These questions will go over basic game-theoretic concepts and some applications. This homework is due during class on week 4.

[1] In this problem (see Fudenberg-Tirole Ex. 1.1) you are asked to play with arbitrary 2 × 2 games just to get used to the idea of equilibrium computation. Specifically, consider the following game:

\[
\begin{array}{cc}
L & R \\
U & a, b & c, d \\
D & e, f & g, h \\
\end{array}
\]

To test whether a pair, say \((U, L)\), is a pure strategy Nash equilibrium is a trivial exercise. For instance, you would simply have to check whether or not the inequalities \(a \geq e\) and \(b \geq d\) hold. The point of the exercise, however, is to draw reaction curves for understanding mixed strategy equilibria. To this end, let \(p\) be the probability that player 1 plays \(U\) and \(q\) the probability that player 2 plays \(L\), and do the following with important locations on the axes properly computed and labeled:

(i) Plot each person’s best response correspondence as a function of the other’s mixed strategy (as summarized by the randomization probability). Pay special attention to the points at which the player is indifferent between her own two pure strategies.

(ii) For which parameters does each player have a strictly dominant strategy?

(iii) Show that if neither player has a strictly dominant strategy and the game has a unique equilibrium, the equilibrium must be in mixed strategies. Make sure you understand why both premises are assumed for the result.

[2] Find all equilibria of the following games:

**Battle of the Sexes**

\[
\begin{array}{cc}
L & R \\
U & 2, 1 & 0, 0 \\
D & 0, 0 & 1, 2 \\
\end{array}
\]

“Hawk-Dove” or “Chicken”

\[
\begin{array}{cc}
L & R \\
U & -1, -1 & 2, 1 \\
D & 1, 2 & 0, 0 \\
\end{array}
\]

**Coordination**
Risky Coordination

\begin{bmatrix}
L & R \\
U & \begin{bmatrix} 3,3 & 0,0 \\
D & \begin{bmatrix} 0,0 & 1,1 \\
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}


There are \( n \) candidates running for political office. Each announces a \textit{position} on \([0,1]\). Citizens have preferences over positions which look like this: each citizen has an ideal position \(x \in [0,1] \). If a position \(y\) is actually chosen by the winner, our citizen gets a payoff of \(-|x-y|\).

There is an infinite continuum of citizens and their ideal points are distributed according to some unimodal density on \([0,1]\).

Faced with a choice between the candidates, a citizen votes for the one whose announced position is nearest to her ideal point (and randomizes over the favorites if there are more than one). The candidate with the most votes wins.

Set up this model precisely as a game, and prove that there is a pure-strategy Nash equilibrium when \(n = 2\).

Discuss what happens to pure strategy equilibria if \(n > 2\)?


A group of \( n \) agents is engaged in joint production. Each agent \(i\) supplies effort \(e_i\). Let total effort be \(e = \sum_i e_i\). Joint output is given by \(y = e^\alpha\) for some parameter \(\alpha\) strictly between 0 and 1. This output is divided up among the \(n\) agents using a given vector of shares \(\ell \equiv (\lambda_1, \lambda_2, \lambda_n)\), which are nonnegative and add up to 1.

(a) Describe precisely the pure strategy Nash equilibria of this game.

(b) A total effort level \(e^*\) is \textit{efficient} if it maximizes \(e^\alpha - e\) over all possible choices of \(e\). Is there some vector of shares which generates a Nash equilibrium with total effort level \(e^*\)?

(c) [Optional, some philosophy.] Discuss the implications of part (b) for the connections between egalitarian sharing and (in)efficient outcomes.
[5] Here is a variation on [4] that says something about the sociology of the family. Suppose that Mum and Dad both agree that some love and care is good for their only child, but too much of it will spoil him. If \( L \) denotes the total hours of love and care (Mum’s plus Dad’s), let us suppose that Mum feels that the payoff to the child (and hence to Mum) is a function \( m(L) \), which is assumed to go up first and then go down continuously. Likewise for Dad, but he has a different function; call it \( d(L) \). Look at the point where \( m(K) \) is maxed, call it \( L_m \). Likewise define \( L_d \) for the function \( d(L) \). Now prove that even if \( L_d \) is a tiny amount less than \( L_m \), Mum will put in all the love and care and Dad will put in nothing!

[6] Another joint production problem, in which I keep the functional forms intact but ask you to solve the problem by manipulating and comparing first-order conditions.

A family farm with \( n \) members produces a joint output using an increasing, smooth, strictly concave production function \( f(e) \), where \( e \) denotes the sum of individual efforts \( e_i \). Each individual \( i \) has a payoff function \( u(c_i) - v(e_i) \), where \( c_i \) is her consumption. Assume that \( u \) and \( v \) are increasing and smooth, and that \( u \) is strictly concave while \( v \) is strictly convex.

[A] Describe precisely the solution to the social planner’s problem (in which all payoffs are added up and the sum maximized). Prove that it must specify equal effort for all and equal sharing of the output.

[B] Now suppose that \( e_i \) is chosen independently by agent \( i \), under the assumption that the total output will be equally divided. Write down the conditions describing a symmetric equilibrium with positive effort and output. Compare these values with the planner’s solution in part (A), and explain why they are different (and in which direction).

[C] Imagine, now, that the following incentive scheme is in place. A fraction \( \beta \) of the output is divided equally, and the remainder \( 1 - \beta \) is allocated according to “work points” (that is, a share equal to) \( \frac{e_i}{E} \). Continue to look only at symmetric Nash equilibria, where everybody puts in the same effort. Describe what happens (relative to the first best) as \( \beta \) varies from 0 to 1, and provide intuition.

[D] Note that in equilibrium, all players share the output equally anyway regardless of the value of \( \beta \). Explain intuitively why it is that we get different results for different values of \( \beta \), despite this fact.

[7] (From Fudenberg-Tirole and Rabin (1988)): Here is an informal description of a sequential game. There are three players. First player 1 moves \( A \) or \( B \). If she moves \( A \), the game is over with payoffs \( (6,0,6) \). If she chooses \( B \), player 2 gets to move, choosing \( C \) or \( D \). If she chooses \( C \) the game is over with payoffs \( (8,6,8) \). If she chooses \( D \) players 1 and 3 play a simultaneous-move coordination game (payoffs given are for all three players, of course, even though 2 is not active in this subgame):

<table>
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<tr>
<th></th>
<th>( E )</th>
<th>( F )</th>
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<tbody>
<tr>
<td>( G )</td>
<td>7, 10, 7</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>( H )</td>
<td>0, 0, 0</td>
<td>7, 10, 7</td>
</tr>
</tbody>
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(a) Draw a game tree to represent this game.
(b) Prove that in every subgame perfect equilibrium of this game (including those in which behavior strategies are used), player 1 plays $B$ at the beginning.

(c) Describe a situation in which players 1 and 2 both predict Nash equilibria in subgames, but different ones, which justifies player 1 choosing $A$. What sort of assumption rules this out when we define subgame-perfect equilibrium?

[8] Consider the following situation. A risk-neutral individual with income $y$ may decide to pay taxes $t$ or evade them. Assume that if the individual evades taxes and is audited, he pays a fixed fine $F$. An auditor can choose whether or not to audit the individual. Set up game trees for three situations: (i) the auditor can precommit whether or not he will audit, and the individual knows this before making his evasion decision; (ii) the auditor can precommit a probability of audit, and the individual knows this probability before making his evasion decision; (iii) no precommitment is possible. In each case be very careful in defining the strategies.

Assume that the cost of auditing with probability $p$ is $c(p)$. [Define $c(1)$ to be the cost of auditing for sure.] Solve for the subgame perfect equilibria in the three cases.

[9] Consider the following simple model of entry. A monopolist faces a sequence of $N$ entrants. At each stage an entrant can choose to enter or not enter a market. At the end of that stage the entrant dies, to be replaced by another potential entrant who has seen all that has gone before.

Payoffs in each period are as follows. If an entrant decides to stay out, the monopolist earns a monopoly profit of 2, and the entrant earns nothing. If an entrant decides to enter, the monopolist can fight the entrant, or accommodate her. If the former, payoff is 0 to the monopolist and -1 to the entrant. If the latter, payoffs are 1 each.

Set this up as an extensive form game and describe the subgame perfect equilibrium of the game.

[10] [Osborne-Rubinstein Exercise 101.3.] Armies 1 and 2 are fighting over an island initially held by a battalion of Army 2. Army 1 has $K$ battalions and army 2 has $L$. Whenever the island is occupied by one army the opposing army can launch an attack. The outcome of the attack is that the occupying battalion and one of the attacking battalions are destroyed; the attacking army wins and, so long as it has battalions left, occupies the island with one battalion. The commander of each army is interested in maximizing the number of surviving battalions but also regards the occupation of the island as worth more than one battalion but less than two. (If, after an attack, neither army has any battalions left, then the payoff of each commander is 0.) Analyze this situation as an extensive game and, using the notion of subgame perfect equilibrium, predict the winner as a function of $K$ and $L$.

[11] In the Cournot duopoly, it pays for one of the two agents to be able to move first. Why? Discuss the truth of this assertion for general two-player simultaneous move games.
Two players $A$ and $B$ take turns at proposing a division $(a, b)$ of a cake of size 1. First $A$ makes an offer. If $B$ accepts, we are done. Otherwise $B$ rejects, and a unit of time passes, during which both players discount their payoffs (see below for more precision). This process continues for $N$ periods.

If an agreement $(a, b)$ is reached at date $t$ (with $a + b = 1$, presumably), then player $A$’s payoff is $\alpha^t a$ ($\alpha$ is player $A$’s discount factor), and player $B$’s payoff is $\beta^t b$ ($\beta$ is player $B$’s discount factor).

[a] Assuming that both $\alpha$ and $\beta$ are less than one, find the subgame perfect equilibria of this game and discuss what happens as $N \to \infty$.

[b] Explain what happens to this limit result when there is no discounting; i.e., $\alpha = \beta = 1$. 