[1] Adapt class notes.

[2] Let \((m_1, \ldots, m_n)\) be the response vector in any stationary equilibrium. If \(i\) is the proposer, she will obtain \(1 - \delta \sum_{j \neq i} m_j\). If not, she will get \(m_i\). The former happens with probability \(1/n\) upon a rejection; the latter with probability \((n - 1)/n\). This means that when \(i\) is the responder, she should accept any offer that gives her at least

\[
\delta \frac{1}{n} [1 - \sum_{j \neq i} m_j] + \delta \frac{n - 1}{n} m_i.
\]

In other words, \(m_i\) solves the equation

\[
m_i = \delta \frac{1}{n} [1 - \sum_{j \neq i} m_j] + \delta \frac{n - 1}{n} m_i.
\]

Define \(\lambda \equiv (\delta/n)/[(1 - \delta) + (\delta/n)]\). Then \(0 < \lambda < 1\), and

\[
m_i = \lambda [1 - \sum_{j \neq i} m_j].
\]

It is well known that such an equation system has a unique solution.

To answer the second part of this question, the class notes need to be adapted again:

As in those notes, we describe one phase corresponding to each player. Phase \(i\) is described as follows:

Every proposer gives the entire cake to player \(i\).

Now “connect” these phases as follows. Once again we need to recursively define valid and invalid phases and describe what happens in each of them. Let’s first describe what happens in each kind of phase.

[A] Suppose that we are currently in a valid phase \(i\). Then all proposers are required to make the proposal described in that phase, all responders are required to accept the proposal.

[B] Otherwise, the phase is invalid. In this case, the responder \(k\) (among the remaining responders) — if any — is required to reject the proposal if and only of she gets no more than \(1/(n - 1)\), and start phase \(k\).

[C] A phase is valid if it is a going reaction to an invalid phase as specified by [A] or [B], or if it is the phase specified at the very start of the game. Otherwise, it is invalid.

And the initial condition: begin with the specification that some (valid) phase \(i\) is started up. Now follow the rest of the class notes.
[3] Adapt notes in class for a unique equilibrium. Also consult Rubinstein (1982) where such a model was first proposed.

[4] Suppose that bargaining proceeds according to alternating offers. Call the players \( a \) and \( b \) in line with their outside options (if the game breaks down). As in class notes, suppose that \( m_i \) and \( M_i \) are the minimum and maximum subgame perfect payoffs conditional on rejecting a going offer, for \( i = a, b \). Then it is easy to see that

\[
m_a \geq pa + (1 - p)(1 - M_b),
\]

while

\[
M_b \leq pb + (1 - p)(1 - m_a).
\]

Combining these two inequalities, we see that

\[
m_a \geq \frac{a + (1 - p)(1 - b)}{2 - p},
\]

while by the same sort of argument,

\[
M_a \geq \frac{a + (1 - p)(1 - b)}{2 - p}.
\]

Therefore the equilibrium payoff conditional on rejection is unique, and given by

\[
\frac{a + (1 - p)(1 - b)}{2 - p}
\]

for person \( a \), and by symmetry,

\[
\frac{b + (1 - p)(1 - a)}{2 - p}
\]

for person \( b \).

An interesting feature of this solution is seen when \( p \) goes to zero. Then the equilibrium payoffs converge to

\[
a + \frac{1 - a - b}{2} \quad \text{and} \quad b + \frac{1 - a - b}{2}
\]

which is the Nash bargaining solution.

[5] Suppose that \( v \) is balanced. For any integer \( 1 \leq s \leq n - 1 \), define a map \( \delta(S) \) by \( \delta(S) = 0 \) if the cardinality of \( S \) is different from \( s \), and is equal to \( 1/(\binom{n-1}{s-1}) \) if the cardinality of \( S \) is \( s \). Because every \( i \) belongs to precisely \( \binom{n-1}{s-1} \) coalitions of size \( s \), this is a balanced map, and so

\[
v(N) \geq \sum_{S} \delta(S)v(S)
\]

\[
= \binom{n-1}{s-1}^{-1} v(s) \binom{n}{s},
\]

where \( v(s) \) just stands for the worth of any coalition of size \( s \) (all the same, by symmetry). But this means that \( v(n) \geq (n/s)v(s) \).
Conversely, suppose that \( v(n)/n \geq v(s)/s \). Let \( \delta(S) \) be any balancing map. Let \( a(s) \equiv v(s)/s \). Then

\[
\sum_S \delta(S)v(S) = \sum_S \delta(S)a(|S|)|S| \leq a(n) \sum_S \delta(S)|S|.
\]

Note that \( \sum_S \delta(S)|S| = n \), which completes the proof. [Much easier though indirect way: show that the equal division allocation is in the core, which means that the game must be balanced, by Bondareva-Shapley.]

[6] Consider the idea of the credible core. For any characteristic function \( V \), this is a (possibly empty-valued) selection: \( C^*(S) \subseteq V(S) \) for all coalitions \( S \), with the property that for each coalition \( S \), \( C^*(S) \) is precisely the set of all allocations that are unblocked by any coalition \( T \) using allocations in \( C^*(T) \) alone.

(a) This is not a formal definition because it is circular: it uses the credible core of \( T \) to define the credible core of \( S \). However, notice that we only look at blocking by subcoalitions, so we can recursively define this concept starting from singletons and going “up”: we will get precisely the notion introduced above (see Ray (1989, Int. J. Game Theory) for more on this).

(b) The point is that the credible core coincides with the usual core mapping, which we’ll call \( C(S) \) — the set of allocations in \( V(S) \) which are unblocked by any allocation from \( T \subset S \). Because the core requirement is more stringent, it’s obvious that \( C(S) \subseteq C^*(S) \) for all \( S \). Suppose that for some \( S \), equality does not hold. Then there is \( x \in C^*(S) \setminus C(S) \). Because \( x \notin C(S) \), there is \( T \subseteq S \) which blocks \( x \) with \( y \in V(T) \). Because \( x \notin C^*(S) \), \( y \notin C^*(T) \). So \( y \) is in turn blocked by some coalition \( W \) using an allocation \( z \) in \( C^*(W) \). Now \( W \subseteq S \). Moreover, it is easy to check that if \( (W, z) \) blocks \( (T, y) \) while \( (T, y) \) blocks \( (S, x) \), then \( (W, z) \) blocks \( (S, x) \). But this contradicts our presumption that \( f \in C^*(S) \), and the proof is complete.

(c) This argument does not work if the number of players is infinite. The notion \( C^* \) has to be introduced as a solution concept (whose existence has to be established), not as a recursive definition. Here is a small example in case you’re interested:

Let \( N = \{1, 2, 3, \ldots\} \), and \( V(S) \) be a TU characteristic function: \( v(S) = 1 \) if \( S = \{t, t+1, t+2, \ldots\} \) for some \( t \geq 1 \), and is zero otherwise. The core of this game is empty. What is its credible core?

For more on the infinite case, see Einy and Shitovitz (1997, Int. J. Game Theory).

[7] Example of a non-superadditive game which satisfies condition [M]: Let \( v(12) = 0 \), and \( v(i) = 1 \) for \( i = 1, 2 \). This two-person game is not superadditive. Yet \( m_i(i, \delta) = m_i(N, \delta) = \delta \) for each \( i \), so that condition [M] is satisfied.

Example of a superadditive game which fails condition [M]: \( N = \{1, 2, 3\} \), \( v(1j) = 1 \) for \( j = 2, 3 \), \( v(23) = 2 \), \( v(i) = 0 \) for all \( i \), \( v(123) = 2 \). Now we shall check that condition [M] is not satisfied for player 1.
To see this, compute \( m_1(S, \delta) \) for the coalition \( S = \{12\} \). It is easy to see that \( m_1(S, \delta) = \frac{\delta}{1 + \delta} \). However, for the grand coalition, it is easy to verify that

\[
m_1(N, \delta) = \delta \left[ 2 - \frac{4\delta}{1 + \delta} \right],
\]

the idea being that players 2 and 3 can always form their own coalition, so they have response numbers of \( 2\delta/(1 + \delta) \) each. The best player 1 can do is to offer each of them this and take the rest, but it is obvious that the value of this dwindles to zero for \( \delta \) close to 1, while the value of \( m_1(S, \delta) \) goes to 1/2. So condition [M] fails.

It is interesting to note that while [M] fails, there is a unique stationary equilibrium in this example and it is no-delay! That is, [M] is sufficient for no-delay but it is not necessary.

To construct a superadditive example in which [M] fails and there is no equilibrium without delay, use the “superadditive completion” of Example 1 in Chatterjee et al (1993, RES, p.466).

[8] and [9]: Read class notes.

[10] I will prove this result under the assumption that the grand coalition uniquely maximizes the worth of a coalition: i.e., \( v(N) > v(S) \) for all subcoalitions \( S \). I will comment on the other case below.

Under this assumption, there exists \( \epsilon > 0 \) such that \( v(N) > v(S) + \epsilon \) for all subcoalitions \( S \). Let \( K \) be the maximum possible payoff to a person, and define a threshold \( \delta^* \) for the discount factor small enough so that \( \delta^* K(n-1) < \epsilon \), where \( n \) is the total number of players. Now I claim that any proposer must form the grand coalition in any equilibrium. For if not, his payoff is bounded above by \( \max v(S) \), where the max is taken over all proper subcoalitions. But now she can deviate and propose the grand coalition and offer everyone else \( \epsilon/(n-1) \) each. This offer must be accepted if \( \delta < \delta^* \). So player \( i \) picks up \( v(N) - (n-1)\epsilon/(n-1) = v(N) - \epsilon > \max v(S) \), a contradiction.

If the assumption in the first line of this answer does not hold, I don’t believe the result is true. For instance, in a three player game, player 1 will be looking for membership in a coalition which maximizes aggregate worth under the restriction that she is a member of that coalition. But this choice may not maximize aggregate societal worth. Thus if player 1 is chosen to be proposer, the outcome will still be inefficient.


[13] For parts (a)–(d), read the introduction in Ray and Vohra (1999, Games and Economic Behavior). For part (e) you will have to read the full paper and it is outside the scope of this class. (I nevertheless include it in case you are interested.)