associated with investor loss (Example 3).\(^{12}\)

One of the lessons from the Allen and Gale example, that also holds in our model with

\(^{12}\)With linear marginal utilities, profit is invariant to financial structure—any $F \in \mathcal{F}_z$ is revenue maximizing; but among all such financial structures, only those with a full span yield the efficient allocation.
$u'''-0$, is that sufficiently reducing innovation costs offer incentives to complete markets and re-establishes efficiency. Such a cost-reducing policy does not reestablish efficiency when marginal utility is convex.

5 Appendix

Proof: (Lemma: Allocation Equivalence) (Only if) Let $\tilde{p}$ and $(\tilde{t}^1, \ldots, \tilde{t}^K)$ be prices of securities and allocation of securities in a competitive equilibrium. Since $\hat{x}^k - e^k = F\tilde{t}^k$, resulting transfers of numéraire are in the span $\langle F \rangle$ and $\sum_{k=1}^K \hat{x}^k = \sum_{k=1}^K e^k + \sum_{k=1}^K F\tilde{t}^k = \sum_{k=1}^K e^k + \sum_{n=1}^N z_n$. Hence $\hat{x} \equiv (\hat{x}^1, \ldots, \hat{x}^K) \in X(\langle F \rangle)$. In addition, by optimality of each investor’s choice given prices,

$$V(e^k + Ft^k) - \tilde{p} \cdot t^k \leq V(e^k + F\tilde{t}^k) - \tilde{p} \cdot \tilde{t}^k,$$

for all security trades $t^k \in \mathbb{R}^K$. Summing (8) over all $k$ gives $\sum_{k=1}^K u(e^k + Ft^k) \leq \sum_{k=1}^K u(\hat{x}^k)$, for all $(t^1, \ldots, t^K)$ such that $\sum_{k=1}^K Ft^k = z$, where we used that, in a competitive equilibrium, $\tilde{p} = F^T Du(\hat{x}^1)$ and, hence,

$$\tilde{p} \cdot \sum_{k=1}^K t^k = (F^T DV(\hat{x}^1)) \cdot \sum_{k=1}^K t^k = Du(\hat{x}^1)^T \sum_{k=1}^K Ft^k = \tilde{p} \cdot \sum_{k=1}^K \tilde{t}^k. \quad (9)$$

Thus, $\hat{x}$ maximizes planner utility on $X(\langle F \rangle)$.

(If) $F-$feasible set $X(\langle F \rangle)$ can be alternatively written as

$$X(\langle F \rangle) = \left\{(e^1 + Ft^1, ..., e^K + Ft^K) \in \mathbb{R}_{+}^{S \times K} \left| \begin{array}{c} (t^1, ..., t^K) \in \mathbb{R}^{I \times K} \text{ and } \sum_{k=1}^K t^k = t \end{array} \right. \right\}, \quad (10)$$

planners problem $\max_{x \in X(\langle F \rangle)} V^P(x)$ can be equivalently written as

$$\max_{(t^1, ..., t^K)} \sum_{k=1}^K V(e^k + Ft^k) : \sum_{k=1}^K t^k = t, \quad (11)$$

and, by assumption, $\tilde{t}^k$ is a solution to it. By Inada condition and using that return $z$ is strictly positive in all states and lies in $\langle F \rangle$, $\hat{x}^k$ is strictly positive in all components for all individuals. By the Kühn-Tucker Theorem, then, multipliers $\tilde{p}$ must exist such that $F^T DV(e^k + Ft^k) = \tilde{p}$ for every $k$. Since the utility function $u(\cdot)$ is strictly concave in the interior of the domain, we have that $\tilde{t}^k = \arg \max V(e^k + Ft^k) - \tilde{p}t^k$ and hence $\tilde{t}^k$
is a competitive demands given price $\tilde{p}$. Moreover by assumption, security markets clear. Therefore $(\tilde{r}^1, \ldots, \tilde{r}^k)$ and $\tilde{p}$ constitute a competitive equilibrium. \textit{Q.E.D.}

**Proof:** (Lemma: Market Value) Let $x(L)$ be a unique competitive allocation of numéraire for any $(F, t)$ satisfying $\langle F \rangle = L$. Since, prices of securities satisfy $p = F^T D V(x^k(\langle F \rangle))$, for all $(F, t)$ and $k$, taking the average across all investors gives $p^T = \kappa(\langle F \rangle) \cdot F$. In addition, $MV(F, t) = p \cdot t = \kappa(\langle F \rangle) \cdot F t = \kappa(\langle F \rangle) \cdot z$. This also holds in expectations $E[MV(F, t)] = E[\kappa(\langle F \rangle)] \cdot z$. \textit{Q.E.D.}

**Proof:** (Proposition: Existence) Take any linear space $L$ such that $\{z_1, \ldots, z_n\} \subseteq L$. If $\{z_1, \ldots, z_n\}$ contains $M$ linearly independent assets, then the orthogonal complement of $\langle \{z_1, \ldots, z_n\} \rangle$ is a linear subspace of dimension $S - M$, and a basis for $L$ can be constructed by taking the $M$ linearly independent assets and $\dim(L) - M$ linearly independent vectors in $\langle \{z_1, \ldots, z_n\} \rangle^\perp$. It follows that for any $M \leq D \leq S$, the space of $D$-dimensional spaces of trades in revenue that contain $\{z_1, \ldots, z_n\}$ is topologically equivalent to the set of $(D - M)$-dimensional linear subspaces of $\mathbb{R}^{S-M}$. This Grassmannian is a compact manifold (of dimension $(D - M) \times (S - D)$).

Now, take the set of all structures for which the dimension $\langle F \rangle$ is $D$. Over this set, correspondence $X(\langle F \rangle)$ is upper- and lower-semicontinuous. Since $V^p(x)$ is continuous on $X(\langle F \rangle)$ for any $\langle F \rangle$ by the Theorem of the Maximum, the allocation function, $x(L)$, is continuous on the Grassmannian. It follows that expected profit $MV_n(L) = E[\kappa(L)] \cdot z_n$, is continuous as well and, therefore, that a linear space $L^*$ exists that maximizes seller $n$’s expected market value on the set of all linear subspaces of dimension $D$. Denote by $MV_D^*$ the maximized profit over the set of structures that span $D$-dimensional spaces of revenue transfers. Since $S$ is finite, the seller’s program reduces to finding the maximum of $\{MV_M^*, \ldots, MV_S^*\}$. \textit{Q.E.D.}

**Proof:** (Proposition: Profit-Maximizing Financial Structure) With complete financial structure $(F', t')$, the allocation of numéraire is Pareto efficient, each investor consumes $\left( \sum_{h=1}^N z_h + \sum_{k=1}^K e^k \right)/K$, and the resulting profit for the entrepreneur equals $MV(F', t') = DV \left( \frac{1}{K} \left( \sum_{h=1}^N z_h + \sum_{k=1}^K e^k \right) \right) \cdot z_n$. Next, consider a financial structure $(F, t)$ for which $\langle F \rangle \notin \mathbb{R}^S$. Dimension of $\langle F \rangle$ is lower than $S$, so the set of endowment profiles, $(e^1, \ldots, e^K)$, for which $\left( \sum_{h=1}^N z_h + \sum_{k=1}^K e^k \right)/K - e^1 \in \langle F \rangle$ has zero Lebesgue measure (as a subset of $\mathbb{R}^S$). Since $G_e$ is absolutely continuous with respect to the Lebesgue measure, it follows that, for at least two types of individuals, $x^k(\langle F \rangle) \neq x^h(\langle F \rangle)$ almost surely.

For claim (i), notice that since function $DV(\cdot)$ is strictly convex and $\frac{1}{K} \sum_{k=1}^K x^k(\langle F \rangle) =
\[ \kappa(\langle F \rangle) = \frac{1}{K} \sum_{k=1}^{K} DV(x^k(\langle F \rangle)) > DV \left( \frac{1}{K} \left( \sum_{h=1}^{N} z_h + \sum_{k=1}^{K} e^k \right) \right) = \kappa(\langle F' \rangle). \] (12)

Then, it follows that \( MV(F,t) = \kappa(\langle F \rangle) \cdot z > \kappa(\langle F' \rangle) \cdot z = MV((F',t'), G) \)-almost surely. In the (zero \( G \)-probability) set where all investors equate second period consumption, the two profit levels are equal. The arguments for claims (ii) and (iii) can be mimicked. Q.E.D.

Proof: (Proposition: Simultaneous Innovation) Let \( \sigma^k \) be some probability distribution over \( F_{zn} \) and suppose \( \sigma = (\sigma^1, ..., \sigma^N) \) is a mixed strategy Nash equilibrium in which probability of an event that \( \langle F \rangle = \langle \cup_{h=1}^{N} F_n \rangle = \mathbb{R}^2 \) is positive. It follows that at least for one entrepreneur a mass put on financial structures such that \( \langle F_n \rangle = \mathbb{R}^2 \) is nonzero. Since \( \langle F \rangle \neq \mathbb{R}^2 \) gives strictly higher market value, and innovation is costly any \( F'_n \) such that \( \langle F'_n \rangle \neq \mathbb{R}^2 \) (e.g., equity) gives strictly higher profit hence dominates any complete \( F_n \). Thus, deviation from \( \sigma^n \) that reallocates probability from all \( F_n \) for \( \langle F_n \rangle = \mathbb{R}^2 \) to equity increases profit.

Q.E.D.

Proof: (Proposition: sequential Innovation 1) Let \( \bar{\gamma} \equiv \frac{1}{2} (\min \{ \frac{MV^*_S - MV^*_D}{S-D} \})_{D=1} \) where \( MV^*_D \) is the highest attainable profit for all financial structures \( (F,t) \) with dimension of span equal to \( D \) (see the proof of existence of profit maximizing financial structure). By Proposition 1, \( \bar{\gamma} \geq 0 \). Given \( \gamma \leq \bar{\gamma} \) and the choices of other entrepreneurs \( F_n, n = 1, 2, ..., N - 1 \) and hence \( \cup_{n=1}^{N-1} F_n \) it is always optimal to complete the financial structure for entrepreneur \( n = N \).

Q.E.D.

Proof: (Proposition: sequential Innovation 2) Let \( \{F_n\}_{n=1}^{N} \) be equilibrium choices of all entrepreneurs and for any \( n \) define \( \tilde{F}_n \equiv \cup_{h=1}^{n} \{F_h\} \cup \{z_h\}_{h=n+1}^{N} \). We argue by induction that if \( \tilde{F}_n \) is incomplete than \( \tilde{F}_{n+1} \) is incomplete as well. Entrepreneur \( n = N \): if \( \tilde{F}_{N-1} \) is incomplete than by completing financial is suboptimal - closing equity dominates any financial structure that completes \( \cup_{h=1}^{N-1} \{F_h\} \). By analogous argument statement holds for all \( n \). In addition for entrepreneur \( n = 1, \tilde{F}_{n-1} = \{z_h\}_{h=n+1}^{N} \) and hence by assumption is incomplete. The two arguments give the result.

Q.E.D.

Lemma 4. (Optimal Financial Structure) In Example 5: 1) profit maximizing financial structure, \( F'_n \), consists of two independent assets, 2) permutation of the first two rows of \( F'_n \) gives optimal financial structure \( F''_n \). 3) \( \langle F'_n \cup F''_n \rangle = \mathbb{R}^3 \).
Proof: By Proposition 3 optimal financial structure, \( F^* \), exists. In Example 3 and Proposition 1 we demonstrated that equity and any complete financial structure are strictly dominated by financial structures with two-dimensional span. By symmetry of endowments in states one and two, \( F^* \) and \( F^{**} \) give the same profit and hence \( F^* \) is profit maximizing iff \( F^{**} \) is profit maximizing as well. W.l.o.g. assume that \( F^* \) has two securities. Without loss of generality we normalize the supply of all security to one, which gives the following payment structure of securities given riskless asset \( z = (1, 1, 1) \).

\[
F^* = (f_1^*, f_2^*) = \begin{pmatrix} a & 1-a \\ b & 1-b \\ c & 1-c \end{pmatrix} \quad \text{and} \quad F^{**} = (f_1^{**}, f_2^{**}) = \begin{pmatrix} b & 1-b \\ a & 1-a \\ c & 1-c \end{pmatrix}
\]

(13)

Now we show that \( F^* \cup F^{**} = \mathbb{R}^3 \). Suppose not. Then \( f_1^{**} \) can be written as a linear combination of \( f_1^* \) and \( f_2^* \). If \( c \neq \frac{1}{2} \) the only combination that gives \( f_1^{**} \) puts weight one on \( f_1^* \) and zero on \( f_2^* \) (see last row). This implies that \( a = b \). But then unless \( F^* \) is equity, which is suboptimal, portfolio of securities \( F^* \), \( (1, \frac{a}{1-a}) \) gives a state contingent claim in the third state. It follows that there is no distortion in consumption in the third state and \( F^* \) as shown in Example 3 gives (complete market) suboptimal market value, contradicting optimality of \( F^* \). With \( c = \frac{1}{2} \), the only combination of \( f_1^* \) and \( f_2^* \) that gives \( f_1^{**} \) has weights that sum up to one (see last row), which implies that \( b = a \alpha + (1 - \alpha) b \), and hence \( a = b \). By analogous argument \( (1, \frac{a}{1-a}) \) gives a state contingent claim and \( F^* \) cannot be optimal. \( Q.E.D. \)

6 References

**Figure 1. Existence of Profit Maximizing Financial Structure**

Each one-dimensional linear subspace in $\mathbb{R}^2$ - a line passing though origin is uniquely identified by a point on a semicircle (B). A bijection enlarging the distance along the circle by factor two, translates the semicircle into a full circle (C). Given parameterization, the entrepreneur maximizing profit over one dimensional spans is effectively choosing a point on a circle-a compact set. By assumption all linear subspaces parameterized by points in the circle have dimension one; thus $X(L)$ is a continuous correspondence on the circle. By the Maximum Theorem and Lemma 1, allocation $x(L)$ is a continuous function and Arrow prices given by the average marginal utility.

**Figure 2. Equilibrium Allocation and Entrepreneurs Choice**

A. Edgeworth box gives all feasible allocations. With complete $F$, all allocations in the Edgeworth box are in $F$-feasible set, while if only equity is issued, $F$-feasible set comprises only a line segment connecting endowment points. Competitive allocation is determined by planner’s indifference curves: Planner attains unconstrained maximum at the Pareto efficient allocation, at the center of the Edgeworth box and utility decreases (DWL increases) for allocations located further away from the center. Thus with complete $(F,E)$ planner chooses unconstraint maximum, whereas with equity, equilibrium allocation coincides with planner (second-best) choice on $F$-feasible set. B. The curves represent entrepreneur iso-profit map. Whether the Pareto allocation yields a minimum or a maximum, depends on whether the marginal utility and, hence, profit function is convex or concave. With logarithmic utility the entrepreneur maximizes profit by moving away from the Pareto allocation and, hence, prefers the allocation resulting from the incomplete financial structure.

**Figure 3. Allen and Gale Example: Asymmetric Utility**

Allocation maximizing planner’s preferences (Pareto efficient allocation) is different from allocation minimizing market value. With equity only, the equilibrium allocation is given by a point on $F$-feasible set that maximizes the planner preferences, while with a complete financial structure it is given by the planner unconstraint maximum. Given shape of iso-profit curves the efficient allocation is located on outer iso-profit curve and hence gives higher market value. The separation of minimal MV allocation and efficient allocation makes complete financial structure more profitable even with convex marginal utility.