Econometrics I

Midterm Examination

Fall 2005

Answer Key

Please answer all of the questions and show all of your work. If you think that a question is ambiguous, clearly state how you interpret it before providing an answer. All question parts have equal weight.

1. Consider the linear regression specification

$$y_i = X_i\beta + \varepsilon_i,$$

where ε_i has the following distribution for individual i. With probability π_i the disturbance term is drawn from the uniform distribution defined on $[-a, a]$, where $a > 0$ is an unknown parameter, and with probability $(1 - \pi_i)$ the disturbance is drawn from a mean zero normal distribution with unknown variance σ_u^2. Thus the marginal density of ε_i is

$$f_i(\varepsilon_i) = \pi_i \frac{1}{2a} + (1 - \pi_i)\sigma_u^{-1}\phi(\frac{\varepsilon_i}{\sigma_u}),$$

where ϕ denotes the standard normal probability density function. The unknown parameter vector β is of dimension K.You have access to a random sample consisting of N observations with information on y and X. Assume that the rank of $X'X$ is K.

1. Define the ordinary least squares (OLS) estimator of β, $\hat{\beta}$.

Answer: This is simply

$$\hat{\beta} = (X'X)^{-1}X'y.$$

2. Is $\hat{\beta}$ an unbiased estimator of β? Why?

Answer: Yes, this is unbiased since, conditional on π_i, the disturbance term is i.i.d.with

$$\begin{aligned} E(\varepsilon_i|\pi_i) &= \pi_i \times 0 + (1 - \pi_i) \times 0 = 0 \\ &\quad \text{for all } \pi_i. \end{aligned}$$

Thus $E(\varepsilon_i|X_i) = 0$ for any π_i.

3. If $\pi_i = \pi$ for all i, is $\hat{\beta}$ the best linear unbiased estimator (BLUE) of β? Why?

Answer: Yes, since we know that $\hat{\beta}$ is unbiased for any pattern of π_i, and now we know also that the variance is constant, being given by

$$V(\varepsilon_i) = \pi\frac{(2a)^2}{12} + (1 - \pi)\sigma_u^2. \tag{0.1}$$

Under mean independence and heteroskedasticity, the Gauss-Markov theorem applies and $\hat{\beta}$ is best linear unbiased.

4. If $\pi_i = \pi$ for all i, provide a discussion of whether it would be possible to separately estimate the parameters a, σ_u, and π.

   Answer: The estimator of the variance of ε, given by s^2, will be an unbiased estimator of the quantity given in (0.1), which is a function of all three parameters. Thus from this information alone these three parameters are not estimable. However, by deriving the higher order moments of ε identification may be possible. A natural way to proceed would be to work out the third and fourth population moments, and then determine whether these three moments (second, third, and fourth) were linearly independent functions of π, a, and σ_u^2. If so, one could replace the population moments with the sample analogs and solve for the values of these three parameters. These estimators, if they exist, would be consistent.

5. If π_i varies in the population, is the OLS estimator BLUE? If not, under what conditions would it be possible to find a more efficient linear estimator?

   Answer: No, because now we face a case of heteroskedastic disturbances. We could find a more efficent esitmator if π_i was a function of observable individual characteristics and a finite dimensional parameter vector, such as

   $$\pi_i = \frac{\exp(Z_i\delta)}{1+\exp(Z_i\delta)},$$

   where Z_i is a vector of observable characteristics of i (could be equal to X_i, for example). Could then consider performing OLS in the first stage, and recovering the residuals r_i. Then form the nonlinear regression function

   $$\begin{aligned} r_i^2 &= \pi_i \frac{(2a)^2}{12} + (1-\pi_i)\sigma_u^2 + \xi_i \\ &= \sigma_u^2 + \left[\frac{(2a)^2}{12} - \sigma_u^2\right]\pi_i + \xi_i, \end{aligned}$$

   and estimate the parameters using nonlinear least squares. With consistent estimates of δ (which determines π_i in these parametric assumptions), a, and σ_u^2, we can perform Feasible GLS.

2. Once again, consider the linear regression model

   $$y_i = X_i\beta + \varepsilon_i,$$

   where

   $$E(\varepsilon_i|X_i) = 0, \text{ for all } i.$$

   In the population the variance of ε_i is given by

   $$Var(\varepsilon_i|X_i) = X_i\delta.$$

   We assume that, at the true value of δ, $X_i\delta > 0$ for all i.

   1. If δ is known, define the Generalized Least Squares (GLS) estimator of β, which we will denote by $\tilde{\beta}$, as well as the covariance matrix of $\tilde{\beta}$.

      Answer:

      $$\tilde{\beta} = (X'A^{-1}X)^{-1}X'A^{-1}y,$$

      where A is a diagonal element with $X_i\delta$ as element (i,i). The covariance matrix of the estimator is

      $$Cov(\tilde{\beta}) = (X'A^{-1}X)^{-1}.$$

2. Define the covariance of the OLS estimator of β, .once again under the assumption that δ is known.

   Answer: In this case, with heteroskedasticity, we know that

$$\begin{aligned} Cov(\hat{\beta}) &= E(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1} \\ &= (X'X)^{-1}X'AX'(X'X)^{-1}. \end{aligned}$$

3. Relax the assumption that δ is known. Is it possible to compute a Feasible Generalized Least Squares (FGLS) estimator in this case? If so, describe your estimator. Are there any complications that may arise in forming this estimator?

   Answer: Yes, in principle it is possible. From the OLS residuals we can estimate the linear regression

$$r_i^2 = X_i\delta + \xi_i,$$

   with

$$\hat{\delta} = (X'X)^{-1}X'r^2$$

   being a consistent estimator of δ. Using this estimator we can form $\hat{A}$, with $X_i\hat{\delta}$ in cell $(i.i)$. As in the FGLS estimator of the linear probability model, we may find that the predicted value $X_i\hat{\delta}$ is not positive, so that the estimated $\hat{A}$ is not positive definite. In this case, we couldn't form the FLGS estimator.

4. Modify the assumption on the conditional variance function so that

$$Var(\varepsilon_i|X_i) = X_i\delta_i,$$

   where the (unknown) parameter vector δ_i varies in an unrestricted manner across individuals in the population. Is it possible to form a (consistent) FGLS estimator in this case? If not, is it possible to consistently estimate the covariance matrix of the OLS estimator of β?

   Answer: No, its not possible to consistently estimate a separate parameter vector for each individual i when only one observation is available per individual. It is possible to form a consistent estimator of the covariance matrix of $\hat{\beta}$, however, since we can use the Huber-Eicker-White estimator of the quantity

$$(X'X)^{-1}X'AX(X'X)^{-1}$$

   with diagonal A without any restricitions on the values of the diagonal elements.

3. We saw that if an individual had a Cobb-Douglas utility function

$$U(x_1, x_2) = \alpha \ln(x_1) + (1-\alpha)\ln(x_2),$$

   and if she faced prices p_1 and p_2 and had income Y, then her expenditures on good 1 are given by

$$e_1^* = p_1 x_1^* = \alpha Y.$$

   Assume that you have access to a random sample with information on expenditures on good 1 and income for N individuals, $\{e_{1i}, Y_i\}_{i=1}^N$.

1. Assume that measured expenditures are equal to the sum of true expenditures plus a random variable u that is i.i.d. in the population with mean 0, or

$$e_{1i} = e_{1i}^* + u_i.$$

Define the estimator

$$\hat{\alpha} = \frac{\sum_{i=1}^N (e_{1i} - \overline{e}_1)(Y_i - \overline{Y})}{\sum_{i=1}^N (Y_i - \overline{Y})^2},$$

where $\overline{e}_1$ is the sample mean of measured expenditures and $\overline{Y}$ is the sample mean of incomes. Is $\hat{\alpha}$ an unbiased estimator of α? Define an unbiased estimator of the variance of u, if such an estimator exists.

Answer: First write

$$e_{1i} = \alpha Y_i + u_i.$$

Then $\hat{\alpha}$ is unbiased since u_i is i.i.d. with mean 0, therefore $E(u_i|Y_i) = 0$ for all i. Also because u is i.i.d., it has constant variance σ_u^2, and an unbiased estimator of this variance is

$$s^2 = \frac{\sum_{i=1}^N (e_{1i} - \hat{\alpha} Y_i)^2}{N-1}.$$

2. A different specification of the model posits that there is no measurement error in expenditures (that is, $u_i = 0$ for all i), but that individuals have different tastes. Assume that α is i.i.d. in the population with mean $\overline{\alpha}$ and variance σ_α^2. Define a Best Linear Unbiased estimator for $\overline{\alpha}$, if one exists. Define an unbiased estimator of σ_α^2, if one exists.

Answer: We can write

$$e_{1i} = \overline{\alpha} Y_i + (\alpha_i - \overline{\alpha}) Y_i.$$

Let $\varepsilon_i = (\alpha_i - \overline{\alpha}) Y_i$. Then since α is i.i.d., $E(\varepsilon_i|Y_i) = Y_i E(\alpha_i - \overline{\alpha}) = 0$. So the disturbance term is mean independent of Y. Now

$$\frac{e_{1i}}{Y_i} = \overline{\alpha} + (\alpha_i - \overline{\alpha}),$$

so a regression of e_{1i}/Y_i on '1' (which is simply the mean of e_{1i}/Y_i yields an estimate of $\overline{\alpha}$ that is unbiased and efficient in the sense of the Gauss Markov Theorem (due to the homoskedasticity of $(\alpha_i - \overline{\alpha})$).

3. We maintain the assumptions of Part 3.2 and add an assumption regarding the distibution of α. We assume that α has a power distribution, so the cumulative distribution function of α is given by

$$F(\alpha) = \alpha^\delta, \ \delta > 0, \ \alpha \in [0,1].$$

Write down the log likelihood function for the sample and find the maximum likelihood estimator of the parameter δ.

Answer: Now $e_{1i}/Y_i = \alpha_i$. Then the likelihood of e_{1i}/Y_i is $\delta[e_{1i}/Y_i]^{\delta-1}$, so the log likelihood function of the sample is given by

$$L = N \ln \delta + (\delta - 1) \sum_{i=1}^N \ln(e_{1i}/Y_i).$$

which implies

$$\hat{\delta} = \frac{N}{\sum_{i=1}^N \ln(e_{1i}/Y_i)}.$$

4. Maintain the assumptions underlying Parts 3.2 and 3.3. Say that the government has imposed a rationing scheme on good 1. While the price of the good remains the same, no individual can spend more than 100 dollars on the good. Thus the value of expenditures in the sample to which you have access can never exceed 100. (In the population, some proportion of individuals will spend exactly 100 dollars on the good while there will be a continuous distribution of expenditures on the interval [0,100).) Write down the log likelihood for the sample in this case and define the maximum likelihood estimator of δ.

   Answer: For simplicity let's assume that $p_1 = 1$. Then an individual with income Y will consume 100 units of the good when

$$\begin{aligned} e_{1i} &\geq 100 \\ &\Rightarrow \alpha Y > 100 \\ &\Rightarrow \alpha > \frac{100}{Y}. \end{aligned}$$

   Then the probability that an individual with income Y hits the rationing constraint is

$$p(\alpha > \frac{100}{Y}) = 1 - (100/Y)^{\delta}.$$

   Now the likelihood of a given expenditure level given not at the constraint is

$$p(e|Y, e < 100) = \frac{\delta(e_{1i}/Y_i)^{\delta-1}}{(100/Y_i)^{\delta}},$$

   so that the joint likelihood is

$$p(e, e < 100|Y) = \delta(e_{1i}/Y_i)^{\delta-1}.$$

   Thus the sample log likelihood is

$$\begin{aligned} \ln L &= \sum_{i=1}^{N}\{\chi[e_{1i} = 100]\ln(1 - (100/Y_i)^{\delta}) \\ +\chi[e_{1i} &< 100](\ln\delta + (\delta - 1)\ln(e_{1i}/Y_i))\}. \end{aligned}$$

   The m.l.e. of δ is value that sets the derivative of this function with respect to δ equal to 0. In particular,

$$\begin{aligned} 0 &= \sum_{i=1}^{N}\{\chi[e_{1i} = 100](1 - (100/Y_i)^{\hat{\delta}})^{-1}(-(100/Y_i)^{\hat{\delta}})\ln(100/Y_i) \\ +\chi[e_{1i} &< 100](\hat{\delta}^{-1} + \ln(e_{1i}/Y_i))\}. \end{aligned}$$