[1] (a)–(c). Discussed in class. The important point is that in part (b), beliefs are restricted by subgame perfection. Player 2 must believe that player 1 has played a, whereas in part (a), she is free (by sequential equilibrium) to believe anything. So part (b) has a unique sequential equilibrium, while part (a) does not.

(d) Yes, the belief \(\{0, 1\}\) can be entertained at the information set \(h_1\). Think of the completely mixed strategy (including that of Nature) in which Nature moves with probabilities \(\epsilon_a, \epsilon_b, 1 - \epsilon_a - \epsilon_b\). Take a perturbation for \(A\) in which she plays \(a\) with probability \(1 - \mu_a\) and one for \(B\) in which she plays \(a\) with probability \(\mu_b\). Then by Bayes’ Rule, the probability of the right-hand node in \(h_1\) is

\[
\frac{\epsilon_b \mu_b}{\epsilon_b \mu_b + (1 - \mu_a) \epsilon_a},
\]

and notice that this could converge to 1 as the trembles go to zero, but a necessary condition for that to happen is

\[
\frac{\epsilon_b}{\epsilon_a} \to \infty
\]

(it has to become increasingly likely in a relative sense that \(B\) was the outcome of Nature’s tremble).

How about the belief \(\{1, 0\}\) at the information set \(h_2\)? That works too, but now you need the opposite condition; that

\[
\frac{\epsilon_b}{\epsilon_a} \to 0
\]

as the trembles vanish overall.

So both these beliefs can be part of sequential equilibrium, but they can never be part of the same sequential equilibrium.

[2] Take player 1 who has valuation \(v\) and bids \(b\), where (say) \(b > v\). If player 2’s bid \(b'\) exceeds \(b\), then 1 gets a payoff of 0. If \(b'\) is no greater than \(v\), then 1 gets a payoff of \(v - b'\). But if \(b'\) lies in \((v, b]\), then player 1 wins with some probability (probability 1 when \(b' < b\)) and pays \(v - b'\), which is a loss. This loss can be reduced to 0 by simply bidding \(b = v\), which means that \(b = v\) weakly dominates any bid by player 1 in excess of \(v\). The same kind of argument holds for \(b < v\).

3(a) Omitted.

(b) We show that \(b_i\) is a nondecreasing function of \(\theta_i\). Recall that the payoff of player \(i\) when he bids \(b_i\) and his valuation is \(\theta_i\) is

\[
F_i(\theta_i, b_i) = (\theta_i - b_i) H_i^{-}(b_i) + \sum_{k=1}^{n-1} \frac{\theta_i - b_i}{k + 1} h_i(b_i) p(k; b_i).
\]

Now pick two values \(\theta < \theta'\) and let \(b\) and \(b'\) be corresponding bids. Notice that if \(\theta > b\) (or \(\theta' > b'\)), neither \(b\) nor \(b'\) will be placed at a mass point of \(H_i\): by bidding slightly higher you
can get a discontinuously higher return (avoiding ties). And if \( \theta = b \) (or \( \theta' = b' \)) the payoff is zero anyway. So in both cases, we can write

\[
F(\theta, b) = (\theta - b)H_i(b) \quad \text{and} \quad F(\theta', b') = (\theta' - b')H_i(b')
\]

Now by the revealed preference argument which you’ve seen more than once:

\[
(\theta - b)H_i(b) \geq (\theta' - b')H_i(b')
\]

while

\[
(\theta' - b')H_i(b') \geq (\theta' - b)H_i(b)
\]

Adding and canceling common terms, we have

\[
(\theta - \theta')[H_i(b) - H_i(b')] \geq 0,
\]

which proves that the bid function for \( i \) is nondecreasing.

(c) Next, we show that there are no mass points in the bid function, except possibly at 0. The first thing to notice that 0 must lie in the support of everybody’s bids. For if it didn’t, then \( G_i(b) = 0 \) for all \( b \) in some interval to the right of 0, and then it can be shown that very small valuation types must incur expected losses in equilibrium, a contradiction.

Suppose on the contrary that for some \( j \), \( G_j \) has an atom at some value \( b > 0 \). Then I claim that there is \( \epsilon > 0 \) such that no person \( i \) never bids in the range \([b - \epsilon, b]\).

Suppose not. Then there is a player \( i \), a sequence \( b_k \uparrow b \) and a corresponding sequence of types \( \theta_k \) for player \( i \) such that for each \( k \), \( \theta_k \) finds it a best response to bid \( b_k \).

In what follows, I am always going to use para 1 to look at events in which \( j \) is the top bidder apart from \( i \).

If \( \limsup_k \theta_k > b \), we must have a contradiction for large \( k \). For such a type would earn higher profits by slightly raising the bid above \( b \): she would make a discontinuous gain in the win probability and (for large \( k \)) change her bid by very little.

Conversely, it cannot be that \( \liminf_k \theta_k < b \). For then for large \( k \), these types would make strictly negative profit (use the first para of this part to nail this down).

Therefore \( \lim_k \theta_k = b \). But then it must be that the optimal payoff of type \( k \) converges to 0 as \( k \to \infty \). But these types could easily get a strictly positive expected profit by bidding, say, \( b/2 \) (again use the first para).

So our claim is true: there is a (small) blank region to the left of \( b \) where nobody ever bids. But then all the types of \( j \) that bid \( b \) would be better off by cutting their bid to \( b - (\epsilon/2) \) (no change in win probability, greater win margin). Contradiction.

(d) Next, we show that there cannot be any “gaps” in the range of \( i \)’s bids, or equivalently, that the bid function must be continuous. Suppose not. Then at some \( \theta \) the left hand limit is \( b_- \) and the right hand limit is \( b_+ \), with \( b_+ > b_- \) (monotonicity of the bid function, established earlier, guarantees all this). Notice that no type of \( j \) should ever bid in the gap \((b_-, b_+)\) (there is no gain in win probability at all and win margins are being given up). But if this is the case, then the \( i \)-types with bids slightly above \( b_+ \) (or at \( b_+ \), if there is such a type) will all gain by discretely lowering their bids to just above \( b_- \). Contradiction.
(e) Prove that the maximum bids are the same: \( b_i(\theta) = b_j(\theta) \). Same sort of argument. The person with the higher maximum gains nothing in win probabilities by bidding the higher value.

(f) and (g): Fully explained in the problem. In (g), you should note that you get the same solution in terms of payoffs as the sealed bid second-price auction. This is a special case of the well-known Revenue Equivalence Theorem.


[5a] Suppose that every other player contributes if and only if their valuation is at least \( v^* \). Then the probability that at least one of the other players contributes is the probability that at least one has a valuation exceeding \( v^* \), which is

\[
1 - \left( \frac{v^* - \bar{v} - v}{\bar{v} - \bar{v}} \right)^{n-1}.
\]

So if a particular player \( i \)'s valuation is \( v_i \), her expected payoff is

\[
v_i \left[ 1 - \left( \frac{v^* - v}{\bar{v} - \bar{v}} \right)^{n-1} \right]
\]

if she does not contribute, and it is

\[v_i - c\]

if she does contribute. Thus player \( i \) is exactly indifferent when

\[
v_i \left[ 1 - \left( \frac{v^* - v}{\bar{v} - \bar{v}} \right)^{n-1} \right] = v_i - c,
\]

and because we are looking for a symmetric Nash equilibrium, this indifference condition must be attained when \( v_i = v^* \). Substituting this and rearranging terms in the equation above, we see that

(1)

\[v\left( \frac{v^* - v}{\bar{v} - \bar{v}} \right)^{n-1} = c,
\]

which is what we were asked to prove. Note that the left hand side of this equation is increasing in \( v^* \) while the right hand side is flat, and moreover the left hand side “starts” at 0 and “ends” at a value strictly greater than \( c \). So there exists a unique solution.

[5b] Slight manipulation of the equation above shows that

\[
\left( \frac{v^* - v}{\bar{v} - \bar{v}} \right)^n = \left( \frac{c}{v^*} \right) \left( \frac{v^* - v}{\bar{v} - \bar{v}} \right).
\]

The probability that the good is provided is equal to

\[
1 - \left( \frac{v^* - v}{\bar{v} - \bar{v}} \right)^n.
\]

Combining these two equations, we may conclude that the probability of provision equals

\[
1 - \left( \frac{c}{v^*} \right) \left( \frac{v^* - v}{\bar{v} - \bar{v}} \right).
\]
From equation (1) it is obvious that as \( n \) goes up, the left-hand side falls, so \( v^* \) has to go up to restore equilibrium. So \( v^* \) increases with \( n \). The question is how this affects the probability of provision. By inspecting the above expression the answer is obvious: it all depends on what happens to the ratio

\[
\frac{v^* - \bar{v}}{v^*}
\]

as \( v^* \) goes up. If \( \bar{v} = 0 \) nothing happens, so that the probability of provision is unaffected by a change in \( n \). But if \( \bar{v} > 0 \), then this ratio goes up with \( n \), so that the probability of provision (which is related negatively to this ratio) must decline.

[6a] Let \( H \) and \( L \) be the two possible values of Jim’s wealth. If there to be sequential equilibria which are separating, then notice that the low wealth type will not put any money into conspicuous consumption (what’s the point? he’s identified anyway). So the low type’s payoff is just 0, because it is known for sure that he is low. Suppose that the high type spends an amount \( c^* \). He will then be known to be high, so that his payoff is \( 1 - (c^*/H) \) (the “1” is because he has been identified as high for sure). For each type not to imitate the other, we must have

\[
1 - \frac{c^*}{H} \leq 0,
\]

otherwise the low type will imitate the action of the high type, and

\[
1 - \frac{c^*}{H} \geq 0,
\]

otherwise the high type will imitate the action of the low type.

The two conditions together imply that \( L \leq c^* \leq H \); for any such expenditure of conspicuous consumption between the low and the high wealth there is a separating equilibrium “supporting” the expenditure of 0 by the low type and of \( c^* \) by the high type.

As in class, we need to be careful about specifying beliefs if an expenditure not equal to 0 or to \( c^* \) were to be observed. One set of beliefs that works is that you are believed to be the low type if you spend less than \( c^* \), and the high type otherwise.

[6b] In a pooling equilibrium, both types adopt the same action. This action conveys no fresh information; beliefs remain at \( p \), the prior, regarding the probability that Jim is rich (is of type \( H \)). The reason why this pooling equilibrium may be self-sustaining is that the beliefs surrounding any expenditure distinct from \( c^* \) is that the associated type is low. But of course, this does not mean that every conceivable \( c^* \) can be an equilibrium. It must be the case that \( p - c^*/L \geq 0 \), otherwise the low type will deviate to \( c = 0 \) and earn at least 0. Indeed, this restriction is sufficient as well: any \( c^* \) satisfying this inequality is supportable as a pooling equilibrium.

[6c] In hybrid equilibria, one type or the other (or both) must use mixed strategies. But there are restrictions:

(i) If the low type randomizes between expenditures of \( a \) and \( b \), with \( b > a \), and the high types between expenditures of \( c \) and \( d \), with \( d > c \), then by the single-crossing argument studied in class, it must be the case that \( c \geq b \).

(ii) In a hybrid equilibrium, we must have \( b \) equal to \( c \). For if not, then each type is fully identified by their actions and so neither type will randomize: they will each choose the lowest of their equilibrium actions (which is cheaper).
These general observations are not at all needed for the question but it is useful to see these points. We can easily describe a particular hybrid equilibrium. Consider the values \( a = 0 \) and some \( b > 0 \), and suppose that the low type randomizes 50-50 between \( a \) and \( b \) while the high type chooses \( b \) for sure. Then the payoff of the low type is zero, and by indifference the payoff of the low type must also be 0 when she chooses \( b \). That is,

\[
q - b/L = 0,
\]

where \( q \) is the posterior on observing \( b \). But \( q \) is given by Bayes’ Rule according to the formula

\[
q = \frac{p}{p + (1 - p)(1/2)} = \frac{2p}{p + 1},
\]

and these two equations must therefore pin down the value of \( b \):

\[
b = \frac{2pL}{p + 1}.
\]

The high type gets a strictly positive payoff by announcing \( b \).

[6d] Just as in the class notes, the intuitive criterion knocks out all these equilibria except one.

First we rule out all equilibria in which types \( H \) and \( L \) choose the same value of \( c \) with positive probability. [This deals with all the pooling and all the hybrid equilibria.]

At such an \( c \), the payoff to each type \( \theta \) is

\[
q - c/\theta,
\]

where \( q \) represents the employer’s posterior belief after seeing \( c \). Now, there always exists an \( c' > c \) such that

\[
q - c/L = 1 - c'/L,
\]

while at the same time,

\[
q - c/H < 1 - c'/H.
\]

It is easy to see that if we choose \( c'' \) very close to \( c' \) but slightly bigger than it, it will be equilibrium-dominated for the low type —

\[
q - c/L > 1 - c''/L,
\]

while it is not equilibrium-dominated for the high type:

\[
q - c/H < 1 - c''/H.
\]

But now the equilibrium is broken by having the high type deviate to \( c'' \). By IC, the employer must believe that the type there is high for sure and so must have a posterior belief of 1. But then the high type benefits from this deviation relative to playing \( c \).

Next, consider all separating equilibria in which \( L \) plays 0 while \( H \) plays some \( c > c_1 \), where \( c_1 \) is the lowest threshold for deterring the low type: \( c_1 = L \) (see above). Then a value of \( c' \) which is still bigger than \( c_1 \) but smaller than \( c \) can easily be seen to be equilibrium-dominated.
for the low type but not for the high type. So such values of $c'$ must be rewarded with a belief of 1. But then the high type will indeed deviate, breaking the equilibrium.

This proves that the only equilibrium that can survive the intuitive criterion is the one in which the low type plays 0 and the high type chooses $c_1 = L$.

[7] In both the games under consideration, let $A$ stand for the generic strategy that involves play of $L$ (for Column) or $U$ (for Row), and $B$ for the generic strategy that involves play of $R$ (for Column) or $D$ (for Row). In both cases note that playing $B$ is likely to be “better” under low values of the signal, so that is how we will orient the calculations.

Suppose, then, that we imagine that a player will play $B$ if the signal is some value $X$ or less. Let us calculate the recursion value $\psi(X)$ such that under this assumption, someone will play $B$ if his signal is $\psi(X)$ or less.

These examples have the same general structure. Suppose that the signal space is located on some interval $[\ell,h]$. For signals very close to $\ell$ playing $B$ is dominant. For signals very close to $h$, playing $A$ is dominant. So $\psi(\ell) > \ell$ and $\psi(h) < h$. Finally, we will show that $\psi$ is nondecreasing but has a slope strictly less than one. This yields a unique intersection $x^*$ (which depends on the extent of the noise $\epsilon$). By exactly the same arguments as in Morris-Shin (see my notes), there is a unique equilibrium of the imperfect observation game: play $B$ if the signal falls short of $x^*$. Finally, we describe $x^*$ as $\epsilon \to 0$.

(a) In the first example, suppose that your opponent plays $B$ if his signal is $X$ or less. Suppose you see a signal $x$, and play $B$. if the true state is $\theta$, the chance that your opponent plays $B$ is just the chance that your opponent’s signal falls below the threshold $X$, given $\theta$. This is given by the expression

$$\max\left\{ \frac{X - (\theta - \epsilon)}{2\epsilon}, 0 \right\},$$

and so your expected payoff (now taking expectations over $\theta$ conditional on your signal) is

$$\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (b - \theta) \max\left\{ \frac{X - (\theta - \epsilon)}{2\epsilon}, 0 \right\} d\theta.$$

Likewise, if you play $A$, the chance that your opponent also plays $A$ is

$$1 - \max\left\{ \frac{X - (\theta - \epsilon)}{2\epsilon}, 0 \right\},$$

and so your expected payoff conditional on $x$ is

$$\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (a + \theta) \left[ 1 - \max\left\{ \frac{X - (\theta - \epsilon)}{2\epsilon}, 0 \right\} \right] d\theta.$$

[Above, I am integrating from $x - \epsilon$ to $x + \epsilon$. I should be worrying about the lower and upper bounds on $\theta$ if I am too close to one edge of the signal space. But we can ignore this, because we know the behavior of $\Psi$ at the edges of the signal space without having to write down the exact expressions.]

The equality of expressions (2) and (3) give you the threshold $x$ for which you are indifferent between $A$ and $B$, under the presumption that a signal below $X$ results in a play of $B$ for
your opponent. In other words, \( \psi(X) \) is the solution (in \( x \)) to the equation

\[
(4) \quad \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (b-\theta) \max\{\frac{X-(\theta-\epsilon)}{2\epsilon}, 0\} \, d\theta = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (a+\theta) \left[ 1 - \max\{\frac{X-(\theta-\epsilon)}{2\epsilon}, 0\} \right] \, d\theta.
\]

By inspecting (4) it should be obvious that \( \Psi(X) \) is nondecreasing in \( X \). What is a little less obvious is the assertion that for all \( X' > X \),

\[
(5) \quad \psi(X') - \psi(X) < X' - X.
\]

To prove (5), let \( X \) increase to \( X + \Delta \). We want to show that the required solution to (4) in \( x \) increases by strictly less than \( \Delta \). Suppose this is false, then it must be that after raising \( X \) to \( X + \Delta \), a rise from the previous solution \( x \) to \( x + \Delta \) still does not (weakly) bring the LHS and RHS of (4) into new equality; i.e., we have

\[
\frac{1}{2\epsilon} \int_{x+\Delta-\epsilon}^{x+\Delta+\epsilon} (b-\theta) \max\{\frac{X+\Delta-(\theta-\epsilon)}{2\epsilon}, 0\} \, d\theta \geq \frac{1}{2\epsilon} \int_{x+\Delta-\epsilon}^{x+\Delta+\epsilon} (a+\theta) \left[ 1 - \max\{\frac{X+\Delta-(\theta-\epsilon)}{2\epsilon}, 0\} \right] \, d\theta.
\]

Now make the change of variables \( \theta' \equiv \theta - \Delta \). Then, after all the substitutions, we may conclude that

\[
\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (b-\theta')(\theta' - \Delta) \max\{\frac{X-(\theta'-\epsilon)}{2\epsilon}, 0\} \, d\theta' \geq \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (a+\theta'+\Delta) \left[ 1 - \max\{\frac{X-(\theta'-\epsilon)}{2\epsilon}, 0\} \right] \, d\theta',
\]

but this contradicts (4), the original relationship between \( X \) and \( x \). So the claim in (5) is established. Now we have a unique equilibrium using exactly the same arguments as Morris and Shin.

Call this unique threshold \( x^* \). Then, using this fixed point in (4) and noting that the “maxes” in that equation may now be dropped (why?), we have

\[
\int_{x^*-\epsilon}^{x^*+\epsilon} \frac{(b-\theta)[x^*-(\theta-\epsilon)]}{2\epsilon} \, d\theta = \int_{x^*-\epsilon}^{x^*+\epsilon} \frac{(a+\theta)}{2\epsilon} \left[ 1 - \frac{x^*-(\theta-\epsilon)}{2\epsilon} \right] \, d\theta.
\]

Now pass to the limit as \( \epsilon \to 0 \) (use L’Hospital’s Rule). It is easy to see that at the limit,

\[
x^* = \theta^* = \frac{b-a}{2}.
\]

[b] In the second example, make the same provisional assumption: your opponent plays \( B \) if his signal is \( X \) or less. Suppose you see a signal \( x \), and play \( B \). if the true state is \( \theta \), the chance that your opponent plays \( B \) is just the chance that your opponent’s signal falls below the threshold \( X \), given \( \theta \). This is given by the expression

\[
\max\{\frac{X-(\theta-\epsilon)}{2\epsilon}, 0\},
\]

just as in (a), and so your expected payoff (conditional on your signal) is

\[
\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} 4 \max\{\frac{X-(\theta-\epsilon)}{2\epsilon}, 0\} \, d\theta.
\]

[Again, I am integrating from \( x - \epsilon \) to \( x + \epsilon \) because we can neglect the edges of the state space (see discussion in part (a) above).]
On the other hand, if you play \( A \), you’re guaranteed \( \theta \) (whatever it may turn out to be), so your expected payoff is just \( x \), of course.

The equality of these two expressions give you the indifference threshold \( x \). That is, \( \psi(X) \) solves the equation (in \( x \)):

\[
\frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} 4 \max\left\{ \frac{X - (\theta - \epsilon)}{2\epsilon}, 0 \right\} d\theta = x.
\]

Again, you can show that \( \Psi(X) \) is nondecreasing in \( X \) and has slope less than one; i.e., that (5) holds for the \( \psi \)-function here as well. [Use the same sort of argument we did above; things here are even simpler.]

Call this unique threshold \( x^* \). Then, using this fixed point in (6) and once again noting that the “maxes” may be dropped (why?), we have

\[
\frac{1}{\epsilon} \int_{x^* - \epsilon}^{x^* + \epsilon} \frac{x^* - \theta + \epsilon}{\epsilon} d\theta = x^*.
\]

Now pass to the limit as \( \epsilon \to 0 \). It is easy to see that

\[ x^* = \theta^* = 2. \]

Observe the contrast between parts (a) and (b). In (a), equilibrium selection generally tracks the Pareto-dominant equilibrium. When \( a = b \), the switch point is 0 (how could it be anything else, by symmetry and uniqueness?), and now if \( a \) and \( b \) depart from each other, the switch point moves in the “correct” direction. For example, when, if \( b > a \), \( B \) will be played more often, because the switch point is now positive.

In part (b), the switch point is \( \theta = 2 \) (which is about its midpoint value, given the support of \( \theta \)). At this point, \((4, 4)\) is still much better than \((\theta, \theta) = (2, 2)\). Why does \((4, 4)\) have so little attractive power? It is because the play of \( A \) has “insurance” properties: if your opponent does not play \( A \), you still get something (in this example, you get full insurance in fact). But you get no insurance if you play \( B \) and your opponent does not. Thus the selection device not only looks at payoffs “at the equilibrium”, it looks at payoffs “off the equilibrium” as well to make the selection.

[8] Trivial (really!).

[9] (ii) Suppose, on the contrary, that a mechanism exists that implements the equilibrium \((T, L)\) if the column player is of type \( a \) and the equilibrium \((B, R)\) if she is of type \( b \). By the Revelation principle, we can find a direct truth-telling mechanism which does the same. That is, the outcome function \( g \) of the direct mechanism must have the two entries (The domain is the type announcement)

\[ g(a) = (T, L) \text{ and } g(b) = (B, R). \]

But then type \( b \) can always gain by lying about his type, a contradiction.

It is easy to see that there is only one implementable plan:

\[ g(a) = (T, L) \text{ and } g(b) = (T, R). \]