Fundamental Theorems of Welfare Economics

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1 Pure exchange economies

An exchange economy is defined by the commodities, $h = 1, ..., \ell$ and the **consumers** $i = 1, ..., m$, where each consumer i is characterized by

- a **consumption set** $X_i \subset \mathbf{R}^{\ell}_{+}$;
- an **endowment** $e_i \in X_i$;
- and a **utility function** $u_i : X_i \to \mathbf{R}$.

The m-tuple $\mathcal{E} = \{(X_i, e_i, u_i)\}_{i=1}^{m}$ is called a pure **exchange economy**.

2 Efficiency

An **allocation** is an array $x = (x_1, ..., x_m)$ such that $x_i \in X_i$ for $i = 1, ..., m$. An allocation x is **attainable** if

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i.$$

An attainable allocation x is said to be (**weakly**) **Pareto efficient** if there does not exist an attainable allocation y such that $u_i(y_i) > u_i(x_i)$, for every $i = 1, ..., m$.

An attainable allocation x is said to be (**strongly**) **Pareto efficient** if there does not exist an attainable allocation y such that $u_i(y_i) \geq u_i(x_i)$, for every $i = 1, ..., m$, and $u_i(y_i) > u_i(x_i)$ for at least one i.

2.1 Negishi's Theorem

In this section, we assume that u_i satisfies the following properties:

- for some open set $G \supset X_i \equiv \mathbf{R}^\ell_+$, $u_i : G \to \mathbf{R}$ is C^1;
- the function u_i is **increasing** on $\mathbf{R}^\ell_+$, that is, $x > y$ implies $u_i(x) > u_i(y)$ for any x and y in $\mathbf{R}^\ell_+$;
- the function u_i is **concave** on $\mathbf{R}^\ell_+$, that is, for any x and y in $\mathbf{R}^\ell_+$ and any t in $[0, 1]$,

  $$u_i(tx + (1-t)y) \geq tu_i(x) + (1-t)u_i(y).$$

  To avoid trivial cases we also assume that

- there is a positive endowment of every good in the economy

  $$\sum_{i=1}^{m} e_i \gg 0.$$

The **utility possibility set** is denoted by U and defined to be the set of utility profiles $\bar{u} = (\bar{u}_1, ..., \bar{u}_m) \in \mathbf{R}^m$ such that for some attainable allocation x and every $i = 1, ..., m$,

$$\bar{u}_i \leq u_i(x_i).$$

Proposition 1 *Under the maintained assumptions, U is convex, closed and bounded above.*

An equivalent definition of weak Pareto efficiency can be given in terms of the utility possibility set:

Proposition 2 *An attainable allocation x is weakly Pareto efficient if and only if $\bar{u}$ is a boundary point of U, where*

$$\bar{u} = u(x) \equiv (u_1(x_1), ..., u_m(x_m)).$$

Theorem 3 SUPPORTING HYPERPLANE THEOREM *If $C \subset \mathbf{R}^\ell$ is a closed, convex set and x^* belongs to the boundary of C, there exists a non-zero vector $a \in \mathbf{R}^\ell$ with the property that*

$$a \cdot x \leq a \cdot x^*, \ \forall x \in C.$$

In other words, C lies in the half space defined by the hyperplane $H = \{x : a \cdot x = a \cdot x^*\}$.

Suppose that x^* is a weakly Pareto-efficient allocation. Then $\bar{u}^* = u(x^*)$ belongs to the boundary of U and there exists a non-zero vector a such that

$$a \cdot \bar{u} \leq a \cdot \bar{u}^*, \ \forall \bar{u} \in U.$$

From the definition of the utility possibility set, we must have $a > 0$. Otherwise, $a_i < 0$ for some i and, by choosing $\bar{u}_i$ to be negative and very large in absolute value, we could contradict the inequality above. From the same inequality, it follows that for any attainable allocation x

$$a \cdot u(x) \leq a \cdot u(x^*). \tag{1}$$

Conversely, if x^* is an attainable allocation and (1) holds for all attainable allocations x, then it is easy to see that x^* is weakly Pareto-efficient.

We have established the following theorem demonstrating the very close relationship between the Pareto-optimal allocations and the solutions of a particular optimization problem.

Theorem 4 (NEGISHI) *Under the maintained assumptions, an attainable allocation x^* is weakly Pareto-optimal if and only if, for some set of weights $\alpha_i \geq 0$, with $\alpha_i > 0$ for some i, x^* maximizes the weighted sum of utilities*

$$\sum_{i=1}^{m} \alpha_i u_i(x_i)$$

subject to the constraints

$$\sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} e_i$$

and

$$x_i \geq 0, \quad i = 1, ..., m.$$

Note that we can assume that $\alpha_i > 0$ without loss of generality if $x_i^* > 0$.

3 Walrasian equilibrium

A (**Walrasian**) **equilibrium** consists of an attainable allocation x^* and a price vector $p^* \neq 0$ such that, for every $i = 1, ..., m$, x_i^* maximizes u_i on the budget set

$$B_i(p^*) = \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\}.$$

We call x a **Walras allocation** if (x, p) is a Walrasian equilibrium, for some price vector $p \neq 0$.

Theorem 5 *A Walras allocation is weakly Pareto efficient.*

Proof. Let (x, p) be an equilibrium and suppose, contrary to what we want to prove, that $\hat{x}$ is attainable and is strictly preferred to x by every agent i. Then $p \cdot \hat{x}_i > p \cdot e_i$ for every i, so $\sum_i p \cdot \hat{x}_i > \sum_i p \cdot e_i$, contradicting $\sum_i \hat{x}_i = \sum_i e_i$. ■

Example 6 *The Edgeworth Box with "thick" indifference curves provides an example of an equilibrium that is weakly but not strongly Pareto efficient. There are two commodities* $h = 1, 2$ *and two agents* $i = 1, 2$ *with consumption sets* $\mathbf{R}_+^2$, *endowments* $e_i = (1, 1)$ *for* $i = 1, 2$ *and utility functions* u_i *for* $i = 1, 2$. *Suppose that* U_1 *is a constant function and* U_2 *is an increasing, concave,* C^1 *function. Then the attainable allocation* $(x_1, x_2) = (e_1, e_2)$ *is an equilibrium allocation for the prices* $p = \nabla U_2(1, 1)$ *but it is clearly not efficient in the strong sense.*

Say that agent i is **locally non-satiable** if, for any point x_i in the consumption set X_i and any $\varepsilon > 0$ there is a consumption bundle $x_i' \in X_i$ such that $\|x_i' - x_i\| < \varepsilon$, $x_i' \gg x_i$, and $u_i(x_i') > u_i(x_i)$.

Theorem 7 *A Walras allocation* x *is strongly Pareto efficient if every agent is locally non-satiable.*

Proof. Let (x, p) be an equilibrium and suppose, contrary to what we want to prove, that $\hat{x}$ is attainable and is weakly preferred to x by every agent i and strictly preferred by some agent i. Local non-satiability implies that $p \cdot \hat{x}_i \geq p \cdot e_i$ for every i and the inequality is strict for some i, so $\sum_i p \cdot \hat{x}_i > \sum_i p \cdot e_i$, contradicting $\sum_i \hat{x}_i = \sum_i e_i$. ■

4 Decentralization

An **equilibrium with lump sum transfers** consists of an attainable allocation x^*, a price vector $p^* \neq 0$, and an array of transfers $t = (t_1^*, ..., t_m^*)$ such that, for every agent $i = 1, ..., m$, x_i^* maximizes $u_i(x_i)$ in the budget set

$$B_i(p^*, t_i^*) = \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i - t_i\}.$$

and

$$\sum_{i=1}^{m} t_i^* = 0.$$

Let x be an attainable allocation. We say that x can be **decentralized** if there exists a price vector $p \neq 0$ such that, for every $i = 1, ..., m$,

$$u_i(y_i) > u_i(x_i) \Longrightarrow p \cdot y_i > p \cdot x_i.$$

If an allocation can be decentralized, there exists a price vector p such that every agent i is willing to choose the consumption bundle x_i subject to some budget constraint at the prices p. This means that the allocation can be made into an equilibrium through an appropriate redistribution of income.

In fact, it is clear that *an attainable allocation x^* can be decentralized if and only if there exists a price vector $p^* \neq 0$ and lump-sum taxes and transfers $t^* = (t_1^*, ..., t_m^*)$ such that (x^*, p^*, t^*) is an equilibrium with lump-sum taxes and transfers.*

What we will show next is that all efficient allocations can be decentralized under certain conditions. In other words, the market can be used to attain any efficient allocation.

In order to simplify the arguments, we make a couple of special assumptions that are stronger than necessary.

- X_i is an open set for $i = 1, ..., m$.
- u_i is continuous, quasi-concave, and locally non-satiable for $i = 1, ..., m$.

Theorem 8 *Under the maintained assumptions, every weakly Pareto efficient allocation x^* is decentralizable.*

We shall prove the theorem by establishing a series of claims. Suppose that x^* is a weakly Pareto-efficient allocation and let $P_i(x_i^*)$ denote the set of points that is preferred to x_i^* by agent i, that is,

$$P_i(x_i^*) = \{y_i \in X_i : u_i(y_i) > u_i(x_i^*)\},$$

for every $i = 1, ..., m$.

Claim 9 *$P_i(x_i^*)$ is a non-empty, open and convex set, for $i = 1, ..., m$.*

The fact that $P_i(x_i^*)$ is non-empty follows from the local non-satiability of u_i and the fact that $x_i^* \in X_i$. The openness of $P_i(x_i^*)$ follows from the openness of X_i and the continuity of u_i. Convexity of $P_i(x_i^*)$ follows from the quasi-concavity of u_i and convexity of X_i.

Now, let Z_i be the set of vectors that point from x_i^* into the preferred set $P_i(x_i^*)$, that is, define Z_i by putting

$$Z_i = P_i(x_i^*) - \{x_i^*\}$$

for each $i = 1, ..., m$ and let $Z = \sum_i Z_i$ be the sum of the sets $\{Z_i\}$.

Claim 10 *Z_i is non-empty, open and convex for $i = 1, ..., m$ and so is Z.*

These properties obviously follow from the corresponding properties for $P_i\left(x_i^*\right)$.

Claim 11 *$0 \notin Z$.*

This must be true if x^* is strongly Pareto efficient. To see this, suppose that, contrary to the claim, $0 \in Z$. From the definition of Z, this means that there must be vectors $\{z_i\}_{i=1}^m$ such that $z_i \in Z_i$ and $\sum_{i=1}^m z_i = 0$. Let $x_i \equiv x_i^* + z_i$ for $i = 1, ..., m$. By construction, we know that $x_i \in X_i$ and $u_i(x_i) > u_i\left(x_i^*\right)$ for every $i = 1, ..., m$, and

$$\sum_{i=1}^m x_i = \sum_{i=1}^m \left(x_i^* + z_i\right) = \sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i$$

because x^* is attainable. Hence, there exists an attainable allocation x that every agent i prefers to x^*, contradicting the weak Pareto efficiency of x^*. This contradiction proves the claim.

In order to complete the proof that x^* can be decentralized we shall the need the following result, which is a version of the Minkowski lemma.

Lemma 12 *Let S be a non-empty, open and convex set and suppose that $0 \notin S$. Then there exists a vector $p \neq 0$ such that $0 < p \cdot x$ for any $x \in S$.*

Claim 13 *There exists a price vector* $p^* \neq 0$ *such that* $p^* \cdot z > 0$ *for any* $z \in Z$*.*

This follows directly from the Minkowski lemma, since the set Z satisfies the conditions of the set S in the statement of the lemma.

Claim 14 *For any* $i = 1, ..., m$ *and any* $z_i \in Z_i$*,* $p^* \cdot z_i > 0$*.*

Suppose, contrary to what we want to prove, that $p^* \cdot z_i \leq 0$ for some i and $z_i \in Z_i$. Since Z_i is open, we can assume that $p^* \cdot z_i < 0$. For every $j \neq i$, the local non-satiability of U_j implies that 0 is a limit point of Z_j so we can find a sequence z_j^n in Z_j converging to 0. Let $z^n = z_i + \sum_{j \neq i} z_j^n$ for each value of n. Then $p^* \cdot z^n > 0$ for every n and in the limit,

$$\lim_{n \to \infty} p^* \cdot z^n = p^* \cdot \lim_{n \to \infty} z^n = p^* \cdot z_i \geq 0,$$

a contradiction. This proves the claim.

Claim 15 *For any* $i = 1, ..., m$ *and any* $x_i \in X_i$*,* $u_i(x_i) > U(x_i^*)$ *implies* $p^* \cdot x_i > p^* \cdot x_i^*$*.*

This is simply a restatement of the previous claim with $z_i = x_i - x_i^*$. It shows that p^* is the price vector that decentralizes the weakly Pareto efficient allocation x^*.

The proof of the theorem is now complete.

5 Quasi-equilibrium

We can show, by example, that the Second Fundamental Theorem fails if assumptions X_i is not open or u_i is not continuous, quasi-concave, and l.n.s.

The openness of X_i is restrictive, because it assumes that the indifference curve through x_i does not intersect the boundary of the consumption set X_i. We can weaken this assumption, but only if we add the assumption of **resource relatedness**.

A **quasi-equilibrium** consists of an attainable allocation x^* and a price vector $p^* \neq 0$ such that, for every $i = 1, ..., I$ and every $x_i \in X_i$,

$$U_i(x_i) \geq U_i(x_i^*) \Longrightarrow p^* \cdot x_i \geq p^* \cdot x_i^*.$$

Theorem 16 *If x is a weakly efficient allocation and*

$$\begin{array}{ll} (i) & P_i(x_i) \text{ is convex;} \\ (ii) & x_i \in \overline{P_i(x_i)}; \end{array}$$

for every $i = 1, ..., I$, then there exists a price vector $p \neq 0$ such that (x, p) is a quasi-equilibrium.

Proof. Let $Z_i = P_i(x_i) - \{x_i\}$ for each i and let $Z = \sum_i Z_i$. Then Z is nonempty and convex and $0 \notin Z$ because x is weakly efficient. By the supporting hyperplane theorem, there exists a vector $p \neq 0$ such that $p \cdot z \geq 0$, for any $z \in Z$. Since $0 \in \bar{Z}_i$ for all i, this inequality implies that $p \cdot z_i \geq 0$ for any $z_i \in Z_i$, as required. ■

If (x, p) is a quasi-equilibrium, it does not necessarily follow that x can be decentralized. The definition of quasi-equilibrium ensures that x_i minimizes the cost of achieving $U_i(x_i)$; but this is not the same a maximizing utility subject to the budget $p \cdot x_i$. To show that a quasi-equilibrium can be decentralized, we need to assume that there is a cheaper point in the budget set.

Proposition 17 *Let (x, p) be a quasi-equilibrium and assume that, for every $i = 1, ..., I$,*

$$\begin{array}{ll} (i) & X_i \text{ is convex;} \\ (ii) & U_i \text{ is continuous and l.n.s.} \\ (iii) & p \cdot x_i > \inf p \cdot X_i. \end{array}$$

Then x can be decentralized using the price vector p and transfers t defined by $t_i = p \cdot (x_i - e_i)$ for $i = 1, ..., I$.

5.1 Irreducibility

An exchange economy $\mathcal{E}$ is called **resource-related** or **irreducible** if, for any non-trivial partition $\{I, J\}$ of the agents $i = 1, ..., I$, and any attainable allocation x, there is an allocation x' such that $U_i(x'_i) > U_i(x_i)$ for any $i \in I$ where $\sum_{i \in I} (x'_i - x_i) = -\alpha w$ for some $\alpha > 0$ and $w \in \sum_{j \in J} X_j - \{e_j\}$.

Proposition 18 *Let (p, x) be a quasi-equilibrium and suppose that the aggregate endowment for the economy $\sum_i e_i$ belongs to the interior of the sum of the consumption sets $\sum_i X_i$ and that the economy is resource-related. Then for every $i = 1, ..., I$,*

$$p \cdot x_i > \inf p \cdot X_i.$$

Proof. Because of our assumption about the aggregate endowment,

$$p \cdot \sum_i x_i = p \cdot \sum_i e_i > \inf p \cdot \sum_i X_i,$$

which implies that $p \cdot x_i > \inf p \cdot X_i$ for at least one i. Let I denote the set of agents for whom this inequality is valid and let J denote the complement of I. If J is empty, we are done. Otherwise, there exists an allocation x' such that $U_i(x_i') > U_i(x_i)$ for every $i \in I$, where $\sum_{i \in I} (x_i' - x_i) = -\alpha w$. By the argument in the previous proposition, we can see that $p \cdot x_i' > p \cdot x_i$ for every $i \in I$. Then $p \cdot w < 0$, from which it follows that $p \cdot e_j > \inf p \cdot X_j$ for some $j \in J$, a contradiction. This proves that $J = \varnothing$. ■