

Optimization Theory (A Brief Review)

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1 Separation theorems

A subset C of \mathbf{R}^n is **convex** if for every pair of points x_1 and x_2 in C and any scalar $0 \leq \lambda \leq 1$, the line point $\lambda x_1 + (1 - \lambda)x_2$ is contained in C .

A point x in \mathbf{R}^n is a **convex combination** of points x_1, \dots, x_k if there exists a $\delta = (\delta_1, \dots, \delta_k) \geq 0$ such that $\delta_1 + \dots + \delta_k = 1$

$$x = \delta_1 x_1 + \dots + \delta_k x_k.$$

Theorem 1 *Let C be a closed convex set and let $y \notin C$. Then there exists a vector $a \in \mathbf{R}^n$ and a number α such that*

$$a \cdot x \leq \alpha < a \cdot y, \forall x \in C.$$

Two sets X and Y are **separated** by the hyperplane $H = \{x : a \cdot x = \alpha\}$ if

$$a \cdot x \geq \alpha \geq a \cdot y, \forall x \in X, \forall y \in Y.$$

Theorem 2 *Let X and Y be two disjoint convex sets. Then there exists a hyperplane H that separates them.*

Theorem 3 *Two sets X and Y are **properly separated** by the hyperplane $H = \{x : a \cdot x = \alpha\}$ if*

$$a \cdot x \geq \alpha \geq a \cdot y, \forall x \in X, \forall y \in Y$$

*and at least one of the sets is not contained in H . Two sets X and Y are **strictly separated** by the hyperplane H if*

$$a \cdot x > \alpha > a \cdot y, \forall x \in X, \forall y \in Y.$$

*Two sets X and Y are **strongly separated** by the hyperplane H if there exists $\varepsilon > 0$ such that $X + \varepsilon B(0, 1)$ and $Y + \varepsilon B(0, 1)$ are strictly separated by H . Use these definitions to prove the following results.*

Theorem 4 *The following statements are equivalent:*

- (i) *Two sets are strongly separated by a hyperplane H .*
- (ii) *There exists an $\eta > 0$ such that $a \cdot x > \alpha + \eta$ and $a \cdot y < \alpha - \eta$, for any $x \in X$ and $y \in Y$.*
- (iii) *There exists $\eta' > 0$ such that $\inf\{a \cdot x : x \in X\} \geq \alpha + \eta'$ and $\sup\{a \cdot y : y \in Y\} \leq \alpha - \eta'$.*

Theorem 5 *Let C be a closed convex set and y a vector such that $y \notin C$. Then there exists a hyperplane H that strongly separates y and C .*

Theorem 6 *Let X and Y be two convex sets such that $\text{int}(X) \neq \emptyset$ and $\text{int}(X) \cap Y = \emptyset$. Then there exists a hyperplane H that properly separates X from Y .*

Theorem 7 *Let K be a convex compact set and let C be a closed convex set such that K and C are disjoint. Then K and C can be strongly separated.*

Theorem 8 *Let A be a set contained in some half space. Then the closure of the convex hull of A , $\overline{\text{co}(A)}$ is the intersection of all the closed half spaces containing A .*

A hyperplane $H = \{x : a \cdot x = \alpha\}$ is said to be a **supporting hyperplane** to a set S if, for every $x \in S$, $a \cdot x \leq \alpha$ and there exists at least one point $x_0 \in S$ such that $a \cdot x_0 = \alpha$. The hyperplane H is said to support S at x_0 . The hyperplane is a **nontrivial supporting hyperplane** if there exists a point x_1 in S such that $a \cdot x_1 < \alpha$.

Theorem 9 *Let C be a convex set and let z be a boundary point of C . Then there exists a supporting hyperplane H to C such that $z \in H$. If $\text{int}(C) \neq \emptyset$, the supporting hyperplane is nontrivial.*

2 The Kuhn Tucker theorem

Definition 10 PROBLEM PI *Let X_0 be an open convex set in \mathbf{R}^n . Let f , and g be C^1 functions with domain X_0 and ranges in \mathbf{R} and \mathbf{R}^m , respectively. Let*

$$X = \{x \in X_0 | g(x) \leq 0\}.$$

Minimize f over X .

Definition 11 *The functions g satisfies the constraint qualification CQ at a feasible point x_0 if*

the system $\nabla g_E(x_0)z \ll 0$, has a solution z in \mathbf{R}^n .

Here $E = \{i : g_i(x_0) = 0\}$.

Theorem 12 (Necessity) Let x^* be a solution to the problem PI and let CQ hold at x^* . Then there exists a vector $\lambda \geq 0$ in \mathbf{R}^m such that

$$\begin{aligned} (i) \quad & \lambda \cdot g(x^*) = 0 \\ (ii) \quad & \nabla f(x^*) + \lambda^t \nabla g(x^*) + \mu^t \nabla h(x^*) = 0. \end{aligned}$$

Corollary 13 Let x^* be a solution of problem PI such that the set $E = \{i : g_i(x_0) = 0\}$ is $\{1, \dots, r\}$ and such that the vectors

$$\nabla g_1(x^*), \dots, \nabla g_r(x^*)$$

are linearly independent. Then the conclusion of Theorem 12 holds.

Theorem 14 (Sufficiency) Let f and g be as in the statement of problem PI and let X_0 , f , and g be convex. Let $x^* \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}^m$ be such that

$$\begin{aligned} (i) \quad & g(x^*) \leq 0, \\ (ii) \quad & \lambda \geq 0, \\ (iii) \quad & \lambda \cdot g(x^*) = 0 \\ (iv) \quad & \nabla f(x^*) + \lambda^t \nabla g(x^*) = 0. \end{aligned}$$

Then x^* is a solution of problem PI.

3 Differentiable convex functions

Theorem 15 Let C be an open convex set in \mathbf{R}^n and let f be real-valued and differentiable on C . Then f is convex if and only if for each x_0 in C

$$f(x) \geq f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

for all x in C . Also, f is strictly convex if and only if for each x_0 in C and all $x \neq x_0$ in C

$$f(x) > f(x_0) + \nabla f(x_0) \cdot (x - x_0).$$

Theorem 16 Let f be of class $C^{(2)}$ on an open convex set D . Then f is convex on D if and only if the Hessian matrix

$$H(x) = \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(x) \right)$$

is positive semidefinite at each point x in D . If $H(x)$ is positive definite at each x then f is strictly convex.

4 Existence and continuity

A set K in \mathbf{R}^n is said to be **compact** if it is closed and bounded.

Theorem 17 WEIERSTRASS. *A continuous function on a compact subset K of \mathbf{R}^n achieves a maximum and a minimum on K .*

Suppose there are ℓ commodities, so the **consumption set** (the set of commodity bundles that is feasible for the consumer) is \mathbf{R}_+^ℓ . A consumer's preferences on X are represented by a **utility function**, that is, a real-valued continuous function $u : \mathbf{R}_+^\ell \rightarrow \mathbf{R}$ with the interpretation that for any two commodity bundles x_0 and x_1 in X , x_0 is (weakly) preferred to x_1 if and only if $u(x_0) \geq u(x_1)$. Commodities are purchased on a competitive market. Let $p \neq 0$ denote the price vector at which commodities can be purchased. A consumer is endowed either with an income w to spend on commodities or with an **endowment** of commodities denoted by $e \in \mathbf{R}_+^\ell$, in which case his income is $w = p \cdot e$. In either case, we denote his **budget set** by $B(p, w)$ and define it by

$$B(p, w) = \{x \in X \mid p \cdot x \leq w\}.$$

The consumer's problem is to find the bundle(s) in $B(p, w)$ that maximize $u(x)$.

Proposition 18 *For any $p \neq 0$ and any $w \geq 0$, $B(p, w)$ is closed and non-empty. If, in addition, $B(p, w)$ is bounded, then $B(p, w)$ is compact.*

From the Weierstrass theorem, we know that if $B(p, w)$ is compact and non-empty, $u(x)$ achieves at least one maximum on $B(p, w)$. Let $\varphi(p, w)$ denote the set of maxima of $u(x)$ for each (p, w) and let D denote the set of (p, w) for which $\varphi(p, w)$ is non-empty. Note that D is not necessarily closed. We call the mapping $\varphi : D \rightrightarrows X$ a **correspondence** (set-valued function). The **graph** of φ is denoted by G_φ and defined by

$$G_\varphi = \{(x, p, w) \in X \times D \mid x \in \varphi(p, w)\}.$$

Correspondences have various continuity properties. A particularly useful one is the **closed graph property**.

Proposition 19 *Let $\{x^q, p^q, w^q\}$ be a sequence in G_φ , converging to a point (x^0, p^0, w^0) in $X \times D$. Suppose that x^0 belongs to the closure of the interior of X and $p^0 \cdot x^0 > 0$. Then $(x^0, p^0, w^0) \in G_\varphi$.*

If $\varphi(p, w)$ is a singleton everywhere on D , we denote the single maximizer by $f(p, w)$ and call $f(p, w)$ a demand function.

Proposition 20 *Let $\{p^q, w^q\}$ be a sequence in D , converging to a point (p^0, w^0) in D . Suppose that $p^0 \cdot f(p^0, w^0) > 0$. Then $f(p^q, w^q) \rightarrow f(p^0, w^0)$.*