Uncertainty and Incomplete Markets

November 12, 2007

1 Contingent commodities and the Arrow-Debreu model

Commodities can be distinguished by physical characteristics and by the time and location at which they are to be delivered. They can also be distinguished by the uncertain events on which their delivery is contingent. Uncertainty about endowments, technologies, and preferences are represented by a finite set of **states of nature**.

Definition 1 *If there is a finite number of physically distinct commodities, indexed by $h = 1, ..., \ell$, and a finite number of states of nature, indexed by $s = 1, ..., S$, a unit of the* ***(state-)contingent commodity*** (h, s) *is a title to receive one unit of the physical commodity h if state s occurs and nothing otherwise. A* ***(state-)contingent commodity bundle*** *is an ℓS-tuple*

$$x = (x_{11}, ..., x_{1\ell}, x_{21}, ..., x_{S1}, ..., x_{S\ell}) \in \mathbf{R}^{S\ell},$$

where x_{sh} is the quantity of contingent commodity (s, h) contained in the bundle. If state s occurs the owner of x is entitled to receive the physical commodity bundle $(x_{s1}, ..., x_{s\ell})$.

The definitions of exchange economy, economy with production, and Walrasian equilibrium are essentially the same as before. The only difference is that the commodity space is now $\mathbf{R}^{S\ell}$ rather than $\mathbf{R}^{\ell}$. This illustrates the power of the Walrasian approach to general equilibrium: it allows us to deal with the dimensions of time, space, and uncertainty simply by reinterpreting the commodity space. All of the theorems apply in the new environment; only the interpretation is different.

For the record, we restate the definition of an exchange economy and Walrasian equilibrium for a world with contingent commodities. The (contingent) **commodity space** is $\mathbf{R}^{S\ell}$. There are m **agents** (consumers) indexed $i = 1, ..., m$. Each agent i is characterized by

- a closed, nonempty **consumption set** $X_i \subset \mathbf{R}^{S\ell}$;
- an **endowment** $e_i \in X_i$;

- a **utility function** $u_i : X_i \to \mathbf{R}$.

The m-tuple $\mathcal{E} = \{(X_i, e_i, u_i)\}_{i=1}^m$ is called an **Arrow-Debreu exchange economy**.

An **allocation** is an array $x = (x_1, ..., x_m)$ such that $x_i \in X_i$ for $i = 1, ..., m$. An allocation x is **attainable** if

$$\sum_{i=1}^m x_i = \sum_{i=1}^m e_i.$$

An (**Arrow-Debreu**) **equilibrium** consists of an attainable allocation x^* and a price vector $p^* \neq 0$ such that, for every $i = 1, ..., m$, x_i^* maximizes u_i on the budget set

$$B_i(p^*) = \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\}.$$

It will be useful to have notation for the commodity bundles and price vectors corresponding to each state. Let

$$e_{is} = (e_{is1}, ..., e_{is\ell}), \; x_{is} = (x_{is1}, ..., x_{is\ell}), \; p_s = (p_{s1}, ..., p_{s\ell})$$

denote, respectively, the endowment vector of consumer i in state s, the consumption bundle of consumer i in state s, and the price vector of the physical commodities in state s.

2 Arrow securities

The power of the Arrow-Debreu model rests on the assumption that markets are **complete**, that is, there exists a market on which every commodity can be traded before the world begins. The complete markets assumption is obviously restrictive because it requires an unreasonably large number of markets. The number of markets required can be reduced, however, by the strategy of sequential trade together with **Arrow securities**. An Arrow security is a promise to deliver one unit of account in a particular state s. Suppose there is a full set of Arrow securities (one for each state) and these are traded at some time before the true state is revealed. Agent i purchases a vector $z_i \in \mathbf{R}^S$ of securities subject to the budget constraint $q \cdot z_i \leq 0$. If the true state is revealed to be s, agent i faces the budget constraint

$$p_s \cdot x_{is} \leq p_s \cdot e_{is} + z_{is}.$$

An allocation of securities $z = (z_i)$ is **attainable** if

$$\sum_{i=1}^m z_i = 0.$$

Definition 2 *An* ***equilibrium with Arrow securities*** *is an attainable allocation of commodities x, an attainable allocation of securities z, and a pair of* *price vectors* (p,q) *such that, for every* $i=1,...,m$, (x_i,z_i) *maximizes* u_i *on the budget set defined by the conditions*

$$\begin{array}{ll} (i) & x_i \in X_i, \\ (ii) & q\cdot z_i \leq 0, \\ (iii) & p_s\cdot x_{is} \leq p_s\cdot e_{is} + z_{is},\ \textit{for every}\ s=1,...,S. \end{array}$$

Theorem 3 *Suppose that* (x,p) *is an Arrow-Debreu equilibrium. Then there exists an equilibrium with Arrow securities* (x,z,p,q)*. Conversely, if* (x,z,p,q) *is an equilibrium with Arrow securities, then* (x,p') *is an Arrow-Debreu equilibrium, where* p' *is defined by*

$$p'_s = q_s p_s,\ \textit{for every}\ s=1,...,S.$$

Proof. Let (x,p) be an Arrow-Debreu equilibrium. Define (x,z,p,q) by putting $q_s=1$ and

$$z_{is} = p_s\cdot(x_{is}-e_{is}), \forall i, \forall s.$$

Then it is clear, from the construction and the definition of Arrow-Debreu equilibrium, that $q\cdot z_i \leq 0$ and $\sum_i z_i = 0$. Furthermore, if (x'_i, z'_i) satisfies the budget constraints for the equilibrium with Arrow securities, then it is easy to see that $p\cdot x'_i \leq p\cdot e_i$. Since x_i is optimal in the Arrow-Debreu budget set, (x_i,z_i) must be optimal in the budget set for the equilibrium with Arrow securities. This proves that (x,z,p,q) is an equilibrium with Arrow securities.

Conversely, suppose that (x,z,p,q) is an equilibrium with Arrow securities. Then x is attainable and satisfies the budget constraint:

$$\begin{aligned} 0 &\geq \sum_s q_s(p_s\cdot(x_{is}-e_{is}) - z_{is}) \\ &= \sum_s q_s p_s\cdot(x_{is}-e_{is}) - \sum_s q_s z_{is} \\ &\geq \sum_s q_s p_s\cdot(x_{is}-e_{is}), \end{aligned}$$

for all i, because $\sum_s q_s z_{is} \leq 0$. Defining prices by $p'_s = q_s p_s$ for $s=1,...,S$, (x,p') will be an Arrow-Debreu equilibrium if we can show that x_i maximizes utility in the budget set $B_i(p', p'\cdot e_i)$ for every i. Suppose that x'_i lies in the budget set $B_i(p',p'\cdot e_i)$ and define z'_i by putting $z'_{is} = p_s\cdot(x'_{is}-e_{is})$ for $s=1,...,S$. Then (x'_i,z'_i) satisfies the budget constraints for the equilibrium with Arrow securities and the conditions for equilibrium with Arrow securities then imply that $u_i(x'_i) \leq u_i(x_i)$ as required. ■

3 Arbitrage pricing

A **security** is a claim to a commodity bundle. Trading securities is equivalent to exchanging the corresponding bundles. Suppose there is a finite number of securities, indexed by $j=1,...,n$, and let y_j denote the commodity to which one unit of security j is a claim. A **portfolio** is represented by a vector $\theta = (\theta_1, ..., \theta_j, ..., \theta_n) \in \mathbf{R}^n$, where θ_j denotes the number of units of security j in the portfolio. A portfolio θ is a claim to a commodity bundle, namely the bundle

$$x = \sum_{j=1}^{n} \theta_j y_j.$$

The set of commodity bundles that can be generated by some portfolio is simply the span of $\{y_1, ..., y_n\}$, which we denote by $X = \left\{ x \,\middle|\, x = \sum_{j=1}^{n} \theta_j y_j, \exists \theta \right\}$.

A security is said to be **redundant** if it can be expressed as a linear combination of other securities. Suppose that $\dim X = k < n$. Then there exists a set of k securities that form a basis for X. With an abuse of notation we relabel the securities so that the basis is $\{y_1, ..., y_k\}$. Then the securities $\{y_{k+1}, ..., y_n\}$ are all redundant.

Suppose that the securities can be traded on a competitive market. Let q_j denote the price of security j and $q = (q_1, ..., q_n)$ the vector of security prices. The value of a portfolio is $q \cdot \theta = \sum_{j=1}^{n} q_j \theta_j$. We say that an **arbitrage** exists at the price vector q if there is a portfolio θ such that $\sum_{j=1}^{n} \theta_j y_j = 0$ and $q \cdot \theta > 0$. Otherwise, q satisfies the **no-arbitrage condition**.

Theorem 4 No-Arbitrage Condition*: The security price vector q satisfies the no-arbitrage condition if and only if there is a vector $p \in \mathbf{R}^\ell$ such that $q_j = p \cdot y_j$ for every $j = 1, ..., n$.*

Suppose that securities $\{y_1, ..., y_k\}$ form a basis for X, so that the securities $y_{k+1}, ..., y_n$ are redundant. Then

$$y_i = \sum_{j=1}^{k} \alpha_{ij} y_j$$

for each $i = k+1, ..., n$. Suppose further that the no-arbitrage condition is satisfied. Then the price vector q satisfies

$$q_i = \sum_{j=1}^{k} \alpha_{ij} q_j,$$

for $i = k+1, ..., n$. Conversely, if this condition is not satisfied for some $i > k$ then there exists an arbitrage.

4 The binomial model

The arbitrage pricing principles developed above show how prices of redundant securities can be recovered once the prices of a basic set of securities is known. For practical applications it is useful to have additional structure that can be used to make calculations based on readily available market data. There are many such models used by financial professionals. One simple but useful model, known as the **binomial mode**l, will be used as an example here.

There is a finite number of dates indexed $t = 0, 1, ..., T$ and a single share with initial price S_0. At each date after the first, the share price S_t can take two values conditional on the share price at the previous date. Either the price is $S_t = uS_{t-1}$ or the price is $S_t = dS_{t-1}$, where $u > 1 > d$. A graph with these properties is illustrated in Figure 1.

— Figure 1 here —

The graph illustrated above is called a **recombining** graph, because paths merge or "recombine" after separating at an earlier date. A path is completed defined by the sequence of edges $(0, e_1, e_2, ..., e_T)$ that compose it so we can think of a typical path as being made up of a sequence like $(0, u, u, d, u, d, d, u, u, d)$. Normally, we identify states of nature with the entire path that leads to the terminal node, but in this model we are only interested in the share price and derivatives that are functions of the share price. For this reason, we treat paths that correspond to the same terminal share price as constituting a single state. Now the terminal share price S_T depends only on the number of "up" and "down" terms in a path and not on the order. If there are n "ups" and $T - n$ "downs," the terminal share price will be

$$S(n) = S_0 u^n d^{T-n}.$$

So we can identify a state of nature with the set of paths that have the same number of u and d terms.

We also assume that there is a one-period bond traded at each date and to make things very simple I will assume that the interest rate on the bond is zero. Thus, the price of the bond is always constant and equal to one. We have not said anything about the probability that the share price increases or decreases at each date, nor is there any need to do so. All the relevant information is contained in the market price of the share, more precisely, in the relationship between S_t and the possible values of S_{t+1}.

4.1 Contingent claims prices

Now suppose that we have reached some date at which the current price is S. Suppose that we want to synthesize a security that pays one unit if the share price rises next period and nothing if the share price falls. Let Δ denote the amount of stock that we hold and let B denote the amount of the bond that we hold. The value of the portfolio we hold will be worth $uS\Delta + B$ if the share price rises and $dS\Delta + B$ if the share price falls. Then we must have

$$uS\Delta + B = 1$$

if the share price rises and

$$dS\Delta + B = 0$$

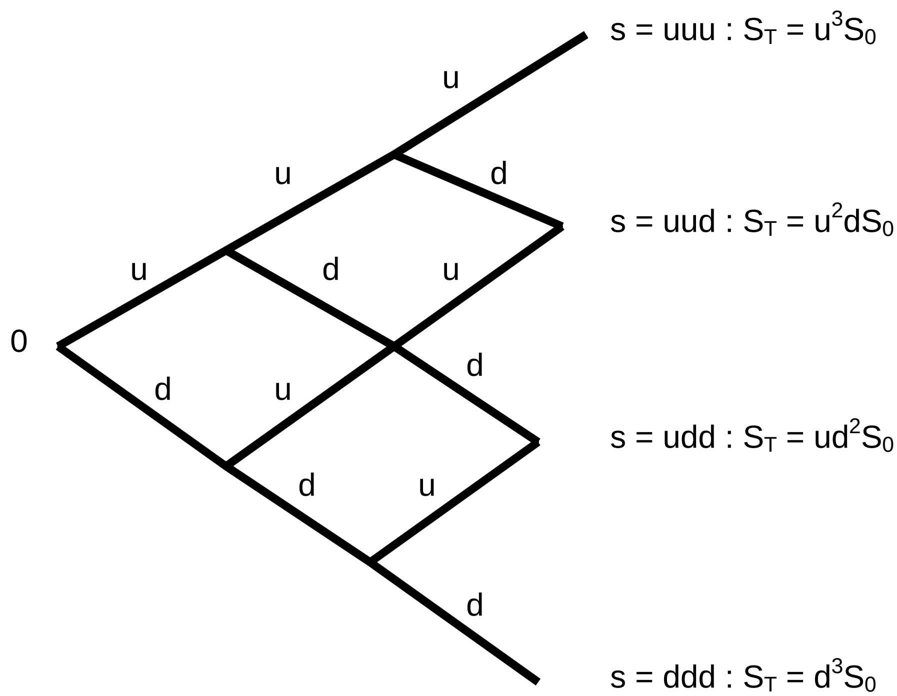

Figure 1
A Binomial Graph with Four Periods

if it falls. We can solve these simultaneous equations to get the values of Δ and B:

$$\Delta = \frac{1}{(u-d)\,S},\;\; B = -\frac{d}{u-d}.$$

The value of the portfolio (Δ, B) in the current period is denoted by p and defined by

$$p = \Delta S + B = \frac{1}{(u-d)\,S} S - \frac{d}{u-d} = \frac{1-d}{u-d}.$$

Here we can interpret p as the price of the contingent claim that pays one unit next period if the share price rises.

In exactly the same way, we can calculate the portfolio (Δ', B') that pays one unit next period if the share price falls and nothing otherwise. This portfolio satisfies

$$uS\Delta' + B' = 0$$

and

$$dS\Delta' + B' = 1$$

and these equations can be solved to give

$$\Delta' = \frac{1}{(d-u)\,S},\;\; B' = -\frac{u}{d-u}.$$

The value of the portfolio (Δ', B') in the current period is denoted by p' and defined by

$$p' = \Delta' S + B' = \frac{1}{(d-u)\,S} S - \frac{u}{d-u} = \frac{1-u}{d-u}.$$

Note that

$$p + p' = \frac{1-d}{u-d} + \frac{1-u}{d-u} = \frac{1-d+u-1}{u-d} = \frac{u-d}{u-d} = 1.$$

Thus, the contingent claims prices can be treated as "probabilities" in what follows.

Note that the definitions of p and p' are independent of S and hence of the date. They depend only on the parameters u and d. We can use these "prices" to construct prices for other contingent claims. For example, suppose that we look two periods ahead and want a portfolio that pays one unit if the share price rises two times in succession and nothing otherwise. There is only one path that leads to the event in which the payoff is one. We can denote this path (u, u). Since the cost of buying one unit after single price rise is p, the cost of one unit after two price rises must be p^2. In general, if a path has n "ups" and $T-n$ "downs," the price of one unit at the end of this path must be $p^n\,(p')^{T-n}$. This allows us to construct prices for contingent claims for all the states (terminal nodes) of the graph. Note that states will have the same share prices if they have the same number of "ups" and "downs." Thus, there is no need to distinguish states with the same share prices as long as we are only interested in derivatives that depend only on the share price. Note also that the contingent claim prices will be the same for states that have the same number of u's and d's. Thus, in what follows we can identify a state with the number n of "ups" (the number of "downs" being $T-n$). The number of different paths of length T with n u's is $\binom{T}{n}$ and the contingent claim price is given by the **binomial probability**

$$B\left(n|T,p\right)=\binom{T}{n}p^{n}\left(1-p\right)^{T-n}.$$

4.2 Pricing a call option

We can price any derivative using these contingent claim prices. For example, consider a European call with strike price K exercisable at date T. Let $\nu\left(K\right)$ denote the smallest value of n such that the share price $S_T\left(n\right)$ is greater than K, that is,

$$\nu\left(K\right)=\min\left\{n:u^{n}d^{T-n}S_0\geq K\right\}.$$

Then the price of the option is

$$\begin{aligned}\sum_{n=0}^{T}B\left(n|T,p\right)\max\left\{0,S\left(n\right)-K\right\} &= \sum_{n=\nu(K)}^{T}B\left(n|T,p\right)\left(S\left(n\right)-K\right)\\ &= \sum_{n=\nu(K)}^{T}B\left(n|T,p\right)\left(u^{n}d^{T-n}S_0-K\right).\end{aligned}$$

We can rewrite this formula as

$$\sum_{n=\nu(K)}^{T}\binom{T}{n}p^{n}\left(p'\right)^{T-n}\left(u^{n}d^{T-n}S_0-K\right)=$$

$$S_0\sum_{n=\nu(K)}^{T}\binom{T}{n}\left(up\right)^{n}\left(dp'\right)^{T-n}-K\sum_{n=\nu(K)}^{T}B\left(n|T,p\right)$$

Now,

$$up=\frac{u\left(1-d\right)}{u-d}$$

and

$$dp'=\frac{d\left(u-1\right)}{u-d}$$

and adding these we note that

$$\begin{aligned}\frac{u\left(1-d\right)}{u-d}+\frac{d\left(u-1\right)}{u-d} &= \frac{u\left(1-d\right)+d\left(u-1\right)}{u-d}\\ &= \frac{u-d}{u-d}=1\end{aligned}$$

so we can treat up and $dp' = 1 - up$ as "probabilities." Hence, the expression $\binom{T}{n}(up)^n (dp')^{T-n}$ can be replaced by the binomial probability $B(n|T, up)$ and the formula for the option price above can be rewritten as

$$S_0 \sum_{n=\nu(K)}^{T} B(n|T, up) - K \sum_{n=\nu(K)}^{T} B(n|T, p).$$

This is the **binomial pricing formula** for the European call option.

4.3 Pricing a put option

We can use similar arguments to derive the price of a Euopean put with an exercise price K and an exercise date of T. The price of the option will be given by

$$\begin{aligned}\sum_{n=0}^{T} B(n|T,p) \max\{0, K - S(n)\} &= \sum_{n=0}^{\nu(K)-1} B(n|T,p)(K - S(n)) \\ &= \sum_{n=0}^{\nu(K)-1} B(n|T,p)\left(K - u^n d^{T-n} S_0\right).\end{aligned}$$

Then the usual re-arrangement of variables leads to the **binomial pricing formula** for the European put option.

$$K \sum_{n=0}^{\nu(K)-1} B(n|T,p) - S_0 \sum_{n=0}^{\nu(K)-1} B(n|T, up).$$

5 Assets

A generalization of the idea of Arrow securities assumes that agents can trade assets that promise delivery of a vector of commodities in each future state. Suppose there are two dates $t = 0, 1$, S states of natures $s = 1, ..., S$, and ℓ physical goods $h = 1, ..., \ell$. The true state of nature is revealed at date 1. Then the commodity space is $\mathbf{R}^{\ell(S+1)}$, because there are ℓ commodities at the first date and ℓ commodities in each of S states at the second date. Let $L = \ell(S+1)$ denote the total number of commodities. For any vector $x \in \mathbf{R}^L$ we use the notation $x = (x(0), x(1), ..., x(S))$, where $x(0) \in \mathbf{R}^\ell$ denotes the vector of commodities at date 0 and $x(s) \in \mathbf{R}^\ell$ denotes the vector of commodities at date 1 in state $s = 1, ..., S$.

Let $\mathcal{E} = \{(X_i, e_i, U_i)\}$ be an exchange economy where $X_i = \mathbf{R}^L_+$, $e_i \in X_i$ and $U_i : X_i \to \mathbf{R}$. We assume there is a finite number of assets, indexed by $k = 1, ..., K$. One unit of asset k promises to deliver a commodity bundle $a_k(s) \in \mathbf{R}^\ell$ in each state of nature $s = 1, ..., S$ at date 1. We assume that the assets are in zero net supply, that is, agents trading these assets are actually trading promises to deliver the specified bundle of commodities in each state at date 1. An **asset economy** is defined by the exchange economy $\mathcal{E}$ and the asset structure $A = \{a_k\}_{k=1}^K$.

In the Arrow-Debreu model, we assume a complete set of markets exists at date 0 so that all trade can be completed then. Here we assume that there are spot markets for goods and assets at each date. This means that agents can trade the K assets and ℓ physical commodities at date 0 and can trade the ℓ physical commodities at date 1. Let $p(0)$ denote the vector of spot prices for commodities at date 0 and $p(s)$ the vector of spot prices for commodities in state s at date 1. Let $p = (p(0), p(1), ..., p(S))$ denote the vector of commodity prices. The vector of asset prices at date 0 is denoted by $q \in \mathbf{R}^K$.

Each agent i chooses a consumption bundle $x_i \in \mathbf{R}^L_+$ and a portfolio $z_i \in \mathbf{R}^K$. An **allocation** for the asset economy consists of an ordered pair (x, z) where $x = (x_1, ..., x_m)$, $z = (z_1, ..., z_m)$, and $(x_i, z_i) \in \mathbf{R}^L_+ \times \mathbf{R}^K$ for every $i = 1, ..., m$. The allocation (x, z) is **attainable** if

$$\sum_{i=1}^m x_i = \sum_{i=1}^m e_i \text{ and } \sum_{i=1}^m z_i = 0.$$

A **Radner equilibrium** consists of an attainable allocation (x^*, z^*) and a price system (p^*, q^*) such that, for every agent i, (x_i^*, z_i^*) maximizes $U_i(x_i)$ subject to the constraints $(x_i, z_i) \in \mathbf{R}^L_+ \times \mathbf{R}^K$ and

$$p^*(0) \cdot (x_i^*(0) - e_i(0)) + q^* \cdot z_i \leq 0$$

$$p^*(s) \cdot (x_i^*(s) - e_i(s)) \leq \sum_{k=1}^K z_{ik} \left(p^*(s) \cdot a_k(s)\right), \forall s = 1, ..., S.$$

Proposition 5 *Suppose that $a_k > 0$ for every $k = 1, ..., K$ and let q^* be the asset price vector for a Radner equilibrium. Then there exist multipliers μ_s, for*

$s = 1, 2, ..., S$, such that

$$q_k^* = \sum_{s=1}^{S} \mu_s p^* (s) \cdot a_k(s),$$

for each asset $k = 1, ..., K$.

We say that the asset structure A is **complete** given the price system p^* if the rank of R is at least S, where R is the $S \times K$ matrix with element (k, s) denoted by r_{ks} and defined by

$$r_{ks} = p^* (s) \cdot a_k(s).$$

If the asset structure is complete in this sense, an equilibrium of the asset economy is equivalent to the equilibrium of an Arrow-Debreu economy and the equilibrium allocation is Pareto-efficient under the usual conditions.

Proposition 6 *Suppose that (p^*, x^*) is a Walrasian equilibrium for $\mathcal{E}$ and that the asset structure A is complete given the price system p^*. Then there exists a Radner equilibrium (p, q, x^*, z) for the corresponding asset economy. Conversely, suppose that (p, q, x^*, z) is a Radner equilibrium for the asset economy $(\mathcal{E}, A)$ and the asset structure A is complete. Then there exist multipliers $(\mu_1, ..., \mu_S) \in \mathbf{R}^S$ such that $\mu_s > 0$, for every s, and (p^*, x^*) is a Walrasian equilibrium for $\mathcal{E}$, where $p^* = (p(0), \mu_1 p(1), ..., \mu_S p(S))$.*

6 Incomplete markets

When the asset structure is incomplete and hence markets are incomplete, many properties of the Arrow-Debreu equilibrium are lost. In particular,

- a Radner equilibrium may fail to exist, even if the usual convexity assumptions are satisfied;
- a Radner equilibrium is not typically Pareto-efficient – in fact, it is not even constrained efficient;
- adding a new market to an economy in which markets are incomplete does not necessarily improve welfare – in fact, it can make everyone worse off.

Consider the special case in which there is a single physical good, that is, $\ell = 1$. Then there is no need for markets at date 1 because there is only one good to trade. In fact, consumption at date 1 is determined entirely by decisions made at date 0:

$$x_i(s) = e_i(s) + \sum_{k=1}^{S} a_k(s) z_{ik}, \ \forall s = 1, ..., S.$$

So we can write agent i's utility as a function of $x_i(0)$ and z_i:

$$U_i^*(x_i(0), z_i) = U_i\left(x_i(0), e_i(1) + a(1) \cdot z, ..., e_i(S) + a(S) \cdot z\right).$$

We can treat this economy as isomorphic to an economy in which the commodities consist of consumption at date 0 and the K assets. An attainable allocation conists of an array $\{(x_i(0), z_i)\}_{i=1}^m$, where $x_i(0) \in \mathbf{R}_+$ and $z_i \in \mathbf{R}^K$, and

$$\sum_{i=1}^m x_i(0) = \sum_{i=1}^m e_i(0) \text{ and } \sum_{i=1}^m z_i = 0.$$

An attainable allocation in this sense is said to be **constrained efficient** if there does not exist another attainable allocation $\{(\hat{x}_i(0), \hat{z}_i)\}$ such that

$$U_i^*(x_i(0), z_i) \geq U_i^*(\hat{x}_i(0), \hat{z}_i)$$

for all i with strict equality for some i. Then it is easy to see that a Radner equilibrium for this economy is constrained efficient if U_i^* is increasing in $x_i(0)$ for every i.

Unfortunately, the one-good economy is a very special case. For the general case, with $\ell > 1$, Geanakoplos and Polemarchakis have shown that a Radner equilibrium is generically constrained inefficient if the asset structure is incomplete.