Games in Extensive Form

Extensive game described by following properties:


2. The order of moves (a tree).

Three types of nodes: initial, (noninitial) decision ($x$), terminal ($z$)

Each node uniquely assigned to a player

Edges are the actions: $A(x)$ set of actions at nonterminal node $x$. 
Formally, a tree is a set of nodes $X$ endowed with a precedence relationship $\succ$.

$\succ$ is a partial order: transitive and asymmetric.

*Initial node* $x_0$: there is no $x \in X$ such that $x \succ x_0$.

*Terminal node* $z$: there is no $x \in X$ such that $z \succ x$.

Additional assumption: *exactly* one immediate predecessor for every noninitial node (*arborescence*).
Node Ownership

Each noninitial node assigned to one player.

\[ \iota : X \setminus Z \mapsto N \cup \{ \text{Nature} \} \]

Information Sets

\[ h \subseteq X. \ H \text{ is collection of all } h \text{'s.} \]

Restriction: If \( x \) and \( x' \) are in same \( h \), then \( \iota(x) = \iota(x') \) and \( A(x) = A(x') \).

So can write things like \( \iota(h) \) and \( A(h) \).
Interpreting Information Sets

Lack of information about something that has already happened simultaneously

Flipping the future and past: moving Nature’s moves up.

Perfect Recall

[1] \( x \) and \( x' \) in same \( h \) implies \( x \) cannot precede \( x' \).

This isn’t enough. Example.

[2] \( x, x' \in h, \ p \succ x \) and \( \iota(p) = \iota(h) \implies \exists p' \) (with \( p = p' \) possibly) s.t. \( p \) and \( p' \) are in the same information set, while \( p' \succ x' \)

Action chosen along \( p \) to \( x \) equals the action taken along \( p' \) to \( x' \).
Strategies

Define $H_i \equiv \{ h \in H | \iota(h) = i \}$ and $A_i \equiv \bigcup_{h \in H_i} A(h)$.

**Pure strategy.** Mapping $s_i : H_i \mapsto A_i$ such that $s_i(h) \in A(h)$ for all $h \in H_i$.

$S_i$ set of pure strategies for $i$.

Can think of it as a mapping or collection of “giant vectors”:

$$S_i = \bigotimes_{h \in H_i} A(h)$$

**Behavior strategies.** Mapping $\sigma_i$ on $H_i$ such that

$$\sigma_i(h) \in \mathcal{M}(A(h))$$ for all $i$

**Mixed strategies.** Probability distribution $m_i$ over all pure strategies.
Are mixed and pure strategies equivalent? Not without perfect recall!

Pure strategy I: play \((L_1, L_2)\).

Pure strategy II: play \((R_1, R_2)\).

Mixed strategy plays these two with equal probability.

No behavior strategy mimics this.
On the other hand, behavior strategies may not be replicable by mixed strategies:

Behavior strategy: play L or R with equal probability.

But then, the path (L, R) is a possible outcome.

This is never possible with a mixed strategy, which can only yield paths (L, L) or (R, R).
This is ruled out as soon as you make [1] of the perfect recall assumption which is often built into the basic definition of a game.

Fix any behavioral strategy \( \sigma_i \). For any pure strategy \( s_i \) simply define \( m_i(s_i) \) by

\[
m_i(s_i) = \prod_{h_i \in H_i} \sigma_i(s_i(h_i))
\]
Kuhn's Theorem. Assume perfect recall. Then for every mixed strategy there is an equivalent behavior strategy.
“Proof.”

Pick some mixed $m_i$ and any $h$ such that $\iota(h) = i$. Define:

$$P_i(h) = \{s_i \in S_i \mid \text{under } s_i \text{ it is possible to reach } h\}$$

If $m_i(s) > 0$ for some $s \in P_i(h)$, define for every $a \in A(h)$,

$$\sigma_i(a) \equiv \frac{\sum_{s \in P(h), s(h) = a} m_i(s)}{\sum_{s \in P(h)} m_i(s)}$$

Otherwise, $m_i(s) = 0$ for all $s \in P_i(h)$. In that case, set

$$\sigma_i(a) \equiv \sum_{s \in S_i, s(h) = a} m_i(s)$$

Check this is ok with perfect recall.
Credibility and Subgame Perfection

Example. Chainstore paradox.

Subgame. Subtree of original tree with following properties:

initial node not a terminal node of original tree (nor its initial node if a proper subgame)

If \( x \) belongs to Subgame and \( x \in h \), then \( y \) belongs to Subgame for all \( y \in h \).

A (behavior) strategy profile is a subgame perfect equilibrium if it is Nash in every subgame.
Theorem. *In every finite extensive game with perfect recall, a subgame perfect equilibrium exists in behavior strategies.*

Proof. By induction on # of nonterminal nodes.

Assume existence in all trees with no more than \( k \) nonterminal nodes, for some \( k \geq 1 \).

For \( k = 1 \), assertion trivial to verify.

Take game with \( k + 1 \) nonterminal nodes.

If it has no proper subgames use standard theorem, then Kuhn’s Theorem.

If it has, append subgame perfect payoffs to any nonterminal non-initial node and use induction. QED

Counterexample to existence (even of Nash equilibrium) in behavioral strategies when no perfect recall.
Backward Induction

*Extensive game of perfect information.* Every information set a singleton.

Subgame perfection reduces to *backward induction* for this case.

For any game $G$, define

$$r(G) = \{x \in X | x \text{ is an immediate predecessor to terminal nodes alone}\}$$

$$s(G) = \{z \in Z | z \text{ has no predecessor in } r(G)\}$$

Begin with $r(G)$. Optimize for $i$ moving at some $x \in r(G)$.

Append resulting payoffs to $r(G)$; redefine as terminal nodes. Consider new game $G'$ with terminal nodes $r(G) \cup s(G)$.

Repeat process until all nodes exhausted. QED
Backward Induction and a Rationality Paradox

Rosenthal’s Centipede

Deviations as trembles

Irrational types