[1] Look at this game:

(a) Find all the sequential equilibria of this game.

Now look at this game:

(b) Find all the sequential equilibria of this game.

(c) Notice that 1 playing $u$ can be part of a sequential equilibrium in (a) but not in (b). Why are these two games different? [Think of our discussion in class regarding off-equilibrium moves as errors.]

(d) Look at this game:
Nature moves first and chooses type \( c \) with probability 1. Strategies are as follows: type \( a \) chooses \( A \), \( b \) chooses \( B \) and \( c \) chooses \( C \). Writing left to right across the nodes in the figure, can the belief \( \{0, 1\} \) be entertained at the information set \( h_1 \) in sequential equilibrium? How about the belief \( \{1, 0\} \) at the information set \( h_2 \)? How about the two together? 

[2] Second-price auction. Two persons are bidding on an object for auction. Each person has independent, private valuations for the object that are drawn from a continuous distribution over the interval \([0, 1]\). The highest bidder gets the object at the second-highest bidder’s bid. Prove that the strategy of bidding one’s true valuation weakly dominates every other strategy (for each type of each agent).

[3] First-price auction with independent private valuations (from Myerson (1991) and F-T). There are \( n \) bidders, and a single good is to be auctioned using sealed bid (first price). Each player has a private valuation for the good, \( \theta_i \), which we assume comes from the compact interval \( \Theta \equiv [0, \bar{\theta}] \). Each \( \theta_i \) is drawn iid from the same density \( g \) over \( \Theta \) (call the cdf \( G \)).

Player \( i \) learns her valuation, then bids \( b_i \). The highest bid wins. Ties are broken equiprobably. Both players are risk-neutral and their utility function is simply given by \( \theta - b \) if they win, or 0 if they lose.

(a) Precisely describe the Bayesian game that is involved here. In writing the payoff function, it will be useful to view each player \( i \)’s bidding strategy as a distribution function \( G_i \) over her bid, as far as the other player is concerned.

Note: In answering (a), observe the following: \( G_i \) cannot be assumed to have a density at this stage. Indeed, there may be mass points at particular bids. Use the notation \( g_i(b) \) to refer to probability mass induced by \( G_i \) at any bid level \( b \). And let \( G_i^{-}(b) \) to denote the probability that \( i \)’s bid is strictly less than \( b \). Finally, define \( H_i \) to be the distribution of the maximum of the bids other than \( i \) (this is induced by the \( G_j \)’s), and define \( h_i(b) \) and \( H_i^{-}(b) \) analogously. Conditional on some bid \( b \) having positive probability, let \( p(k, b) \) denote the probability that \( k \) people other than \( i \) are making that bid.

So the payoff to a type \( \theta_i \) from making a bid \( b_i \) is just

\[
F_i(\theta_i, b_i) = (\theta_i - b_i)H_i^{-}(b_i) + \sum_{k=1}^{n-1} \frac{\theta_i - b_i}{k+1} h_i(b_i)p(k, b_i),
\]

(as you should have verified in part (a) already).

(b) Employ the same revealed preference argument (used for Spence signaling in class) to show that \( b_i \) is a nondecreasing function of \( \theta_i \). That is, pick two values \( \theta < \theta' \) and let \( b \) and \( b' \) be corresponding bids. By optimality,

\[
F_i(\theta, b) \geq F_i(\theta, b')
\]

and

\[
F_i(\theta', b') \geq F_i(\theta', b)
\]

Write these out fully, add both sides, cancel common terms, and examine carefully to obtain the result. But you will have to be careful here. You will first have to show that no bids \( b \) will be placed by \( i \) at mass points of \( H_i \) whenever \( \theta_i > b \).
(c) Next, we show that except possibly at 0, there are no mass points in the bid function. Put another way, we must show that the bid function $b_i(\theta_i)$ is strictly increasing once bids turn positive. [Note: the argument in part (b) does not establish this.] To do this, we must use the full power of an equilibrium argument (not just the optimality argument in part (b)); proceed as follows.

Suppose that $g_i(b_i) > 0$ for some $b_i > 0$. Then first prove that there exists $\epsilon > 0$ such that no person $j$ will bid in the interval $[b_i - \epsilon, b_i)$. Then complete the proof by showing that in this case, a bid of $b_i$ is not optimal for player $i$.

(d) Next, show that there cannot be any “gaps” in the range of $i$’s bids, or equivalently, that the bid function must be continuous. This uses the same style of argument as in part (c): if there is a gap in $i$’s bid, first, prove something about the other bids, and then return to $i$ to complete a proof by contradiction.

(e) Prove that the maximum bids are the same: $b_i(\bar{\theta}) = b_j(\bar{\theta})$ for all $i$ and $j$.

This much one can easily prove and one can go quite a bit further in this manner (see F-T p.224–225). But in what follows we will assume a little more: that everybody uses the same bid function, which we shall denote by $\beta$. It is strictly increasing, and so differentiable a.e., but we shall also assume that it is everywhere differentiable.

(f) By using first-order conditions, prove that for all $\theta \in \Theta$, the equilibrium bid function must satisfy the equation

$$\beta(\theta)G(\theta)^{n-1} = \int_0^\theta x(n-1)G(x)^{n-2}G'(x)dx.$$ 

To do this problem, first write down what will happen if $i$ makes another bid in $[0, B]$ (other than his equilibrium bid $\beta(\theta_i)$). This is the same as trying to mimic some other valuation $\theta$ (see Revelation Principle below), so that his payoff is

$$\theta_i - \beta(\theta)G(\theta)^{n-1}.$$ 

Write down the first order conditions and set $\theta = \theta_i$ to solve.

(g) See what you get when $G$ is uniformly distributed.

[5] In this question, we study a problem of public goods provision with incomplete information. Suppose that a well is to be dug in a village with $n$ people. It costs $c$ to get the well running. Each person has a valuation for the well. Each person $i$ knows her true valuation $v_i$. But when $i \neq j$, $i$ only knows that $j$’s valuation is drawn independently from a uniform distribution (the same one for everybody) over the range $[\underline{v}, \bar{v}]$. Assume that $0 \leq \underline{v} < c < \bar{v}$.

The following mechanism is to determine whether the well will be dug. All the individuals simultaneously submit sealed envelopes to an arbitrator. The envelopes can either contain a contribution of $c$ or contain nothing (no intermediate contributions are allowed).

If everyone submits nothing, the well is not dug and everyone gets 0. If at least one person submits $c$, then each person $i$ who did contribute $c$ gets $v_i - c$ (no excess contributions are returned), and each person $i$ who contributed nothing gets free use of the well, with payoff $v_i$. 

(a) By comparing the costs and benefits of contributing \( c \), find a symmetric Bayesian Nash equilibrium in which each person contributes if and only if her valuation is above some threshold \( v^* \). Prove that \( v^* \) is the unique solution to the equation

\[
v^* \left( \frac{v^* - \bar{v}}{\bar{v} - v} \right)^{n-1} = c.
\]

(b) The probability that the good is provided is obviously equal to

\[
1 - \left( \frac{v^* - \bar{v}}{\bar{v} - v} \right)^n
\]

(because this is the chance that at least one person has a valuation exceeding \( v^* \)). Using equation (2), prove that

\[
1 - \left( \frac{v^* - \bar{v}}{\bar{v} - v} \right)^n = 1 - c F(v^*),
\]

and using this equation, what can you say about the probability of the public good being provided as the number of individuals \( n \) increases? Does it rise? Fall? Stay the same? Explain your answer.

[6] This is a formalization of the idea of conspicuous consumption. Suppose that Jim’s wealth is either high \( H \) or low \( L \), with \( H > L > 0 \). Jim knows which one it is; the world at large doesn’t. But Jim would like people to think that he has high wealth; this makes him happier. Assume that if people think he has high wealth with probability \( q \), his payoff equals \( q \).

Initially the world has a prior on Jim: it thinks Jim is rich with probability \( p \). Now, suppose that Jim has the option to buy flashy things to signal his wealth. Let \( c \) be the “conspicuous consumption” of these flashy items and suppose that they bring no intrinsic pleasure to Jim. There is a cost, though. Let the cost of \( c \) units be \( c/w \), where \( w \) is Jim’s wealth, which can take on one of two values.

The world observes \( c \) and updates its priors on Jim. Jim’s final net payoff is \( q - c/w \), where \( q \) is the updated posterior and \( w \) is Jim’s private wealth.

(a) Are there separating sequential equilibria in this model?

(b) Are there pooling sequential equilibria in this model?

(c) Are there hybrid equilibria?

(d) Describe what happens if the intuitive criterion is applied to these equilibria.

[7] Here is some more practice along Carlsson-Van Damme and Morris-Shin lines of reasoning. Consider the following two-player game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>( a + \theta, a + \theta )</td>
<td>0, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, 0</td>
<td>( b - \theta, b - \theta )</td>
</tr>
</tbody>
</table>

where \( a \) and \( b \) both lie strictly between 0 and 1, and \( \theta \) is a random variable distributed uniformly on \([-1, 1]\). Use the Carlsson-van Damme / Morris-Shin construction: \( \theta \) is observed with some uniform noise on \([\theta - \epsilon, \theta + \epsilon]\), where \( \epsilon \) is a tiny positive number. The noise is iid across the two players.
[a] Solve the equilibrium strategy of the perturbed game using the techniques studied in class. Find the limit value of the switch point $\theta^*$ as $\epsilon \to 0$ and evaluate this limit relative to the values of $a$ and $b$.

By the way, take special note of this: in the Morris-Shin world, a lot of the argument works because of reasoning like this: player $i$ thinks that player $j$ thinks that player $k$ thinks that ... But in this model there are only two players! Explain why the above sort of reasoning still matters.

[b] Apply the same logic to the game

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>$\theta, \theta$</td>
<td>$\theta, 0$</td>
</tr>
<tr>
<td>D</td>
<td>0, $\theta$</td>
<td>4, 4</td>
</tr>
</tbody>
</table>

where $\theta$ is a random variable on some interval $[\underline{\theta}, \overline{\theta}]$, with $\underline{\theta} < 0$ and $\overline{\theta} > 4$. Is it true that (as $\epsilon > 0$) the critical switch point involves the Pareto-dominant equilibrium being played? Is this in contrast to the game of part [a], and why or why not?

[8] Revelation Principle. Consider a Bayesian game in which $\Omega = T_1 \times \cdots \times T_n$, where $T_i$ is each player’s type set. All parameters — conditional posteriors and payoff functions — are defined exactly as in class.

Suppose that before the play of the game each player can communicate to an impartial mediator by choosing a message $m_i$ in some message space $M_i$. The mediator sees the vector of messages and then makes a private recommendation of action, $a_i$ to each player $i$. The planner makes such recommendations according to a conditional distribution $\mu(a|m)$ over the product action space $A$. The collection $\{M_1, M_2, \ldots, M_n; \mu\}$ is called a mechanism.

The mechanism induces a game in the obvious way. Player $i$ first learns her type, then announces $m_i$, and then, following the recommendation $a_i$, takes some action $b_i \in A_i$. There are therefore two potential ways in which any player can distort the outcome. Explain.

Prove that for any equilibrium of any mechanism, there exists another mechanism with message space $M_i = T_i$ for all $i$, and an equilibrium for that mechanism, such that

(a) Each player announces her true type and takes the recommended action.

(b) For each type realization, the outcome of this equilibrium is the same as the outcome of the earlier equilibrium.

[9] Consider the following game taken from Myerson (1991, page 128): The row player’s type is known, but the column player can be of type $a$ (with probability $\alpha$) or $b$ (with probability $1 - \alpha$). If he is of type $a$, the situation is

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 4</td>
<td>1, 3</td>
</tr>
</tbody>
</table>

while if he is of type $b$, the situation is
(i) Make a note of the equilibria in this game if types were commonly known.

(ii) Use the Revelation Principle to prove that there is no mechanism which implements the equilibrium mapping in (i) as a function of type.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1,3</td>
<td>0,4</td>
</tr>
<tr>
<td>B</td>
<td>0,1</td>
<td>1,2</td>
</tr>
</tbody>
</table>