Microeconomic Theory I
Final Examination

Wednesday, December 19, 2007

Answer the following questions completely, clearly and concisely. You will be graded on how clearly you explain your answer as well on the content. Write the answers to Questions #1 and #2 in one blue book and the answers to Question #3 in another. Write the number of the questions, your name in BLOCK LETTERS, and your ID number on the front of each blue book. **Each question will receive equal weight**.

1. On the planet Dystopia, there are two types of toxic waste, denoted by $h = 1, 2$, and two types of individuals, denoted by $i = 1, 2$. An agent i has a consumption sets $X_i = \mathbf{R}^2_+$, a utility function

$$u_i(x_i) = \begin{cases} -\max\{2x_{11}, x_{12}\} & \text{if } i = 1, \\ -\max\{x_{21}, 2x_{22}\} & \text{if } i = 2, \end{cases}$$

and an endowment

$$e_i = \begin{cases} (0,2) & \text{if } i = 1, \\ (1,0) & \text{if } i = 2. \end{cases}$$

(a) Give a formal definition of a Pareto efficient allocation for this economy.
(b) Derive the set of all Pareto-efficient allocations for this economy.
(c) Illustrate your answer to (b) with an Edgeworth Box diagram.

Solution An allocation is an ordered pair $(x_1, x_2) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+$. An allocation is attainable if $x_1 + x_2 = (1, 2)$. An attainable allocation x is Pareto efficient if there does not exist an attainable allocation x' such that $u_i(x'_i) \geq u_i(x_i)$ for $i = 1, 2$ and the inequality is strict for at least one $i = 1, 2$.

The necessary conditions for Pareto efficiency are

$$2x_{11} \geq x_{12} \Longrightarrow x_{21} \geq 2x_{22}$$

$$2x_{11} \leq x_{12} \Longrightarrow x_{21} \leq 2x_{22}.$$

If we let (c_1, c_2) be agent 1's consumption bundle $(1 - c_1, 2 - c_2)$ be agent 2's, then the necessary and sufficient conditions are

$$2c_1 \geq c_2 \text{ and } c_1 \leq 2c_2 - 3$$

$$2c_1 \leq c_2 \text{ and } c_1 \geq 2c_2 - 3.$$

The first pair of inequalities implies $c_1 \geq 1$ and hence can be ignored. We are left with the second pair, which is equivalent to

$$2c_1 \leq c_2 \leq 1.5 + 0.5c_1$$

where $0 \leq c_1 \leq 1$. This completely characterizes the set of Pareto efficient allocations.

2. Consider an Arrow-Debreu economy in which there are two dates $t = 0, 1$, a finite number of states $s = 1, ..., S$, and a single good at each date. At date 0, every agent has a common prior probability distribution $\{\pi_s\}_{s=1}^{S}$ on the set of states. At date 1, the true state is revealed. There are m agents with identical utility functions

$$U(x_i) = \ln x_{i0} + \sum_{s=1}^{S} \pi_s \ln x_{is}$$

and distinct endowments $e_i = (e_{i0}, e_{i1}, ..., e_{iS})$.

(a) Solve for the unique equilibrium of the Arrow-Debreu economy.

(b) Now consider an economy that is identical to the preceding one except that it possesses a complete set of real assets traded at date 0. The returns of asset k are represented the vector $a_k = (a_{k1}, ..., a_{ks}, ..., a_{kS})$ and for convenience let

$$A = [a_{ks}]_{K \times S}.$$

What is the price of asset $k = 1, ..., K$?

(c) Now consider an economy that is identical to the preceding one except that it has only spot markets for goods and assets. Assume that the number of assets is just equal to the number of states. Assuming that each agent's consumption is the same as in the Arrow-Debreu equilibrium, what is the portfolio of assets held by each agent $i = 1, ..., m$? Prove that this portfolio satisfies the agent's budget constraint at date 0.

Solution (a) Since agents have identical, homothetic preferences, we can solve the equilibrium problem as if there were a representative agent holding the aggregate endowment. In that case, the equilibrium prices will be proportional to the gradient vector. Normalizing prices, we can put

$$p^* = (p_0^*, p_1^*, ..., p_S^*) = \left(\frac{1}{e_0}, \frac{\pi_1}{e_1}, ..., \frac{\pi_S}{e_S} \right),$$

where $e_s = \sum_{i=1}^{m} e_{is}$ is the aggregate endowment of good $s = 0, 1, ..., S$.

With these prices, we can define the wealth of agent i to be $w_i^* = p^* \cdot e_i$ and, using the properties of the Cobb-Douglas function, write the demand vector x_i^* as

$$x_i^* = \frac{1}{2} w_i^* e,$$

where $e = (e_0, e_1, ..., e_S)$. Then (x^*, p^*) is the unique equilibrium of the economy.

(b) Let $q^* = (q_1^*, ..., q_K^*)$ denote the vector of asset prices. The no-arbitrage condition then implies that

$$q_k^* = p^* \cdot a_k,$$

for $k = 1, ..., K$.

(c) We have complete markets if and only if the asset return matrix has maximum rank. If $K = S$ this implies that A is invertible. The excess demand of agent i in state s is defined by

$$z_{is} = x_{is}^* - e_{is} = \frac{1}{2} w_i^* e_s - e_{is},$$

for every $s = 1, ..., S$. Then the required portfolio must satisfy $\theta_i A = z_i$ or

$$\theta_i = z_i A^{-1}.$$

The value of this portfolio is

$$\begin{aligned} q^* \cdot \theta_i &= \sum_{k=1}^{K} q_k^* \theta_{ik} \\ &= \sum_{k=1}^{K} \theta_{ik} \sum_{s=1}^{S} p_s^* a_{ks} \\ &= \sum_{s=1}^{S} p_s^* \sum_{k=1}^{K} \theta_{ik} a_{ks} \\ &= \sum_{s=1}^{S} p_s^* z_{is} = \sum_{s=1}^{S} p_s^* \left(x_{is}^* - e_{is} \right) \leq -p_0^* (x_{i0}^* - e_{i0}), \end{aligned}$$

where the last inequality follows from the A-D budget constraint.

3. Imagine a small, open economy that has a 2×2 production sector with endowments of capital K and labor $L = 1$. The production functions $f_h(k_h)$ satisfy the usual properties: f_h is increasing, strictly concave, and C^1 and satisfies $f_h(0) = 0$ and $\lim_{k \to 0} f_h'(k) = \infty$. Both goods can be traded on the world market at the prices $p = (p_1, p_2)$. Assume that good 1 is capital intensive.

(a) Write down the problem of maximizing the value of output [not profit!] in this economy.

(b) Write down the first-order conditions for a solution of this problem. [You may assume an interior solution.]

(c) Suppose that $f_h(k_h) = k_h^{\alpha_h}$ for $h = 1, 2$. Show that the capital labor ratio in each sector is uniquely determined given the prices p and the technologies.

(d) What is the significance of the result in (c) for factor prices in different countries?

Solution (a) The planner's problem is

$$\begin{array}{ll} \max & \lambda p_1 f_1(k_1) + (1-\lambda)\, p_2 f_2(k_2) \\ \text{s.t.} & \lambda k_1 + (1-\lambda)\, k_2 \le K \\ & k_1, k_2 \ge 0,\ 0 \le \lambda \le 1, \end{array}$$

where λ is the amount of labor in sector 1 and k_h is the capital-labor ratio in sector h.

(b) We assume that this problem has an interior solution (k_1, k_2, λ) for the given values of (p_1, p_2):

$$k_1, k_2 > 0,\ 0 < \lambda < 1.$$

Then the concavity of the objective function and the convexity of the feasible set implies that the following first-order conditions are necessary and sufficient:

$$\begin{aligned} p_1 f_1'(k_1) = \mu = p_2 f_2'(k_2) \\ p_1 f_1(k_1) - p_2 f_2(k_2) = \mu\,(k_1 - k_2)\,. \end{aligned}$$

Eliminating μ, these equations are equivalent to

$$\frac{f_1'(k_1)}{f_1(k_1) - f_1'(k_1)k_1} = \frac{f_2'(k_2)}{f_2(k_2) - f_2'(k_2)k_2}.$$

(c) Substituting for the production functions we get

$$\frac{\alpha_1 k_1^{\alpha_1 - 1}}{k_1^{\alpha_1} - \alpha_1 k_1^{\alpha_1 - 1} k_1} = \frac{\alpha_2 k_2^{\alpha_2 - 1}}{k_2^{\alpha_2} - \alpha_2 k_2^{\alpha_2 - 1} k_2}$$

which simplifies to

$$\frac{1-\alpha_1}{\alpha_1} k_1 = \frac{1-\alpha_2}{\alpha_2} k_2$$

or $k_1 = \gamma k_2$ where $\gamma = \frac{\alpha_1}{1-\alpha_1}\frac{1-\alpha_2}{\alpha_2}$. Then the first-order conditions imply that

$$p_1 \alpha_1 (\gamma k_2)^{\alpha_1 - 1} = p_2 \alpha_2 k_2^{\alpha_2 - 1}$$

or

$$k_2 = \left[\frac{p_1 \alpha_1}{p_2 \alpha_2} \gamma^{\alpha_1 - 1}\right]^{\frac{\alpha_1 - 1}{\alpha_2 - 1}}.$$

(d) The factor prices are determined by the good prices and the capital labor ratios via the relations

$$r = p_h f_h'(k_h) \text{ and } w = p_h \left(f_h(k_h) - f_h'(k_h)\, k_h\right)$$

for $h = 1, 2$. Any countries in which both goods are produced and both goods are traded at the world prices $p = (p_1, p_2)$, must have the same factor prices.