Long Swings in Currency Markets: Imperfect Knowledge and I(2) Trends*

Roman Frydman,[1] Michael D. Goldberg,[2] Søren Johansen,[3] and Katarina Juselius[4]

April 6, 2012

Abstract

Using multivariate unit-root tests, the paper finds that real and nominal exchange rates are more persistent than univariate unit-root studies suggest. It shows that an imperfect knowledge economics model of currency swings and risk is able to account for this greater persistence, even though it recognizes that market participants revise their forecasting strategies in non-routine ways, and thus, does not impose a fixed probability distribution on how these strategies unfold over time. The model provides the micro-foundations for a persistent segmented trends process, thereby explaining the tendency for asset prices to move in one direction for long stretches of time. The paper shows that if one were prepared to assume away the importance of non-routine revisions of forecasting strategies and impose a Markov chain on such change, the Engel and Hamilton (1990) segmented-trends model of long swings would result.

*The authors are grateful to the Institute for New Economic Thinking (INET) for supporting the research on which this paper is based.

[1]New York University and INET, roman.frydman@nyu.edu
[2]University of New Hampshire and INET, michael.goldberg@unh.edu
[3]University of Copenhagen and INET, soren.johansen@econ.ku.dk
[4]University of Copenhagen and INET, katarina.juselius@econ.ku.dk

1 Introduction

The global financial crisis that began in 2007 has led to much discussion about the inability of macroeconomics and finance theory based on the rational expectations hypothesis (REH) to account for outcomes in financial markets and the broader economy. Much attention has been devoted to asset prices' tendency to undergo wide swings around estimates of commonly-used benchmark values. Nowhere is this tendency more apparent than in currency markets. Figure 1, which plots the German mark-US dollar nominal exchange rate and a measure of its purchasing power parity (PPP) value, shows that these markets are characterized by stretches of time, sometimes lasting years, during which the exchange rate moves persistently away from or back toward the PPP benchmark.

Stochastic versions of Dornbusch's (1976) seminal sticky-price model, including new open-economy macroeconomics extensions (e.g., Obstfeld and Rogoff, 1998), were developed to account for such fluctuations. In these REH models, the exchange rate tends to move back toward PPP at every point in time. To produce currency movements away from this benchmark, therefore, the models rely on monetary or other shocks and nominal rigidities, which imply that the impact of shocks on the level of the real exchange rate is persistent over many periods. The models generally imply a persistent random walk or unit-root ($I(1)$) process for the log of the nominal exchange rate and, under a plausible degree of price sluggishness, a much less persistent $I(0)$ log real exchange rate. Dozens of studies using univariate unit-root tests find support for the $I(1)$ characterization of nominal rates. But they report that real rates are near-$I(1)$ and almost as persistent as nominal rates, indicating that swings away from PPP, such as those in figure 1, tend to be much longer lasting than those implied by conventional sticky-price theory.[1]

In this paper, we present evidence that nominal and real exchange rates are persistent not only in terms of levels but also in first differences, indicating that REH models' difficulties in accounting for persistence and long swings in currency markets are greater than most empirical studies suggest. We also show that Frydman's and Goldberg's (FG) (2007, 2013a) imperfect knowledge economics (IKE) model of currency swings and risk can account for the observed persistence of nominal *and* real exchange rates.

Although international macroeconomists often associate nominal exchange-rate fluctuations with a driftless random walk, there is evidence in the

[1] A near-$I(1)$ variable has one large root with modulus close to one. For review articles on the near-$I(1)$ persistence of real exchange rates, see Rogoff (1996) and Taylor and Taylor (2004).

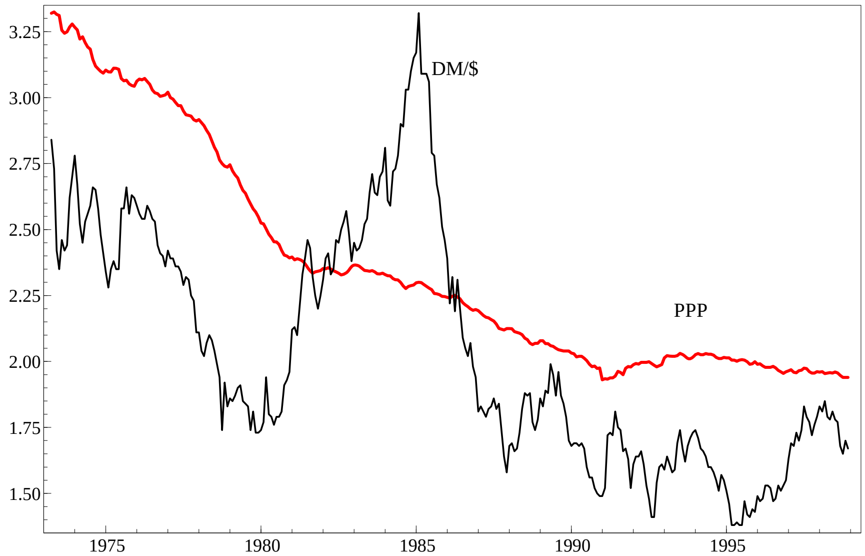

Figure 1: The DM/$ exchange rate and PPP: 1973-1998

literature that these fluctuations are more persistent than $I(1)$ characterizations suggest. Engel and Hamilton (1990) estimate a segmented-trends model in which exchange rate movements are driven by a drift that switches between preset positive and negative values according to a Markov chain. They find that switches occur infrequently, implying that the *change* in the exchange rate is persistent and tends to take on the same sign for long stretches of time.[2] By contrast, a random walk displays no such persistence. Consequently, Engel and Hamilton (1990) reject the random walk model in favor of a Markov-switching process that produces both upswings and downswings that tend to last much longer.[3] Real exchange rates' nearly parallel movements with nominal rates suggest that deviations from PPP also display greater persistence than a random walk, let alone the $I(0)$ processes implied by REH sticky-price models.

In the first part of the paper, we present evidence of such persistence in real exchange rates. Using the $I(2)$ framework (see Johansen, 1997, 2006) to estimate a CVAR for the German mark-US dollar exchange rate,

[2] A switch in the exchange-rate regime can be thought of as a shock to the drift, which is persistent when switches are infrequent.

[3] Other studies that reject the random walk in favor of a persistent Markov-switching process include Engel (1994) and Cheung and Erlandsson (2005). See also Hsieh (1989), which finds non-linear dependence in exchange rate movements.

goods prices, and interest rates, we reject $I(1)$-type characterizations for the real and nominal exchange rate at high significance levels in favor of $I(2)$-type characterizations.[4] $I(2)$-type processes are persistent in terms of levels and changes, implying that exchange-rate movements tend to take on the same sign over many periods. Consequently, like Engel and Hamilton's (1990) segmented-trends process, $I(2)$-type processes have a tendency to display longer-lasting swings than $I(1)$-type processes.[5]

An $I(2)$-type characterization and a persistent segmented-trends specification are compatible ways to model empirically persistence and long swings in nominal and real exchange rates. Engel and Hamilton (1990), Kaminsky (1993), and others appeal to shifts in monetary policy or bubbles to provide theoretical underpinning for their segmented-trends specifications of the nominal exchange rate. But, although shifts in monetary policy can generate long swings in nominal rates in the context of an REH model, this is not the case with real rates. Bubble models attempt to explain long swings in asset prices away from benchmark values by according market participants' expectations an autonomous role. But, in many bubble models, the asset price jumps immediately back to the fundamental value when a bubble bursts, which is inconsistent with the pronounced persistence that Engel and Hamilton (1990) and others find during *both* upswings and downswings.[6]

Researchers have also portrayed an active role for expectations with algorithmic learning models.[7] They have shown that learning and bubble models can generate more persistence than conventional REH models, with some finding price deviations from benchmark values that display near-$I(1)$ persistence.[8] A key outstanding question is how to ac-

[4] The persistence displayed by a near-$I(1)$ process and that of an $I(1)$ process are nearly indistinguishable in all but very long samples. This is also the case with near-$I(2)$ and $I(2)$ processes. (A near-$I(2)$ variable has two large roots with modulus close to one.) We thus refer to near-$I(1)$ and $I(1)$ processes on the one hand, and near-$I(2)$ and $I(2)$ processes on the other hand, as $I(1)$-type and $I(2)$-type, respectively.

[5] This evidence of I(2)-type persistence is consistent with the results of Evans's (1986) runs tests, which indicate that the real exchange rate is too persistent to be explained by a random walk.

[6] See Brunnermeier (2001) and references therein for bubble models based on both REH and behavioral considerations. Johansen and Lange (2011) show that Blanchard and Watson's (1982) REH bubble model (s) implies at most near-$I(1)$ persistence owing to its bust dynamics. Some behavioral bubble models – for example, the seminal noise-trader model of Frankel and Froot (1995) – imply persistence during both upswings and downswings.

[7] See, for example, Adam and Marcet (2011) for stock prices and Lewis (1989) and Mark (2009) for exchange rates.

[8] See, for example, Adam et al. (2010) and DeGrauwe and Grimaldi (2006). See also Mark (2009), which estimates a least squares learning model and finds that swings are more persistent with learning than they are with REH.

count for the observed persistence of both nominal and real exchange rates, whether characterized as persistent segmented-trends or $I(2)$-type processes.

In the second part of the paper, we show that the FG model provides a theoretical foundation for both characterizations. This portfolio-balance model is part of the literature that accords market participants' expectations an autonomous role in driving asset price fluctuations. But, unlike bubble and learning models, the FG model relies on both the fundamental variables emphasized by REH models and the psychological considerations stressed by behaviroral models in portraying forecasting behavior.[9] Strikingly, the assumption of sluggish goods prices is not needed to explain long swings away from and back towards PPP: the model generates such fluctuations even if goods prices are fully flexible.

We show that the FG model provides micro-foundations for a segmented-trends process for both the real and nominal exchange rate. The time-varying drift depends on trends in fundamental variables and participants' forecasting strategies, which map these trends into their exchange rate predictions. The drift moves when participants revise their strategies or monetary policy or other features of the social context change. But, unlike learning models, IKE does not impose fixed rules that govern when or how strategies or policy are altered. Instead, it constrains such change with qualitative and contingent conditions that imply long stretches of time in which the drift tends to take on values of the same algebraic sign; thus, real and nominal exchange rates move persistently in one direction or the other.[10] We also show that if we were to constrain forecasting behavior and policy to two preset regimes and impose a Markov chain on switches, we would obtain Engel and Hamilton's (1990) segmented-trends specification.

It would appear that without a fixed probability rule, the FG model has no implications for the order of integration or other properties of time-series data. However, we propose a way to derive such implications from the model. We show that the model implies an $I(2)$-type process for the real and nominal exchange rate without restricting with a probability rule when or how strategies or policy are revised. We also show that the

[9] By allowing for non-rotuine change, the FG model is able to recognize the importance of both fundamental and psychological considerations without presuming that market participants systematically forego profit opportunities. See Frydman and Goldberg (2013b).

[10] Qualitative conditions allow for a range of outcomes conditional on movements of the causal variables. Qualitative conditions are contingent if they do not hold at every point in time and no fixed probability distribution describes when they begin or cease to be relevant. For a discussion of such conditions and how they place sufficient structure on the analysis, see section 3.3.

model implies a persistent drift in the real and nominal exchange rate process that is small relative to the variance of the differenced series.

This second result helps to explain why we and others who use multivariate unit root tests find $I(2)$-type persistence in exchange rate data,[11] while those relying on univariate tests generally fail to do so. Jensen and Juselius (2012) find that univariate unit-root tests have low power to detect a second large root in data that involve a small but persistent drift, whereas multivariate tests do not. Our empirical analysis suggests that exchange rate data are characterized by such a drift.

The remainder of the paper is structured as follows. In section 2, we present the results of our CVAR-based unit root tests. We leave some of the details and underlying results of our analysis to an appendix. Section 3 provides a sketch of the FG model, while section 4 shows how it provides a theoretical rationale for a segmented-trends process with long-lasting upswings and downswings. Section 5 shows how the implication of near-$I(2)$ trends can be derived from this IKE model. We again leave some of the details of this analysis to an appendix. Section 6 discusses broader implications of our analysis.

2 Exchange Rate Persistence: $I(1)$- or $I(2)$-type?

The pronounced persistence in exchange rates and other macroeconomic time series suggests the presence of unit roots in the data. Much of the testing for $I(1)$- and $I(2)$-type persistence has relied on univariate Dickey-Fuller (DF) type models. These and other univariate tests are well known to have low power to discriminate between the null of a unit-root and stationary, but near-unit-root alternatives.[12] Here, we are concerned with a different problem: the low power of univariate tests to detect the presence of a second large root in data that are characterized by a persistent drift that is small relative to the variance of first differences, which we shall refer to as a small "signal-to-noise ratio."

2.1 Persistence and the Signal-to-Noise Ratio

To illustrate the problem, we examine the persistence of several simulated real exchange rate series generated from the following process:

$$\Delta q_t = \mu_t + \varepsilon_t \qquad \varepsilon_t \sim N(0,1) \tag{1}$$

[11] See Johansen (1992), Juselius (1994, 2009), Kongsted (2003, 2005), Kongsted and Nielsen (2004), Bacchiocchi and Fanelli (2005), and Johansen et al. (2010). Adam and Marcet use what is essentially an I(2) specification of stock returns to characterize learning in their model. For an alternative to new Keynesian models that replaces the short-run Phillip's curve with an $I(2)$ specification of nominal income, see Farmer (2013).

[12] See Sarno and Taylor (2002) and references therein for a review of this literature.

$$\mu_t = \rho\mu_{t-1} + v_t, \quad v_t \sim N(0, \sigma_v^2) \tag{2}$$

where we consider different specifications of the drift term, $\rho = \{0.0, 0.95, 1.0\}$, corresponding to an $I(1)$, near-$I(2)$, and $I(2)$ real exchange rate, respectively, and $\sigma_v/\sigma_\varepsilon = \{1.00, 0.15\}$, corresponding to a large and small signal-to-noise ratio, respectively.[13] The length of the simulated sample is set to 500.

Figure 2 illustrates in the upper panel the swings that are produced by a random walk series ($\rho = 0$) and a near-$I(2)$ process ($\rho = 0.95$) and, in the lower panel, those produced by the same near-$I(2)$ process ($\rho = 0.95$) and an $I(2)$ process ($\rho = 1.00$) in the lower panel.[14] All three series have been generated from the same random shocks ε_t and v_t to exclude any influence from ordinary sample variation. The signal-to-noise ratio is 0.15 for both processes. The upper panel shows that both series exhibit persistent swings, although this behavior is much more pronounced for the near-$I(2)$ process compared to the random walk. The two processes in the lower panel exhibit swings that are similarly persistent, but the swings of the $I(2)$ series tend to drift off much more, signifying the absence of significant mean reversion in its changes. This becomes apparent at the end of the sample.

When the signal-to-noise ratio is small, differenced $I(2)$-type processes will display what looks like strong mean-reversion and thus little persistence. The small, but persistent drift term of $I(2)$-type processes tends to be hidden by the high volatility of first differences, which often look similar to a differenced random walk.

By smoothing out the highly volatile short-term movements, a moving average can provide a first rough indication of a persistent drift in the data, as shown in Figure 3. The upper panel displays the differenced near-$I(2)$ series from Figure 2 and its 12-period moving average, together with the drift term, μ_t. The figure shows that the moving average component co-moves closely with the drift term, displaying persistent swings. To be sure, a moving average is a time dependent process by design, so that a moving average of a purely white noise process will tend to show some indication of swings. Such behavior can be seen in the middle panel, which displays a differenced random walk (generated from (1) with $\mu_t = 0$) and its 12-period moving average. As expected, the moving average exhibits swings despite the absence of drift in the data. But, these swings occur within fairly narrow bands around the mean, in contrast to the much more pronounced swings of the moving

[13] For simplicity, we ignore lagged differences in (1).

[14] The difference in appearance of the same near-$I(2)$ process in the upper and lower panels of figure 2 arises because the range of variation of the near-$I(2)$ process is 50 compared to 220 for the $I(2)$.

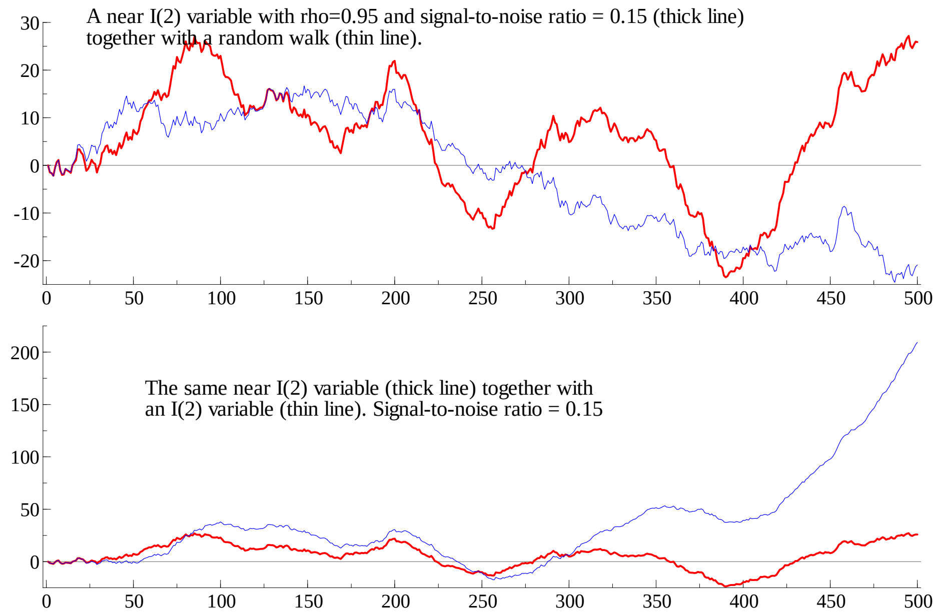

Figure 2: The graph of a near-$I(2)$ variable together with a random walk (upper panel) and the same near-$I(2)$ variable together with an $I(2)$ variable

average for the differenced near-$I(2)$ series.

To illustrate the difference between a small and a large signal-to-noise ratio, the lower panel displays the graph of a differenced near-$I(2)$ series that was generated with a large $\sigma_v/\sigma_\varepsilon$. The persistence of the drift in this series is identical to that in the near-I(2) series in the upper panel with a small $\sigma_v/\sigma_\varepsilon$; both were generated with a $\rho = 0.95$. As such, the drifts driving movements in the two panels differ only in terms of the magnitude of their shocks. A comparison of the upper and lower panels shows that this difference is striking: when the shocks are large, no moving average is needed to see the persistent drift in the data.

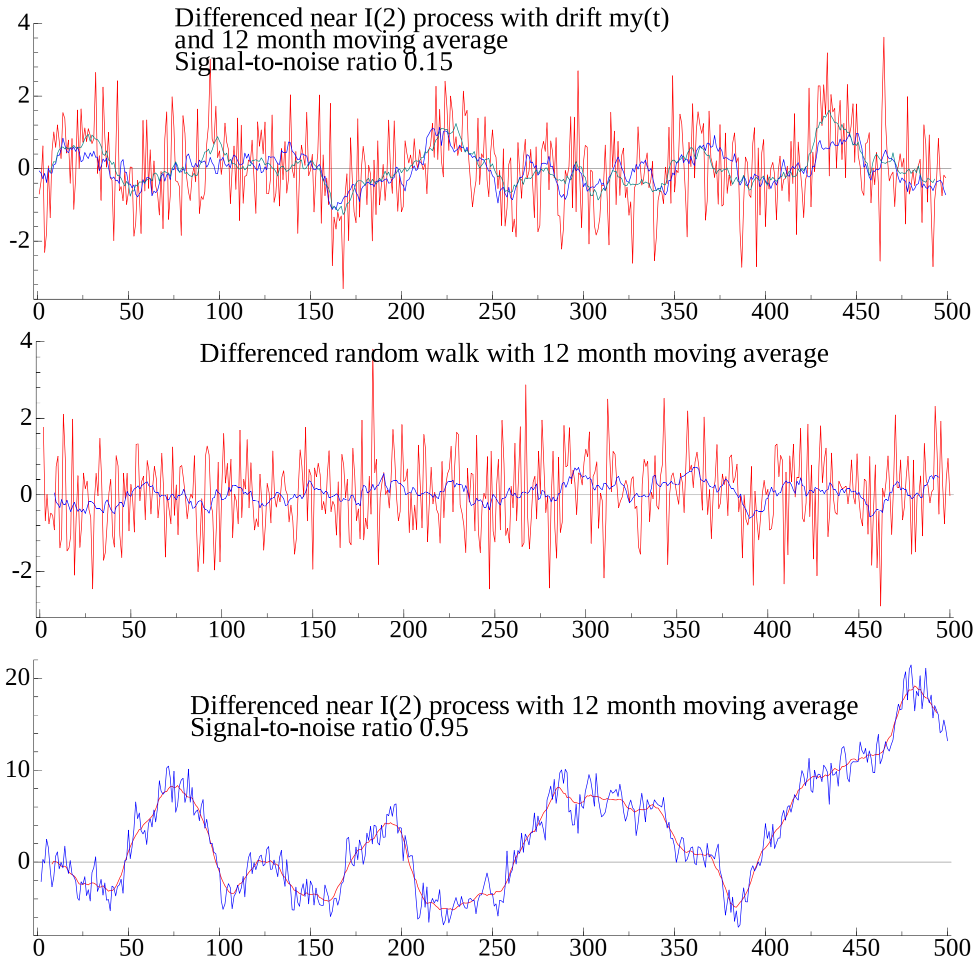

Figure 3: The graphs of Δq_t together with a 12 months MA when $\rho = 0.95$ (upper panel), $\rho = 0$ (middle panel) and $\rho = 1$ (lower panel)

2.2 Univariate Tests

Although the moving average in the upper panel of Figure 3 shows some indication of the large second root present in the near-$I(2)$ series, the power of univariate unit root tests to detect it is very low. This is because the estimated error term in a univariate Dickey Fuller type model is an average of ε_t and v_t. With a small signal-to-noise ratio, ε_t will completely dominate v_t and the small but persistent drift that is associated with the second large root becomes hard to detect.

Table 2 presents the results of testing the null of a unit root in the levels ($\gamma_1 = 0$) and first differences ($\gamma_2 = 0$) of the simulated series using

Table 1: Testing the order of integration with a Dickey-Fuller test

DF tests of	$q_t \sim I(2),\ \rho = 1.0$		$q_t \sim nearI(2),\ \rho = 0.95$		$q_t \sim I(1),\ \rho = 0.0$	
	$\hat{\gamma}_i$	$\tau - ratio$	$\hat{\gamma}_i$	$\tau - ratio$	$\hat{\gamma}_i$	$\tau - ratio$
$\sigma_v/\sigma_\varepsilon$	0.15		0.15		-	
$I(1) : \gamma_1 = 0$	0.007	5.57	-0.004	-0.93	-0.0008	-0.23
$I(2) : \gamma_2 = 0$	-0.31	-6.48	-0.72	-12.8	-0.96	-21.3
$\sigma_v/\sigma_\varepsilon$	1.0		1.0			
$I(1) : \gamma_1 = 0$	0.0006	1.96	0.0006	1.94		
$I(2) : \gamma_2 = 0$	-0.021	-1.90	-0.031	-2.74		

Dickey-Fuller regressions.[15] The results in the upper part of the table are for the simulated $I(2)$, near $I(2)$, and $I(1)$ series that are displayed in Figure 2, which are based on a signal-to-noise ratio of 0.15, whereas those in the lower part are for the $I(2)$-type processes with a signal-to-noise ratio of 1.0. We find that the null of a second unit root is strongly rejected for both the $I(2)$ and near-$I(2)$ cases when $\sigma_v/\sigma_\varepsilon$ is small, whereas it cannot be rejected when $\sigma_v/\sigma_\varepsilon$ is large.[16]

By contrast, Jensen and Juselius (2012) shows that CVAR-based multivariate tests have adequate power to discriminate between $I(1)$- and $I(2)$-type persistence with both small and large signal-to-noise ratios, because they allow for more than one error process.

2.3 Multivariate Unit Root Tests of Exchange Rate Data

Figure 4 displays the graphs of the log difference of the monthly U.S. dollar-German mark real and nominal exchange rates. The graphs look almost identical, showing that the nominal and real \$/Dmk rates move in nearly-parallel fashion. The graphs also illustrate that the series are highly volatile and display what looks like little if any persistence. However, their 12 month moving averages suggest a small but persistent drift term beneath the noisy changes, that is, the data are characterized by a small signal-to-noise ratio. As we have seen, univariate DF tests are unlikely to detect a second large root in the data even if one is present. We thus test for a double unit root using the multivariate trace test within the cointegrated VAR model.

Our results are based on estimating an unrestricted VAR with two lags, which is conveniently formulated in acceleration rates, changes and

[15] The regressions for levels and first differences were $\Delta q_t = \gamma_1 q_{t-1} + \gamma_2 \Delta q_{t-1} + \gamma_0 + \varepsilon_t$ and $\Delta^2 q_t = \gamma_2 \Delta q_{t-1} + \varepsilon_t$, respectively.

[16] While this conclusion is based on just one realization for all series, Juselius et al (2012) obatains similar results in a much more comprehensive simulation analysis.

levels:

$$\Delta^2 x_t = \Gamma \Delta x_{t-1} + \Pi x_{t-2} + \mu_0 + \mu_1 t + \varepsilon_t, \;\; t = 1975{:}6, ..., 1998{:}12 \qquad (3)$$

where $x_t' = [p_{1,t},\ p_{2,t},\ s_{12,t},\ b_{1,t},\ b_{2,t}]$, p and s denote the log levels of consumer prices and the nominal dollar-mark exchange rate (the average over the month), respectively, b is the long-term (10-year) bond rate, and the subscripts 1 and 2 denote U.S. and Germany, respectively.[17]

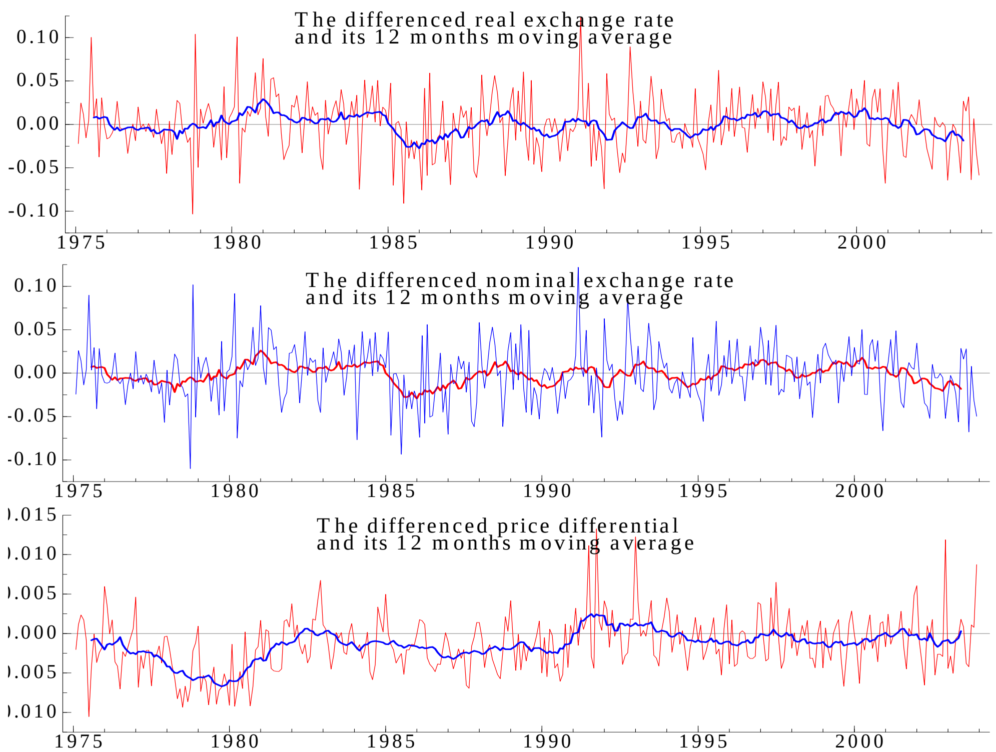

Figure 4: The graph of the differenced real exchange rate (upper panel) and the nominal exchange rate (lower panel) together with their 12 months moving averages

As a first step, we use the multivariate trace test to determine the cointegration rank and the number of $I(2)$ versus $I(1)$ trends among the

[17]The matrices $\{\Gamma, \Pi\}$ are variation free, but μ_0 and μ_1 are restricted to exclude the possibility of quadratic trends in the data. We included several dummy variables in the model to account for the German reunification and the excise taxes that were levied during this period as explained in Johansen et al. (2010) and Juselius (2010) where more detailed results can be found. All series are from the IMF's *International Financial Statistics*.

common stochastic trends. The details of this analyses and the results are provided in appendix A. To summarize, the $I(2)$ trace tests and the number of estimated large characteristic roots strongly support the case for rank two with three common stochastic trends, of which two are order 2 and one is order 1. We conclude that the vector process x_t is $I(2)$.

As a second step, we determine the order of integration of individual variables by testing whether a specific variable is a unit vector in the space spanned by the cointegration relations (see Juselius [2006, Chapter 18] for further details). The null hypothesis is that the variable is at most $I(1)$ against the alternative that it is $I(2)$. Table 2 reports the results of these tests for the individual series in the system, as well as for some relevant transformations of these series. The results show that, except for the German bond rate, all null hypotheses were strongly rejected, indicating that nominal and real exchange rates, the price differential, and the bond rate differential exhibit sufficiently pronounced persistence to reject the $I(1)$ hypothesis in favor of the $I(2)$-type characterization.[18]

These results contrast sharply with those of the dozens of studies that employ univariate DF-type tests, which generally show no evidence of a unit root in the differenced series. As the simulated results above demonstrated, a small signal-to-noise ratio can explain this contrast.

3 An IKE Model of Currency Swings and Risk

In order to show that the FG model can account for the $I(2)$ trends in the data, we first provide a sketch of the model, focusing on IKE's approach to according market participants' expectations an autonomous role in driving outcomes. We highlight how the use of qualitative and contingent conditions to restrict revisions of forecasting strategies enables the model to portray the importance of both fundamental and psychological considerations in this process. We show that the tendency in the model for both the nominal and real exchange rate to undergo long swings depends in part on market participants' tendency to revise their forecasting strategies in moderate ways, a qualitative and contingent regularity that was emphasized by Keynes (1936) and that is consistent with empirical findings in behavioral economics. We use this result in the next two sections in showing that the FG model is able to account for the observed persistence of currency fluctuations.

[18] The inability to reject the I(1) hypothesis for the German bond rate with a p-value of 0.20 indicates that the German bond rate has moved in a slightly less persistent manner than the other variables. See Juselius (2011) for more discussion of this result.

Table 2: Testing hypotheses of I(1) versus I(2)

		$p_{1,t}$	$p_{2,t}$	$s_{12,t}$	$b_{1,t}$	$b_{2,t}$	t_{91}	t	$\chi^2(v)$	$p-val$
Is the price differential $I(1)$?										
$\mathcal{H}_1$	β_1'	1.0	-1.0	-	-	-	-	-	64.09 (4)	0.00
Is the nominal exchange rate $I(1)$?										
$\mathcal{H}_2$	β_1'	-	-	1.0	-	-	-	-	23.6 (4)	0.00
Is the US trend-adjusted price $I(1)$?										
$\mathcal{H}_3$	β_1'	1.0	-	-	-	-	*	*	39.1 (3)	0.00
Is the German trend-adjusted price $I(1)$?										
$\mathcal{H}_4$	β_1'	-	1.0	-	-	-	*	*	48.02 (3)	0.00
Is the bond rate differential $I(1)$?										
$\mathcal{H}_8$	β_1'	-	-	-	1.0	-1.0	-	-	11.2 (4)	0.02
Is the US bond rate $I(1)$?										
$\mathcal{H}_9$	β_1'	-	-	-	1.0	-	-	-	12.4 (4)	0.01
Is the German bond rate $I(1)$?										
$\mathcal{H}_{10}$	β_1'	-	-	-	-	1.0	-	-	5.5(4)	0.24
Is the real exchange rate $I(1)$?										
$\mathcal{H}_{11}$	β_1'	1.0	-1.0	-1.0	-	-	-	-	10.4 (4)	0.03

3.1 The Model's Structure

The FG model's structure at each point in time is no different from that of standard macroeconomics and finance models. It consists of representations of individuals' preferences, forecasting behavior, constraints, and decision rule, as well as an aggregation rule and specifications of the processes governing informational variables. Where it differs is in how it characterizes revisions of market participants forecasting strategies and changes in the policy environment.

3.1.1 Foreign Exchange Market

As is the case in traditional portfolio-balance models,[19] individuals are assumed to decide in each period how much of their non-monetary wealth they want to hold in domestic and foreign bonds, in order to maximize the expected utility of next-period's wealth. The FG model relies on endogenous prospect theory and the assumption of loss aversion to characterize individuals' decision rule and preferences instead of expected utility theory and the assumption of risk aversion.[20] This alternative

[19] See, for example, Kouri (1976) and Dornbusch (1983).

[20] Endogenous prospect theory, which is developed in Frydman and Goldberg (2007), provides a way to represent the experimental findings of Kahneman and Tversky (1979) and others in a world of imperfect knowledge. It also addresses problems that arise when applying prospect theory to model asset markets.

characterization of preferences leads to the following equilibrium condition for the foreign exchange market:

$$\widehat{s}_{t|t+1} - s_t = i_t^r + \widehat{up}_{t|t+1} \tag{4}$$

where s_t denotes the spot exchange rate, $\widehat{s}_{t|t+1}$ represents the aggregate of market participants' point forecasts of s_{t+1} at time t, conditional on their information sets and forecasting strategies, i_t^r denotes the relative (domestic minus foreign) nominal interest rate, and $\widehat{up}_{t|t+1}$ is a time-varying aggregate "uncertainty premium" on foreign exchange, which arises because individuals are endogenously loss averse and must expect an excess return before they are willing to hold a long or short position.

In modeling the aggregate premium, Frydman and Goldberg (2007, 2013a) relate the risk of capital loss from an open position not to the variance of foreign exchange returns, but to the gap between the exchange rate and its PPP benchmark value,

$$\widehat{up}_{t|t+1} = \sigma \left(s_t - \hat{s}_t^{\text{PPP}} \right) \tag{5}$$

where $\hat{s}_t^{\text{PPP}} = p_t^r + \hat{q}_t^{\text{PPP}}$, p_t^r denotes the relative level of domestic and foreign goods prices, and $\hat{q}_t^{\text{PPP}}$ is an aggregate of participants' assessments of the PPP real exchange rate. The specification in (5) captures the idea that participants who hold long (short) positions in an overvalued (undervalued) currency raise their forecasts of the potential losses from speculating, and thus the premium that they demand to stay in the market, as the gap from the benchmark and the overvaluation (undervaluation) grows.[21]

3.1.2 Money and Goods Markets

To represent behavior in the money and goods markets, the FG model follows Dornbusch (1976) and Frankel (1979):

$$m_t^r = p_t^r + \phi y_t^r - \lambda i_t^r \tag{6}$$

$$\Delta p_{t+1}^r = \delta \left[\alpha \left(s_t - p_t^r - q^{\text{PPP}} \right) - \eta \left(i_t^r - \hat{\pi}_t^r \right) \right] + \Delta \bar{p}_{t+1}^r \tag{7}$$

where m_t^r and y_t^r denote relative money supplies and income levels, respectively, q^{PPP} is the PPP real exchange rate, π_t^r is the aggregate of participants' assessments concerning the relative inflation rate that would prevail if good markets cleared, which is assumed to be exogenous to the model, and an overbar denotes a goods-market-clearing level.

[21] Relating assessments of risk in financial markets to the gap between the asset price and its historical benchmark can be traced to Keynes (1936) and Tobin (1958). Frydman and Goldberg (2007) and Stillwagon et al. (2011) provide evidence of an equlibrium gap effect in currency markets.

The FG model implies that longs swings in the real exchange rate stem from imperfect knowledge rather from sticky goods prices. To highlight this result and simplify our analysis, we will focus on the implications of assuming flexible goods prices, which sets excess demand (the term in square brackets in (7)) to zero at every point in time:[22]

$$\bar{s}_t - \bar{p}_t^r - q^{\text{PPP}} = \frac{\eta}{\alpha}\left(\overline{i^r}_t - \hat{\pi}_t^r\right) \tag{8}$$

3.1.3 Autonomous Expectations

As in REH models, the FG model supposes that individuals' point forecasts of s_{t+1} depend on macroeconomic fundamentals:

$$\widehat{s}_{t|t+1} = \beta_t x_t + \rho s_t \tag{9}$$

where the vector x_t represents the union of informational variables that individuals use in forming their forecasts and β_t and $0 < \rho < 1$ are weighted averages of the parameters that they attach to these variables.[23]

If we were to restrict the specification in (9) so that x_t included only those variables contained in the non-expectational components of the model, that is, to m_t^r, y_t^r, q^{PPP}, and their rates of change, and β_t and ρ were constrained to be particular functions of the non-expectational parameters of the model, we could obtain the REH representation:

$$\widehat{s}_{t|t+1}^{\text{RE}} = s_t + \theta\left[\bar{s}_t^{\text{PPP}} - s_t\right] + \hat{\pi} \tag{10}$$

where $\bar{s}_t^{\text{PPP}} = \bar{p}_t^r + q^{\text{PPP}}$, $\bar{p}_t^r = m_t^r - \phi y_t^r + \lambda\hat{\pi}^r$, $\hat{\pi}^r = \mu^m - \phi\mu^y$, θ is one minus the stable root of the system, and money and income are assumed to follow random walks with constant drift,[24]

$$m_t^r = \mu^m + m_{t-1}^r + \varepsilon_t^m \text{ and } y_t^r = \mu^y + y_{t-1}^r + \varepsilon_t^y \tag{11}$$

However, the autonomous representation in (9) does not impose these restrictions. It recognizes that some market participants may very well base their exchange rate forecasts solely on the PPP exchange rate, but this variable is merely one of many fundamental factors on which participants might reasonably rely in forming their forecasts. The model,

[22] For the implications of the model under sticky goods prices, see Frydman and Goldberg (2007).

[23] The FG model uses wealth shares as aggregation weights. The vector x_t includes expected one-period-ahead changes in informational variables. Frydman and Goldberg (2007, 2013a) simplify by setting these expected changes to constants. A more general IKE model would impose qualitative and contingent contraints on how they unfold over time.

[24] We consider the implications of shifts in μ^m and μ^y in section 4.

therefore, does not constrain $\widehat{s}_{t|t+1}$ to imply necessarily the prediction that s_t will move back toward PPP in the coming time period.

Whether the exchange rate moves back toward or farther away from PPP between t and $t+1$ depends not on the market's forecast at t, but on how $\widehat{s}_{t|t+1}$ moves over time. The main driver of this aggregate forecast are the movements in its exogenous component:

$$\widehat{s}^{x}_{t|t+1} = \beta_t x_t \tag{12}$$

where the informational variables in x_t are assumed to be exogenous. The t-subscript on β_t reflects the model's recognition that participants in financial markets revise their forecasting strategies, at least intermittently, over time.

3.2 Momentary Equilibrium With Flexible Prices

Before we sketch how the model portrays such change, it is useful to express the model's implications for the nominal and real exchange rate in terms of $\widehat{s}^{x}_{t|t+1}$, assuming that the goods markets clear at every point in time. Under REH, goods market equilibrium in (8) is associated with PPP and real interest rate parity–$\overline{i^r}_t = \hat{\pi}^r_t$. However, once the REH link between $\widehat{s}_{t|t+1}$ and PPP is broken, goods market equilibrium is characterized by neither PPP nor real interest rate parity.

Goods-market-equilibrium values can be expressed as follows:

$$s_t = s^{\mathrm{RE}}_t + \frac{\eta + \alpha\lambda}{G}\left(\widehat{s}^{x}_{t|t+1} - \widehat{s}^{\mathrm{RE}x}_{t|t+1}\right) \tag{13}$$

$$q_t = q^{\mathrm{PPP}} + \frac{\eta}{G}\left(\widehat{s}^{x}_{t|t+1} - \widehat{s}^{\mathrm{RE}x}_{t|t+1}\right) \tag{14}$$

where $\widehat{s}^{\mathrm{RE}x}_{t|t+1}$ is the autonomous component of $\widehat{s}^{\mathrm{RE}}_{t|t+1}$[25], $G = \alpha + \sigma\eta + (\eta + \alpha\lambda)(1-\rho) > 0$, and the overbars are omitted for simplicity.[26] Like other asset market models, the primary factor that underpins fluctuations is the aggregate of participants' expectations. The time paths in (13) and (14) show that swings in the nominal and real exchange rate away from and back toward PPP values occur during stretches of time in which the market's forecast undergoes such swings.

Even with flexible goods prices, the model is consistent with nearly parallel swings in the nominal and real exchange rate. Such behavior

[25] From equation (10), $\widehat{s}^{\mathrm{RE}x}_{t|t+1} = (1-\rho)\,\bar{s}^{\mathrm{PPP}}_t + \hat{\pi}^r$, where we use $(1-\rho)$ instead of θ.

[26] To simplify, we omit from equations (13) and (14) terms that depend on $\hat{\pi}^r_t - \hat{\pi}^{r\,\mathrm{RE}}$ and $\hat{q}^{\mathrm{PPP}}_t - q^{\mathrm{PPP}}$, which we assume are constants and thus play no role in how the system moves over time. Allowing $\hat{\pi}^r_t$ or $\hat{q}^{\mathrm{PPP}}_t$ to vary would provide an additional channel through which currency swings could occur in the model. Doing so would not alter the model's implications for the degree of persistence.

depends on the relative sensitivity of excess demand in the goods markets to real exchange rate and real interest rate movements, that is, on the size of α relative to η, respectively. Research on the J-curve effect suggests that α is small relative to η, so that fluctuations of $\widehat{s}^{\mathrm{RE}_x}_{t|t+1}$ would be associated with near-parallel fluctuations of s_t and q_t.[27]

3.3 IKE Restrictions on Change

The restrictions that the FG model places on forecasting behavior enable expectations to play such an autonomous role in accounting for currency swings. The unfolding of participants' exchange rate forecasts depends on movements of the fundamental variables and revisions of forecasting strategies. From equation (12), we have:

$$\Delta\widehat{s}^{x}_{t|t+1} = \beta_{t-1}\Delta x_t + \Delta\beta_t x_t \tag{15}$$

3.3.1 Movements of Fundamentals

The FG model allows for shifts in the processes that underpin the informational variables. But, in order to simplify our analysis of the time series implications of the model, we abstract from this change. We assume that these variables, which can include m^r_t and y^r_t, follow random walks with constant drift:[28]

$$x_t = \mu^m + x_{t-1} + \varepsilon^x_t \tag{16}$$

3.3.2 Guardedly Moderate Revisions of Strategies

The FG model presumes that when and how market participants revise their forecasting strategies do not conform to any mechanical rule. Its use of qualitative and contingent conditions to characterize the revisions explores the possibility that they nonetheless exhibit some regularity, at least during stretches of time, that can be built into a mathematical model.

The model formalizes a regularity that was emphasized by Keynes (1936, p.152): participants in financial markets, regardless of whether they are bulls or bears, tend to assume that the "existing state of affairs will continue indefinitely, except in so far as we have specific reasons to expect a change." Even when a participant does have specific reasons to expect a change, it is entirely unclear what new forecasting strategy,

[27] See Meade (1988), Moffet (1989), Marquez (1991), and Hooper and Marquez (1995). The VAR estimates in Johansen *et al.* (2010) and Frydman *et al.* (2011) indicate that $\frac{\alpha}{\eta}$ is roughly 0.01.

[28] As our discussion in section 4 makes clear, we could allow for infrequent and small changes in the drifts of one or more of the variables in x_t without affecting our results.

if any, she should adopt. Faced with this uncertainty, participants tend to revise their thinking about how fundamentals matter in what Frydman and Goldberg (2007, 2013a) call "guardedly moderate" ways: there are stretches of time during which they either stick with their current forecasting strategies or revise them gradually. Such revisions do not generally alter, in substantial ways, the set of fundamentals that participants consider relevant or their interpretations of how changes in these variables affect future outcomes.

The model formalizes guardedly moderate revisions with two qualitative conditions. One of these conditions restricts changes of β_t so that their impact on the level of $\widehat{s}^x_{t|t+1}$ is smaller in size than the impact from trends in the informational variables:

$$|\Delta\beta_t x_t| < \delta_t \tag{17}$$

where $|\cdot|$ denotes an absolute value and $\delta_t = \left|\beta_{t-1}\mu^x\right|$. We note that δ_t, called the "baseline" trend, determines the change in $\widehat{s}^x_{t|t+1}$ that would occur if participants decided not to revise their forecasting strategies at all between $t-1$ and t, that is, $\Delta\beta_t = 0$. The second condition restricts revisions of β_t so that the baseline trends in two consecutive periods have the same sign:

$$|\Delta\beta_t\mu^x| < \delta_t \tag{18}$$

The constraints in (17) and (18) formalize the idea that when an individual decides to revise her strategy, she is reluctant to do so in ways that would alter her forecast and its baseline trend too much, in the sense that the impact of her revisions on $\widehat{s}^x_{t|t+1}$ and $\beta_{t-t}\mu^x$ does not outweigh the influence stemming from her time-$t-1$ interpretation of the trends in fundamentals.

We are unaware of any direct empirical evidence that the constraints in (17) and (18) are empirically relevant in real-world markets. But psychologists have uncovered experimental evidence indicating that when individuals alter their forecasts about uncertain outcomes in the face of new evidence, they tend to do so gradually, relative to some baseline.[29] This finding of "conservatism" is consistent with Keynes's insight and the FG model's characterization of guardedly moderate revisions.[30]

3.3.3 Fundamentals and Psychology in Forecasting

A market participant's decision about when and how to revise her forecasting strategy depends on many factors, including news about funda-

[29] See Edwards (1968) and Shleifer (2000) and references therein.

[30] The behavioral model of Barberis et al. (1998) also formalizes the experimental evidence on conservatism with restrictions that allow such behavior to vary over time. But, unlike the FG model, it imposes an overarching probability distribution on when conservatism begins and ceases to be relevant for individual behavior.

mental variables and the performance of her current strategy. But it also depends on the confidence that she attaches to her current or any new strategy and her intuition concerning whether the process driving fluctuations has changed or is about to change. By imposing qualitative conditions on revisions, the FG model is open to the non-routine ways that fundamentals and psychology matter for forecasting behavior.

Like asset-price swings themselves, market participants' tendency for moderate revisions occurs in irregular ways. Moreover, there are occasions when news and psychological considerations lead participants to revise their forecasting strategies in non-moderate ways. Such revisions can have a dramatic impact on prices and spell the end of a price swing in one direction and the start of a new one in the opposite direction. The model acknowledges this contingency by assuming that the constraints in (17) and (18) often, but not always, characterize market participants' revisions.

3.4 Contingent Predictions of Long Swings

The FG model's implications for currency fluctuations are contingent on how forecasting behavior unfolds over time: if revisions of participants' strategies satisfied the qualitative constraints in (17) and (18) over a stretch of time, the trends in fundamentals would lead participants to change $\widehat{s}^{x}_{t|t+1}$ in one direction on average, either away from or back toward PPP. According to the time paths in (13) and(14), this swing in the market's forecast would tend to lead to swings in the nominal and real exchange rate in the same direction.[31] If this swing were initially toward PPP, then $\bar{s}_t$ and $\bar{q}_t$ would eventually shoot through this benchmark and begin trending away from the other side for as long as revisions remained moderate.

Reversals in the direction of exchange rate swings occur at points in time that are associated with a change in the sign of the baseline trend driving participants' forecasts, that is, with non-moderate revisions. Once a reversal occurs, the tendency toward guardedly moderate revisions can lead to a stretch of time in which the exchange rate continues to trend in the new direction.

However, in order to account for the long swings and near-$I(2)$ persistence that we observe in currency markets, the FG model has to imply not only swings away from and back toward PPP, but also that these swings are sometimes long lasting. Its implication that currency swings

[31] We are simplifying by assuming that $\hat{\pi}^{r_{\mathrm{RE}}} = 0$. Alternatively, Frydman and Goldberg (2007) define $\delta_t = \left|\beta_{t-1}\mu^{x} - (1-\rho)\,\hat{\pi}^{r_{\mathrm{RE}}}\right|$, so that guardedly moderate revisions imply a tendency for $\left(\widehat{s}^{x}_{t|t+1} - \widehat{s}^{\mathrm{RE}x}_{t|t+1}\right)$ to move in one direction.

stem in part from guardedly moderate revisions implies that the model's ability to account for currency fluctuations depends on how pronounced the tendency is for such revisions.

4 Imperfect Knowledge and Segmented Trends

The connection between the tendency for guardedly moderate revisions and long swings can be seen by expressing the time paths in (13) and(14) as segmented trends processes. For the real exchange rate, we have:[32]

$$\bar{q}_t = \bar{q}_{t-1} + \xi_t + \upsilon_t \tag{19}$$

$$\xi_t = \frac{\eta}{G}\left(\Delta\beta_t x_t + \beta_{t-1}\mu^x\right) \tag{20}$$

where υ_t depends on ε_t^x, ε_t^m, and ε_t^y and we see that the drift, ξ_t, depends on how individuals revise their forecasting strategies and trends in fundamentals.

Equations (19) and (20) imply that stretches of time in which participants' forecasting strategies remain unchanged in the aggregate ($\Delta\hat{\beta}_t = 0$) are associated with a distinct drift, that is, $\beta_{t-1}\mu^x$. Points in time at which participants revise their strategies, therefore, lead to shifts in the drift. But, as long as these revisions satisfy the qualitative constraints (17) and (18), changes in the ξ_t's will tend to be sufficiently small and the drift will tend to take on values of the same algebraic sign.[33] During these periods, the real exchange rate would tend to move persistently in one direction, up if the succession of ξ_t's were positive or down if they were negative. Reversals would be associated with non-moderate revisions leading to changes in ξ_t that were large enough to imply a change in its sign.

The greater the tendency of market participants' to revise their forecasting strategies in guardedly moderate ways, the greater the model's ability to account for long swings in real and nominal exchange rates. Keynes's (1936) insight that individuals tend to assume that the "existing state of affairs will continue," as well as behavioral findings on conservatism, suggest that the tendency for moderate revisions is high. According to the segmented trends process in equations (19) and (20), such behavior would lead to long stretches of time in which the exchange rate tended to move in one direction. It would be difficult to fit a Markov chain to such a process; there would be too many ξ_t's to estimate. But,

[32]We continue to assume that $\hat{\pi}^{r_{\mathrm{RE}}} = 0$.

[33]The first constraint in (17) ensures that the baseline trend, $\hat{\beta}_{t-1}\mu^x$, will determine the direction of change of $\widehat{s}^x_{t|t+1}$, while the second constraint in (18) implies that the sign of the baseline trend remains unchanged between $t-1$ and t.

as we show in the next section, if participants are assumed to have a pronounced tendency toward guardedly moderate revisions, the IKE process in equations (19) and (20) would imply near-$I(2)$ trends and thus long swings in exchange rates.

4.1 Shifts in the Policy Regime

We have assumed that the drifts underpinning movements of the informational variables, μ^x, are constant. However, equations (19) and (20) show that changes in ξ_t could also arise from shifts in monetary policy or other changes in the social context that would lead to shifts in μ^x. If we were to assume that the "policy regime" tended to be long lasting and that market participants' forecasting strategies were either fixed or changed only when the policy regime changed, the model would imply long swings in the real and nominal exchange rate.

We do not need to assume that change in the model is governed by a Markov chain, as in Engel and Hamilton (1990) and Kaminsky (1993), to account for long swings. As we show in the next section, the near-$I(2)$ result that we derive for the more general case with guardedly moderate revisions also applies when long-lasting policy regimes are assumed.

4.2 A Persistent Markov-Switching Model

If we were prepared to impose a Markov chain on change in the model, we could obtain Engel and Hamilton's (1990) persistent segmented-trends process not only for the nominal exchange rate, but also for the real exchange rate. Engel and Hamilton suppose that the exchange rate process involves a deterministic trend, ξ_t, that switches between preset values. To introduce some notation, assume that we have a number, n, of swings and let $0 = T_0 < T_1^* < T_2^* < \cdots < T_n^* = T$ denote the points in time at which the currency swing changes direction. Then $T_i = T_i^* - T_{i-1}^*$ denotes the length of the ith swing. Focusing on the real exchange rate, the Engel-Hamilton econometric model of swings can be written as:

$$\Delta q_t = \xi_t + \varepsilon_t....t = T_{i-1}^* + 1, ..., T_i^* \tag{21}$$

where ε_t is $N(0, \sigma_\varepsilon)$, the trend is constrained to take on two values

$$\xi_t = \xi_1 > 0 \quad \text{or} \quad \xi_t = \xi_2 < 0 \tag{22}$$

and switches in ξ_t are assumed to follow a Markov chain with fixed transition probabilities.

Equations (19) and (20) show that converting the FG model into the Engel-Hamilton specification requires imposing three restrictions on forecasting behavior and policy. First, during each upswing and downswing, market participants leave their forecasting strategies unaltered, that is, $\Delta\beta_t = 0$, and the policy environment remains unchanged. Second, when strategies or policies are revised, it leads to one of two baseline trends, so that the drift,

$$\xi_i = \frac{\eta}{G}\beta_i\mu_i^x \quad i = 1, 2$$

can take on one of only two values, $\xi_1 > 0$ and $\xi_2 < 0$. Finally, the timing and frequency of revisions is governed by a Markov chain.

But, imposing a Markov chain on change that implies long-lasting regimes is not needed for the model to account for long swings and near-$I(2)$ trends in currency markets. What is needed is long-lasting policy regimes and a pronounced tendency for market participants to revise their forecasting strategies in guardedly moderate ways.

5 Imperfect Knowledge, $I(2)$ Trends, and the Signal-To-Noise Ratio

The IKE process in equations (19) and (20) does not follow any fixed stochastic process. In order to derive implications for the degree of persistence of currency fluctuations from the model, therefore, we must define a notion of persistence that recognizes the importance of non-routine change and imperfect knowledge. We also need to impose additional IKE restrictions on forecasting behavior that formalize the assumption that market participants have a pronounced tendency to revise their strategies in guardedly moderate ways. We sketch these assumptions here, leaving analysis of their implications for appendix B.

5.1 Defining Persistence for IKE Models

Although IKE models do not imply any fixed stochastic process for market outcomes, we can still examine the following question: if the real and nominal exchange rate process were adequately characterized by the FG model and an $AR(1)$ regression in first differences were estimated,

$$\xi_t = \gamma\xi_{t-1} + u_t \tag{23}$$

would the limit of the autoregressive estimator,

$$\hat{\gamma} = \frac{\sum_{t=1}^{T}\xi_t\xi_{t-1}}{\sum_{t=1}^{T}\xi_{t-1}^2} \tag{24}$$

be close to but less than one? We take such a result to imply near-$I(1)$ persistence in the drift and thus near-$I(2)$ persistence in the exchange rate.[34]

5.2 Two Regularity Conditions

Like before, assume that we have a number, n, of swings, and let $0 = T_0^* < T_1^* < T_2^* < \cdots < T_n^* = T$ be given positive numbers denoting points in time at which a reversal in the algebraic sign of the drift occurs.[35] We assume that the drift in equation (20) can be expressed as,

$$\begin{aligned}
\xi_t &= a_1 + \delta_t, \quad t = 1, \ldots, T_1^* \\
\xi_t &= a_2 + \delta_t, \quad t = T_1^* + 1, \ldots, T_2^* \\
\xi_t &= a_3 + \delta_t, \quad t = T_2^* + 1, \ldots, T_3^* \\
&\quad etc.
\end{aligned}$$

so that,

$$\xi_t = a_i + \delta_t, \ t \in \mathcal{I}_i, i = 1, \ldots, n, \tag{25}$$

where $\mathcal{I}_i$ is the i'th interval of length $T_i = T_{i+1}^* - T_i^*$, a_i is the average value taken in the i'th interval, $\mathcal{I}_i$, in the sense that

$$E(T_i^{-1} \sum_{t \in \mathcal{I}_i} \xi_t) = a_i, i = 1, \ldots, n.$$

$\delta_t = \Delta\beta_t x_t$ is a disturbance and $a_i = \beta_i \mu_i^x$. Thus ξ_t switches between the values $a_1, a_2, \ldots, a_n$ of alternating sign, and given that the T_i^*'s denote points at which ξ_t switches sign, we have $a_i a_{i+1} < 0$.

In appendix B, we show that the limit of $\hat{\gamma}$ is close to, but less than, one if two qualitative conditions beyond those in (17) and (18) are imposed on forecasting behavior. First, we assume that revisions are sufficiently moderate so that the variance of the disturbance, σ_δ^2, is small relative to a magnitude that depends on the average size of the separate drifts over all intervals. We note that this condition is consistent with guardedly moderate revisions and places no constraint on revisions across intervals, rendering it consistent with non-mechanical behavior. Second, we assume that the tendency toward guardedly moderate revisions is pronounced enough to imply that the number of switches, n, is small compared to the total time, T.

These two assumptions also portray the case that we discussed in section 4.1 involving long-lasting policy regimes and fixed forecasting strategies. In this case, $\delta_t = 0$, and ξ_t switches between deterministic values. It is clear from the analysis in appendix B that the near-$I(2)$ result would also obtain in this case.

[34] This way of uncovering the time-series implications of models is also used in Johansen and Lange (2011) and Frydman et al. (2012).

[35] We could allow T_i to be random without changing the basic result.

Although we have focused on modeling exchange rate fluctuations in this paper, the FG model can also account for the near-$I(2)$ persistence in the interest rate differential and relative goods prices. It is straightforward to show that these variables can also be expressed as a persistent segemented-trends process as in equations (19) and (20).

5.3 A Small Signal-to-Noise Ratio

We saw in section 1 that currency fluctuations over our sample entailed a small signal-to-noise ratio. Although the IKE process in equations (19) and (20) is not stochastic, we can ask, as before, what estimate of the signal-to-noise ratio, $\frac{\sigma_\xi^2}{\sigma_v^2}$, we would be likely to obtain using historical data if this model provided an adequate account of fluctuations.

Equation (20) implies that movements of the drift between consecutive points in time depend on how individuals revise their forecasting strategies at each of these points:

$$\xi_t = \xi_{t-1} + \frac{\eta}{G}\left[\Delta^2\beta_t x_{t-1} + \left(\Delta\beta_t + \Delta\beta_{t-1}\right)\mu^x + \Delta\beta_t\varepsilon_t^x\right] \qquad (26)$$

since revisions at time t affect the drift not only at time t, but also at $t+1$, owing to their impact on the baseline trend, $\beta_{t-1}\mu^x$. This specification shows that the greater the tendency toward guardedly moderate revisions, the greater the tendency for changes in ξ_t, and thus estimates of σ_ξ^2, to be small relative to estimates of σ_v^2. Consequently, our analysis not only connects the IKE process implied by the FG model to $I(2)$-type persistence; it also shows how the model can account for the small signal-to-noise ratio found in the data.

6 Concluding Remarks

We have shown in this paper that the degree of persistence associated with real and nominal exchange rate fluctuations is better approximated as $I(2)$-type, and that the FG model is able to account for this persistence. Like bubble and learning models, the FG model accords market participants' expectations an autonomous role in driving currency fluctuations. But, unlike these models, the FG model makes use of qualitative and contingent conditions to portray this autonomous role. This leads us to propose a definition of persistence for IKE models that enables us to connect the implications of the FG model to estimates of the properties of time series using historical data.

In this paper, we focused on the FG model's implications for the degree of persistence of real and nominal exchange rates. In Frydman et al. (2012), we explore its implications for the PPP puzzle. These implications follow in part from the specification of goods market equilibrium in equation (8). Under IKE, this equilibrium is characterized not by PPP, but by a relationship between the real exchange rate and the real interest rate differential. In the model, autonomous movements in participants' exchange rate forecasts lead to swings in the exchange rate away from and back toward PPP, and thus to a real exchange rate and real interest rate differential that are separately near-$I(2)$. However, goods market equilibrium implies that these two variables should co-move over time, that is, the real exchange rate and real interest differential should be cointegrated. The CVAR studies of Juselius (1995) and others find evidence of such co-movement.

In considering the PPP puzzle, Engel and Morley (2001) and Cheung, Lai, and Bergman (2004) find that the adjustment of goods prices to equilibrium levels is much faster than the adjustment of exchange rates to PPP. This has led Benigno (2004) and others to develop REH models in which endogenous monetary policy leads to a de-linking of exchange rate persistence from goods market adjustment. However, because expectations are unable to play an autonomous role in these REH models, they continue to imply at most a near-$I(1)$ real exchange rate.

The FG model also de-links exchange rate persistence and goods market adjustment; it generates swings in the real exchange rate regardless of the speed of adjustment in goods markets. And because this de-linking follows from according expectations an autonomous role, the model is able to account for the greater persistence of currency fluctuations. This suggests that the model provides a basis for explaining the PPP puzzle.

Table 3: Determination of the two rank indices

Rank Test Statistics								
$p-r$	r	$s_2=5$	$s_2=4$	$s_2=3$	$s_2=2$	$s_2=1$	$s_2=0$	
3	2			140.4 [0.00]	**56.96** [0.36]	**42.85** [0.35]	**37.22** [0.17]	
Six largest characteristic roots:								
Unrestricted VAR			**0.99**	**0.99**	**0.98**	**0.98**	**0.82**	0.50
$r=2, p-r=3$			1.0	1.0	1.0	**0.96**	**0.96**	0.50
$r=2, s_1=2, s_2=1$			1.0	1.0	1.0	1.0	**0.96**	0.51
$r=2, s_1=1, s_2=2$			1.0	1.0	1.0	1.0	1.00	0.51

7 Appendix A: Multivariate Tests of $I(2)$ Trends

The hypothesis that $x_t \sim I(2)$ is formulated as two reduced rank hypotheses:

$$\Pi = \alpha\beta' \tag{27}$$

and

$$\alpha'_{\perp}\Gamma\beta_{\perp} = \xi\eta' \tag{28}$$

where α and β are $p \times p-r$ and $r < p$, respectively, p is the number of variables in the VAR model, r is the cointegration rank, ξ and η are both $p-r \times p-r-s_2$, and $p-r = s_1+s_2$ is the number of common stochastic trends of which s_1 are integrated of order one and s_2 of order two. If $s_2 = 0$, (28) is a full rank matrix and $x_t \sim I(1)$. Testing whether $x_t \sim I(2)$ amounts to testing $s_2 > 0$ for $r < p$.

Table 3 reports the $I(2)$ trace tests as well as the characteristic roots of the model. For simplicity, we report only the results for $r = 2$ as $r = 0$, 1 was rejected by the trace tests. The estimated characteristic roots in the unrestricted VAR suggest a total of five large roots, four of which are almost exactly on the unit circle, while the fifth is large (0.82), but not equally close to one. The choice of rank indices should, therefore, be consistent with five large roots. With a p-value of 0.36, the trace test supports the choice of $\{r = 2, s_1 = 1, s_2 = 2\}$ which corresponds to $s_1 + s_2 \times 2 = 5$ unit roots in the characteristic polynomial. The lower part of Table 3 shows that for this choice, the largest unrestricted root is 0.51. Instead, if we were to set $s_2 = 0$ (and treat the variables in the model as $I(1)$), our model would contain two large roots of size 0.96, whereas $s_2 = 1$ would leave a large root of size 0.96 in the model. In both cases the large roots would render any inference of stationarity totally unreliable. Consequently, the case $\{r = 2, s_1 = 1, s_2 = 2\}$ is strongly supported by the tests and is able to account for all five large roots in the unrestricted VAR. We conclude that the vector process x_t is $I(2)$.

8 Appendix B: Near-$I(2)$ Trends

We investigate the estimator $\hat{\gamma} = \sum_{t=1}^{T} \xi_t \xi_{t-1} / \sum_{t=2}^{T} \xi_{t-1}^2$, see (24), and apply the representation in (25). The product moments for the first period can be found as

$$\sum_{t=2}^{T_1} \xi_{t-1}^2 = \sum_{t=2}^{T_1} (a_1 + \delta_{t-1})^2 = a_1^2 (T_1 - 1) + \sum_{t=2}^{T_1} \delta_{t-1}^2 + 2a_1 \sum_{t=2}^{T_1} \delta_{t-1}$$

$$\begin{aligned} \sum_{t=1}^{T_1} \xi_t \xi_{t-1} &= \sum_{t=2}^{T_1} (a_1 + \delta_t)(a_1 + \delta_{t-1}) = a_1^2 (T_1 - 1) + 2a_1 \sum_{t=2}^{T_1} \delta_{t-1} \\ &+ \sum_{t=2}^{T_1} \delta_t \delta_{t-1} - a_1 (\delta_1 - \delta_{T_1}) \end{aligned}$$

For the second period we find, noting that $\xi_{T_1} = a_1 + \delta_{T_1}$, the expression

$$\begin{aligned} \sum_{t=T_1+1}^{T_1+T_2} \xi_{t-1}^2 &= (a_1 + \delta_{T_1})^2 + \sum_{t=T_1+2}^{T_1+T_2} (a_2 + \delta_{t-1})^2 \\ &= a_1^2 + a_2^2 (T_2 - 1) + \sum_{t=T_1+1}^{T_1+T_2} \delta_{t-1}^2 + 2a_2 \sum_{t=T_1+1}^{T_1+T_2} \delta_{t-1} + [2(a_1 - a_2)\delta_{T_1}] \end{aligned}$$

$$\begin{aligned} \sum_{t=T_1+1}^{T_1+T_2} \xi_t \xi_{t-1} &= (a_2 + \delta_{T_1+1})(a_1 + \delta_{T_1}) + \sum_{t=T_1+2}^{T_1+T_2} (a_2 + \delta_t)(a_2 + \delta_{t-1}) \\ &= a_1 a_2 + a_2^2 (T_2 - 1) + 2a_2 \sum_{t=T_1+1}^{T_1+T_2} \delta_{t-1} \\ &+ \sum_{t=T_1+1}^{T_1+T_2} \delta_t \delta_{t-1} + [(a_1 - a_2)\delta_{T_1+1} - a_2(\delta_{T_1} - \delta_{T_1+T_2})] \end{aligned}$$

With similar expressions for $\sum_{t\in\mathcal{I}_i}\xi_{t-1}^2$ and $\sum_{t\in\mathcal{I}_i}\xi_t\xi_{t-1}$, where $T_i^*=T_1+\cdots+T_i$, $T=T_n^*$ we then find

$$\sum_{t=2}^{T}\xi_{t-1}^2 = \sum_{i=1}^{n}a_i^2T_i+\sum_{t=2}^{T}\delta_{t-1}^2+2\sum_{i=1}^{n}a_i\sum_{t\in\mathcal{I}_i}\delta_{t-1}+nR_n^{(1)}$$

$$R_n^{(1)} = n^{-1}\sum_{i=2}^{n}[2(a_{i-1}-a_i)\delta_{T_{i-1}^*}]$$

$$\sum_{t=2}^{T}\xi_t\xi_{t-1} = \sum_{i=1}^{n}a_i^2(T_i-1)+\sum_{i=1}^{n-1}a_ia_{i+1}+2\sum_{i=1}^{n}a_i\sum_{t\in\mathcal{I}_i}\delta_{t-1} + \sum_{t=2}^{T}\delta_t\delta_{t-1}+nR_n^2$$

$$R_n^{(2)} = n^{-1}\sum_{i=2}^{n}[(a_{i-1}-a_i)\delta_{T_{i-1}^*+1}-a_i(\delta_{T_{i-1}^*}-\delta_{T_i^*})]$$

$$\sum_{t=2}^{T}\xi_t\xi_{t-1}-\sum_{t=2}^{T}\xi_{t-1}^2 = -\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n-1}a_ia_{i+1}+\sum_{t=2}^{T}\delta_t\delta_{t-1}-\sum_{t=2}^{T}\delta_{t-1}^2 +n(R_n^{(2)}-R_n^{(1)}),$$

and hence,

$$\hat{\gamma}-1=\frac{\sum_{t=1}^{T}\xi_t\xi_{t-1}-\sum_{t=1}^{T}\xi_{t-1}^2}{\sum_{t=1}^{T}\xi_{t-1}^2}=\frac{A_T}{B_T}, \tag{29}$$

$$A_T = -T^{-1}\sum_{i=1}^{n}a_i^2+T^{-1}\sum_{i=2}^{n}a_ia_{i-1}+T^{-1}\sum_{t=2}^{T}\delta_t\delta_{t-1}-T^{-1}\sum_{t=2}^{T}\delta_{t-1}^2 +nT^{-1}(R_n^{(2)}-R_n^{(1)})$$

$$B_T = T^{-1}\sum_{i=1}^{n}a_i^2T_i+T^{-1}\sum_{t=2}^{T}\delta_{t-1}^2+2T^{-1}\sum_{i=1}^{n}a_i\sum_{t\in\mathcal{I}_i}\delta_{t-1}+nT^{-1}R_n^{(1)}$$

We can now show that $\hat{\gamma}<1$ for large T under reasonable assumptions that are consistent with the guardedly moderate conditions in (17) and (18). We assume that $n^{-1}\sum_{i=1}^{n-1}a_ia_{i+1}<0$ is bounded, that there are few switches so that $T^{-1}n\to 0$, and that $T^{-1}\sum_{i=1}^{n}a_i^2T_i\to a^2>0$. We also assume that δ_t is a stationary ergodic process with $E(\delta_t)=0$, $E(\delta_t^2)=\sigma_\delta^2>0$, $\rho=Corr(\delta_t,\delta_{t-1})<1$, and finite fourth moment so we can apply the law of large numbers. More precisely we can show that $\hat{\gamma}-1\xrightarrow{P}-(1-\rho)\sigma_\delta^2/(a^2+\sigma_\delta^2)<0$, $T\to\infty$, so that $\hat{\gamma}$ is close to, but less than, 1 for large T, if the variation of δ_t is small compared to the variation of a, that is σ_δ^2/a^2 is small.

To prove this we consider the expression (29).

A_T: Because $a_i a_{i+1} < 0$ we have that the first two terms are negative $-T^{-1}\sum_{i=1}^n a_i^2 < 0$ and $T^{-1}\sum_{i=2}^n a_i a_{i-1} < 0$ and tend to zero. The next two terms converge by the law of large numbers to $\sigma_\delta^2(1-\rho) < 0$ and the last satisfies $nT^{-1}(R_n^{(2)} - R_n^{(1)}) \xrightarrow{P} 0$.

B_T: The first term measures the variation of a_i around zero and is assumed to converge:

$$T^{-1}\sum_{i=1}^n a_i^2 T_i = \frac{\sum_{i=1}^n a_i^2 T_i}{\sum_{i=1}^n T_i} \to a^2 > 0.$$

We also find

$$T^{-1}\sum_{t=1}^T \delta_t^2 \xrightarrow{P} \sigma_\delta^2 > 0$$

and

$$2T^{-1}\sum_{i=1}^n a_i T_i [T_i^{-1}\sum_{t\in\mathcal{I}_i} \delta_t] \xrightarrow{P} 0,$$

because we assumed that $E(\delta_t) = 0$ and $nT^{-1}R_n^{(1)} \xrightarrow{P} 0$.

References

[1] Adam, Klaus and Albert Marcet (2011), "Booms and Busts in Asset Prices," CEPR Working Paper.

[2] Adam, Klaus, Albert Marcet, and Juan Pablo Nicolini (2009), "Stock Market Volatility and Learning," CEPR Working Paper.

[3] Bacchiocchi, Emanuele and Luca Fanelli (2005), "Testing the Purchasing Power Parity through $I(2)$ Cointegration Techniques," *Journal of Applied Econometrics* 20, 749–770.

[4] Barberis, Nicholas, Andrei Shleifer, and Robert Vishny (1998), "A Model of Investor Sentiment," *Journal of Financial Economics* 49, 307–43.

[5] Beckman, Joscha, Ansgar Belke, and Michael Kuhl (2011), "How Stable Are Monetary Models of the Dollar-Euro Exchange Rate?: A Time-Varying Coefficient Approach," *Review of World Economics*, 147, 1, 11–40.

[6] Benigno, Gianluca (2004), "Real Exchange Rate Persistence and Monetary Policy Rules," *Journal of Monetary Economics* 51, 473–502.

[7] Brunnermeier, Markus K. (2001), *Asset Pricing Under Asymmetric Information: Bubbles, Crashes, Technical Analysis, and Herding*, Oxford, UK: Oxford University Press.

[8] Cheung, Yin–Wong , K.S. Lai, and Michael Bergman (2004), "Dissecting the PPP Puzzle: The Unconventional Roles of Nominal Exchange Rate and Price Adjustments," *Journal of International Economics* 64, 135–150.

[9] Cheung, Yin-Wong and Ulf G. Erlandsson (2005), "Exchange Rates and Markov Switching Dynamics," *Journal of Business and Economic Statistics*, 23, 3, 314–320.

[10] De Grauwe, Paul and Marianna Grimaldi (2006), *The Exchange Rate in a Behavioral Finance Framework*, Princeton, NJ: Princeton University Press.

[11] Dornbusch, Rudiger (1976), "Expectations and Exchange Rate Dynamics," *Journal of Political Economy*, December, 1161–1174.

[12] Dornbusch, Rudiger (1983), "Exchange Rate Risk and the Macroeconomics of Exchange Rate Determination," in R. Hawkins, R. Levich and C. G. Wihlborg (eds.), *The Internationalization of Financial Markets and National Economic Policy*, Vol. 3, Greenwich, CT: JAI Press, 3–27.

[13] Edwards, Ward (1968), "Conservatism in Human Information Processing," in B. Kleinmuth (ed.), *Formal Representation of Human Judgement*, New York: John Wiley and Sons.

[14] Burnsidey, Craig, Martin Eichenbaum, and Sergio Rebelo (2011),

"Understanding Booms and Busts in Housing Markets," working paper.

[15] Engel, Charles (1994), "Can the Markov Switching Model Forecast Exchange Rates," *Journal of International Economics*, 36, 151–165.

[16] Engel, Charles and James D. Hamilton (1990), "Long Swings in the Exchange Rate: Are They in the Data and Do Markets Know It?," *American Economic Review*, September, 689–713.

[17] Engel, Charles and James C. Morley (2001), "The Adjustment of Prices and the Adjustment of the Exchange Rate," NBER Working Paper 8550.

[18] Evans, George (1986), "A Test for Speculative Bubbles in the Sterling-Dollar Exchange Rate: 1981–84," *American Economic Review* 76, 621–36.

[19] Farmer, Roger (2013), "Animal Spirits, Persistent Unemployment, and the Belief Function," forthcoming in Roman Frydman and Edmund S. Phelps (eds.), *Rethinking Expectations: The Way Forward for Macroeconomics*, Princeton , N.J.: Princeton University Press.

[20] Frankel, Jeffrey. A. (1979), "On the Mark: A Theory of Floating Exchange Rate Based On Real Interest Differentials," *American Economic Review* 69, September, 610–22.

[21] Frankel, Jeffrey. A. and Kenneth. A. Froot (1995), "Understanding the U.S. Dollar in the Eighties: The Expectations of Chartists and Fundamentalists," in Jeffrey A. Frankel (ed.), *On Exchange Rates*, Cambridge, MA: MIT Press, 295–316.

[22] Frömmel, Michael, Ronald MacDonald, and Lukas Menkhoff (2005), "Markov Switching Regimes in a Monetary Exchange Rate Model," *Economic Modeling*, vol. 22, May, 485–502.

[23] Frydman, Roman and Michael D. Goldberg (2007), *Imperfect Knowledge Economics: Exchange rates and Risk*, Princeton, NJ: Princeton University Press.

[24] Frydman, Roman and Michael D. Goldberg (2011), *Beyond Mechanical Markets: Asset Price Swings, Risk, and the Role of the State,* Princeton, NJ.: Princeton University Press.

[25] Frydman, Roman and Michael D. Goldberg (2013a), "Opening Models of Asset Prices and Risk to Non-Routine Change," forthcoming in Roman Frydman and Edmund S. Phelps (eds.), *Rethinking Expectations: The Way Forward for Macroeconomics*, Princeton , N.J.: Princeton University Press.

[26] Frydman, Roman and Michael D. Goldberg (2013b), "The Imperfect Knowledge Imperative in Modern Macroeconomics and Finance Theory," Roman Frydman and Edmund S. Phelps (eds.), *Rethinking Expectations: The Way Forward for Macroeconomics*, forthcoming

Princeton, NJ.: Princeton University Press.

[27] Hsieh, David A. (1989), "Testing for Nonlinear Dependence in Daily Foreign Exchange Rates," *Journal of Business,* 62, 3, 339–368.

[28] Hooper, Peter and Jaime Marquez (1995), "Exchange Rates, Prices, and External Adjustment in the United States and Japan," in Peter B. Kenen (ed.), *Understanding Interdependence*, Princeton, NJ: Princeton University Press.

[29] Johansen, Søren (1992), "An I(2) Cointegration Analysis of the Purchasing Power Parity between Australia and the United States," in Hargreaves, C. ed., *Macroeconomic Modelling of the Long Run*, 229–248. London: Edward Elgar.

[30] Johansen, Søren (1995), *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models,* Oxford: Oxford University Press.

[31] Johansen, Søren (1997), "Likelihood Analysis of the $I(2)$ Model," *Scandinavian Journal of Statistics* 24*,* 433–462.

[32] Johansen, Søren (2006), "Statistical Analysis of Hypotheses on the Cointegrating Relations in the $I(2)$ Model," *Journal of Econometrics* 132, 81–115.

[33] Johansen, Søren, Katarina Juselius, Roman Frydman, and Michael D. Goldberg (2010), "Testing Hypotheses in an I(2) Model With Piecewise Linear Trends. An Analysis of the Persistent Long Swings in the Dmk/$ Rate," *Journal of Econometrics,* 158, 117–29.

[34] Johansen, Søren and Theis Lange (2011), "Some Econometric Results for the Blanchard-Watson Bubble Model," University of Copenhagen Working Paper.

[35] Juselius, Katarina (1995), "Do the Purchasing Power Parity and the Uncovered Interest Rate Parity Hold in the Long Run? - An Application of Likelihood Inference in a Multivariate Time Series Model," *Journal of Econometrics* 69, 211–240.

[36] Juselius, Katarina (2007), *The Cointegrated VAR Model: Methodology and Applications*, Oxford: Oxford University Press.

[37] Kahneman, Daniel and Amos Tversky (1979), "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica* 47, 263–291.

[38] Kaminsky, Graciela L. (1993), "Is There a Peso Problem?: Evidence From the Dollar/Pound Exchange Rate 1976-1987," *American Economic Review*, 83, 450–472.

[39] Keynes, John M. (1936), *The General Theory of Employment, Interest and Money*, New York: Harcourt, Brace and World, 1964.

[40] Kongsted, Hans C. (2003), "An I(2) Cointegration Analysis of Small Country Import Price Determination," *Econometrics Journal* 12, 163–178.

[41] Kongsted, Hans C. (2005), "Testing the Nominal-to-Real Transfor-

mation," *Journal of Econometrics* 124, 205–225.

[42] Kongsted, Hans C. and Heino B. Nielsen (2004), "Analyzing $I(2)$ Systems by Transformed Vector Autoregressions," *Oxford Bulletin of Economics and Statistics* 66, 379–397.

[43] Kouri, Pentti J.K. (1976), "The Exchange Rate and the Balance of Payments in the Short Run and in the Long Run," *Scandinavian Journal of Economics,* 2, 2280–304.

[44] Lewis, Karen K. (1989), " Can Learning Affect Exchange Rate Behavior?: The Case of the Dollar in the Early 1980's," *Journal of Monetary Economics*, 23, 79–100.

[45] Mark, Nelson C. (2009), "Changing Monetary Policy Rules, Learning, and Real Exchange Rate Dynamics," *Journal of Money, Credit and Banking*, 41, 6, 1047–1070.

[46] Marquez, Jaime (1991), "The Dynamics of Uncertainty or the Uncertainty of Dynamics: Stochastic J-Curves," *Review of Economics and Statistics* 73, 125–133.

[47] Meade, Ellen E. (1988), "Exchange Rates, Adjustment, and the J-Curve," *Federal Reserve Bulletin,* 74, 633–644.

[48] Moffet, Michael (1989), "The Lagged Adjustment of U.S. Trade to Prices and Income," *Journal of International Money and Finance* 8, 425–444.

[49] Obstfeld, Maurice and Kenneth Rogoff (1998), "Risk and Exchange Rates," NBER Working Paper 6694.

[50] Rogoff, Kenneth (1996), "The Purchasing Power Puzzle," *Journal of Economic Literature*, 34, June, 647–68.

[51] Sarno, Lucio and Mark P. Taylor (2002), *The Economics of Exchange Rates*, Cambridge, UK: Cambridge University Press.

[52] Shleifer, Andrei (2000), *Inefficient Markets*, Oxford: Oxford University Press.

[53] Stillwagon, Joshua (2010), "Currency Risk: Long Swings and Volatility," University of New Hampshire Working Paper.

[54] Taylor, Alan and Mark P. Taylor (2004), "The Purchasing Power Parity Debate," *Journal of Economic Perspectives*, 18, 4, 135–158.

[55] Tobin, James (1958), "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies*, 25 (1), 15–29.