# Dynamic Economies

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## 1 Intertemporal efficiency

Time is divided into an infinite sequence of discrete periods or **dates** $t = 0, 1, 2, ....$ There are assumed to be $\ell$ (non-durable) **commodities** at each date. The production is assumed to take two periods: inputs at date $t$ produce outputs at date $t+1$. The vector of inputs is generically represented by $x \in \mathbf{R}^{\ell}$ and the vector of outputs by $y \in \mathbf{R}^{\ell}$. Let $Y$ denote the production set. We typically assume that

- $Y$ is a closed, convex cone;
- no outputs are possible without inputs

$$(x, y) \in Y \text{ and } x \leq 0 \text{ implies } y \leq 0;$$

- there is free disposal of commodities

$$[(x, y) \in Y,\ x' \geq x,\ y' \leq y] \Longrightarrow (x', y') \in Y;$$

- and production plans can be truncated

$$(x, y) \in Y \Longrightarrow (x, 0) \in Y.$$

Note that this technology allows for durable goods (storage) and capital goods (a special kind of commodity that appears as input and output).

A **production path** or **trajectory** is a sequence $\{(x_t, y_t)\}$ such that $(x_t, y_{t+1}) \in Y$ for every $t$. A production path $\{(x_t, y_t)\}$ is **efficient** if there does not exist another path $\{(x'_t, y'_t)\}$ such that

$$y_t - x_t \leq y'_t - x'_t, \forall t,$$

and the inequality is strict for some $t$.

A **price system** is a sequence $\{p_t\}$ in $\mathbf{R}^{\ell}$. Given a production path $\{(x_t, y_t)\}$ and a price system $\{p_t\}$, the profit associated with the path at each date is

$$p_t \cdot (y_t - x_t).$$

The production path $\{(x_t, y_t)\}$ is **short-run profit-maximizing** (SRPM) for the price system $\{p_t\}$ if

$$p_{t+1} \cdot y_{t+1} - p_t \cdot x_t \geq p_{t+1} \cdot y' - p_t \cdot x', \forall (x', y') \in Y,$$

for every $t$.

**Proposition 1** *Suppose that the production path $\{(x_t, y_t)\}$ is SRPM for the price system $\{p_t\}$ where $p_t \gg 0$ for every $t$. Suppose also that the transversality condition*

$$\lim_{t \to \infty} p_t \cdot y_t = 0$$

*is satisfied. Then the path $\{(x_t, y_t)\}$ is efficient.*

**Proof.** Suppose that the production path $\{(x_t, y_t)\}$ is SRPM for the price path $\{p_t\}$, where $p_t \gg 0$ for every $t$, and suppose that the production path $\{(x'_t, y'_t)\}$ dominates $\{(x_t, y_t)\}$, that is, $y_t - x_t \leq y'_t - x'_t$ for every $t$ with strict inequality for some $t$. Then, for some $T$ sufficiently large,

$$\sum_{t=0}^{T} p_t \cdot (y'_t - x'_t) > \sum_{t=0}^{T} p_t \cdot (y_t - x_t) + \varepsilon,$$

for some $\varepsilon > 0$. For all $T$ sufficiently large the transversality condition implies that $p_{T+1} \cdot y_{T+1} < \varepsilon$, so

$$\sum_{t=0}^{T} p_t \cdot (y'_t - x'_t) > p_{T+1} \cdot y_{T+1} + \sum_{t=0}^{T} p_t \cdot (y_t - x_t).$$

Re-arranging the terms and recalling that $y'_0 = y_0 = \bar{y}_0$, we have

$$-p_T \cdot x'_T + \sum_{t=0}^{T-1} \left(p_{t+1} \cdot y'_{t+1} - p_t \cdot x'_t\right) > \sum_{t=0}^{T} \left(p_{t+1} \cdot y_t - p_t \cdot x_t\right),$$

which implies that either

$$p_{t+1} \cdot y'_t - p_t \cdot x'_t > p_{t+1} \cdot y_t - p_t \cdot x_t$$

for some $t = 0, ..., T-1$, which contradicts SRPM, or else

$$-p_T \cdot x'_T > p_{T+1} \cdot y_{T+1} - p_T \cdot x_T.$$

Now $(x'_T, y'_T) \in Y$ implies $(x'_T, 0) \in Y$ so the last inequality contradicts SRPM too, so we have established the desired result. ∎

Suppose that the production path $\{(x_t, y_t)\}$ is efficient. Do there exist prices $\{p_t\}$ such that $\{(x_t, y_t)\}$ is SRPM? If such prices exist, is the transversality condition satisfied?

[Discuss nontightness condition.]

### 1.1 The Cass criterion

Cass (1972) studies an aggregative (one good) economy with a neoclassical production function $f(x)$. The function $f$ is assumed to be increasing, strictly concave and $C^2$ for $x \geq 0$ and satisfies the boundary conditions

$$f(0) = 0,\ 0 \leq f'(\infty) < 1 < f'(\bar{x}) < \infty,$$

for some constant $\bar{x} \geq 0$. There is an infinite horizon and time is divided into periods $t = 0, 1, ....$ The initial stock of capital is denoted by $\hat{x}_0$. A feasible path $\{x_t\}$ satisfies the basic law of motion of the economy

$$x_{t+1} = f(x_t) - c_t$$

where $c_t \geq 0$ and $x_t \geq \bar{x}$ for every $t$. A feasible path $\{x_t\}$ is **efficient** if there is no feasible path $\{x'_t\}$ which provides at least as much consumption in every period and more consumption in some period.

**Theorem 2** *The feasible path $\{x_t\}$ is **inefficient** if and only if*

$$\lim_{t\to\infty} \sum_{s=0}^{t} \pi_s < \infty,$$

*where $\pi_t = \prod_{s=0}^{t} f'(x_s)$ for every $t$.*

## 2 The one-consumer case

The economy is defined by a production set $Y \subset \mathbf{R}^{2\ell}$, a utility function $u(\cdot)$ defined on $\mathbf{R}^{\ell}_{+}$, a discount factor $0 < \delta < 1$, and a bounded sequence of endowments $(e_0, ..., e_t, ...)$ where $e_t \in \mathbf{R}^{\ell}_{+}$. The production set $Y \subset \mathbf{R}^{2\ell}$ satisfies the usual properties:

- $Y$ is closed and convex;
- $(x, y) \in Y$ and $x \leq 0$ implies $y \leq 0$;
- if $(x, y) \in Y$ and $(-x', y') \leq (-x, y)$ then $(x', y') \in Y$;
- if $(x, y) \in Y$ then $(x, 0) \in Y$.

A production path $\{(x_t, y_t)\}$ is **feasible** if the induced consumption stream is denoted by $\{c_t\}$ and defined by

$$c_t = y_t - x_t + e_t$$

is non-negative for every $t$. In what follows, we assume that production paths and consumption streams are **bounded**.

Given a production path $\{(x_t, y_t)\}$ and a price sequence $\{p_t\}$, the induced stream of profits is denoted by $\{\pi_t\}$ and defined by

$$\pi_t = p_{t+1} \cdot y_{t+1} - p_t \cdot x_t$$

for every $t$. Note that

$$\begin{aligned}
c_t &= y_t - x_t + e_t \\
&\Longrightarrow \sum_{t=0}^{T} p_t \cdot c_t = \sum_{t=0}^{T} p_t \cdot (y_t - x_t + e_t) \\
&\Longrightarrow \sum_{t=0}^{T} p_t \cdot c_t = \sum_{t=0}^{T} (\pi_t + p_t \cdot e_t) - p_{T+1} \cdot y_{T+1} \\
&\Longrightarrow \sum_{t=0}^{T} (\pi_t + p_t \cdot e_t) - \sum_{t=0}^{T} p_t \cdot c_t = p_{T+1} \cdot y_{T+1}.
\end{aligned}$$

So the transversality condition is equivalent to a present value budget constraint holding with equality, i.e., the present value of consumption is exactly equal to wealth.

**Definition 3** *The (bounded) production path $\{(x_t^*, y_t^*)\}$ and the (bounded) price sequence $\{p_t^*\}$ constitute a Walrasian equilibrium if*

*(i) $c_t^* = y_t^* - x_t^* + e_t$ for all $t$.*
*(ii) For every $t$, $\pi_t = p_{t+1}^* \cdot y_{t+1}^* - p_t^* \cdot x_t^* \geq p_{t+1}^* \cdot y - p_t^* \cdot x$ for any $(x, y) \in Y$.*
*(iii) The consumption sequence $\{c_t\}$ solves the problem*

$$\begin{array}{ll} \max & \sum_t \delta^t u(c_t) \\ s.t. & \sum_t p_t^* \cdot c_t \leq \sum_t \pi_t + \sum_t p_t^* \cdot e_t. \end{array}$$

**Proposition 4** *Suppose that the (bounded) production path $\{(x_t^*, y_t^*)\}$ and the (bounded) price sequence $\{p_t^*\}$ constitute a Walrasian equilibrium. Then the transversality condition $p_{t+1}^* \cdot y_{t+1}^* \to 0$ holds.*

**Definition 5** *A consumption stream $\{c_t\}$ is short-run utility maximizing (SRUM) in the budget set determined by the price sequence $\{p_t\}$ and wealth $w < \infty$ if utility cannot be increased by a new consumption stream that merely transfers purchasing power between two consecutive periods.*

**Proposition 6** *If the consumption stream $\{c_t\}$ satisfies the budget constraint $\sum_t p_t \cdot c_t = w < \infty$ and the first-order conditions for SRUM*

$$\delta^t \nabla u(c_t) = \lambda p_t, \ \forall t,$$

*then it is utility maximizing in the budget set defined by $\{p_t\}$ and $w$.*

**Proof.** Suppose, contrary to what we want to prove, that a consumption stream $\{c'_t\}$ satisfies the budget constraint defined by $\{p_t\}$ and $w$ and

$$\sum_t \delta^t u(c_t) < \sum_t \delta^t u(c'_t) - 2\varepsilon.$$

Now consider the consumption stream $\{c^T_t\}$ defined by

$$c^T_t = \begin{cases} c'_t & \text{for } t = 0, ..., T \\ c_t & \text{for } t = T+1, .... \end{cases}$$

We can choose $T_0$ such that for every $T > T_0$,

$$\sum_t \delta^t u(c_t) < \sum_t \delta^t u(c^T_t) - 2\varepsilon.$$

Since $\{c_t\}$ is SRUM,

$$\sum_{t=0}^{T} p_t \cdot c_t < \sum_{t=0}^{T} p_t \cdot c'_t,$$

for $T > T_0$. However, since the consumption streams $\{c_t\}$ and $\{c'_t\}$ both satisfy the budget constraint, for any $\delta > 0$, there exists $T_1$ such that for $T > T_1$,

$$\sum_{t=T}^{\infty} p_t \cdot c_t < \delta \text{ and } \sum_{t=T}^{\infty} p_t \cdot c'_t < \delta.$$

This implies that since both $\{c_t\}$ and $\{c'_t\}$ satisfy the budget constraint,

$$\sum_{t=0}^{T} p_t \cdot c'_t - \sum_{t=0}^{T} p_t \cdot c_t < 2\delta.$$

Since $\delta > 0$ is arbitrary, we can choose it so that, in order to balance the budget with $\{c^T_t\}$ it is sufficient to reduce consumption at the first date by a small amount proportional to $\delta$ that will decrease utility by less than $\varepsilon$. Then, using the same notation, we have found a sequence $\{c^T_t\}$ such that

$$\sum_{t=0}^{T} p_t \cdot c'_t \leq \sum_{t=0}^{T} p_t \cdot c_t$$

and

$$\sum_{t=0}^{T} \delta^t u(c_t) < \sum_{t=0}^{T} \delta^t u(c^T_t) - \varepsilon,$$

contradicting SRUM. ■

**Proposition 7** *Any Walrasian equilibrium path $\{(x^*_t, y^*_t)\}$ solves the planning problem*

$$\begin{array}{ll} \max & \sum_t \delta^t u(c_t) \\ s.t. & c_t = y_t - x_t + e_t \geq 0, \ \forall t, \\ & (x_t, y_{t+1}) \in Y, \ \forall t. \end{array}$$

**Proof.** To prove this, it is enough to show that any path $\{(x_t, y_t)\}$ that satisfies the constraints of the planner's problem also satisfies the consumer's budget constraint. Suppose that $\{(x_t, y_t)\}$ satisfies the constraints of the planner's problem. Then

$$\begin{aligned}
\sum_{t=0}^{T} p_t^* \cdot c_t &= \sum_{t=0}^{T} p_t^* \cdot (y_t - x_t + e_t) \\
&= \sum_{t=0}^{T} (\pi_t + p_t^* \cdot e_t) + p_0^* \cdot y_0 - p_{T+1}^* \cdot y_{T+1} \\
&\leq \sum_{t=0}^{T} (\pi_t^* + p_t^* \cdot e_t) + p_0^* \cdot y_0 + \varepsilon
\end{aligned}$$

for any positive $\varepsilon > 0$ and $T$ sufficiently large. Since $\varepsilon$ is arbitrary,

$$\sum_{t=0}^{\infty} p_t^* \cdot c_t \leq \sum_{t=0}^{\infty} (\pi_t^* + p_t^* \cdot e_t) \leq w,$$

as required. ■

**Proposition 8** *Suppose that the (bounded) path $\{(x_t^*, y_t^*)\}$ solves the planning problem above and that it yields strictly positive consumption in the sense that for some $\varepsilon > 0$ and all $t$, $c_{ht}^* > \varepsilon$ for $h = 1, ..., \ell$. Then the path is Walrasian with respect to some price sequence $\{p_t^*\}$.*

# 3 Overlapping generations

Many of the properties of the single-consumer model extend to an infinite horizon model with a finite number of consumers as long as we continue to assume **complete markets**. The same is not true if we assume a countable number of agents $i = 1, 2, ....$ There is a countable number of dates indexed by $t = 0, 1, 2, ...$ and each agent $i$ is born at a finite date $b_i$ and dies at a finite date $d_i$, where $0 \leq b_i < d_i < \infty$. The number of agents alive at any date is assumed to be finite, i.e., for any date $t$

$$\#\{i : b_i \leq t \leq d_i\} < \infty.$$

We assume that there are $\ell$ goods available at each date, so the commodity space is the set of sequences $\{x_t\}$ in $\mathbf{R}^\ell$, sometimes denoted $X = \left(\mathbf{R}^\ell\right)^\infty$. The consumption set for agent $i$ is denoted by $X_i \subset X$ and defined by

$$X_i = \{x_i \in X : x_i \geq 0,\ x_{it} = 0,\ \forall t \notin [b_i, d_i]\}.$$

Agent $i$'s preferences are represented by a utility function $u_i : X_i \to \mathbf{R}$ and his endowment is denoted by $e_i \in X_i$.

An **allocation** is a sequence of consumption bundles $x = \{x_i\}_{i=1}^{\infty}$ such that $x_i \in X_i$ for each $i$. An allocation $x$ is **attainable** if

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} e_i,$$

where the infinite sums are defined pointwise. A price system $p$ is a non-zero element of $X$. The **Walrasian equilibrium** consists of an attainable allocation $x^*$ and a price system $p^*$ such that, for each $i$, $x_i^*$ maximizes $u_i(x_i)$ in the budget set

$$B_i(p^*, e_i) = \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\},$$

where the inner product $a \cdot x = \sum_{t=0}^{\infty} a_t \cdot x_t$.

## 3.1 A 2-period example

At each date a generation of identical agents is born and lives for two periods ($d_i - b_i = 1$). There is one good in each period ($\ell = 1$). Let $p_t$ and $x_t$ denote the price of the good and the amount of the good at date $t$, respectively. The utility function of the representative individual born at date $t$ is denoted by $u_t(x) = v(x_t, x_{t+1})$, where $v : \mathbf{R}_+^2 \to \mathbf{R}$ represents the common preferences over first- and second-period consumption. There is also an "old" generation at date 0, which is labelled $-1$, which dies at the end of that period. Their endowment consists of date 0 goods and their utility is increasing in consumption at that date.

**Proposition 9** *Suppose that* $e_t = (0, ..., 0, 1, 1, 0, ...)$ *for each* $t$ *and that* $v$ *is* $C^1$ *in a neighborhood of* $(1, 1)$. *Then there is a unique Walrasian equilibrium* $(x^*, p^*)$ *for this economy such that* $x_i^* = e_i$ *and*

$$\frac{p_{t+1}}{p_t} = \frac{\frac{\partial v}{\partial x_2}(1,1)}{\frac{\partial v}{\partial x_1}(1,1)}.$$

## 3.2 First welfare theorem

**Theorem 10** *Suppose that* $(x^*, p^*)$ *is a competitive equilibrium for a pure exchange OLG economy with local non-satiation. If* $p \cdot \sum_{t=0}^{\infty} e_i < \infty$ *then* $x$ *is a Pareto-efficient allocation.*

Note that the assumption $p \cdot \sum_{t=0}^{\infty} e_i < \infty$ is critical. The two-period example above does not satisfy this assumption if $\frac{\partial v}{\partial x_2}(1,1) \geq \frac{\partial v}{\partial x_1}(1,1)$. Can you find a Pareto improvement for the case $v(c_1, c_2) = c_1 + c_2$?

The lack of curvature in the indifference curve is important for the inefficiency result. Consider a one-good, two-period-lived OLG model example presented above, but suppose that the utility function $v(c_1, c_2)$ is strictly quasi-concave. Suppose that at date 0, the young agent gives $\varepsilon > 0$ units of consumption to the old. In order to be no worse off than before, the young agent must receive at least $\phi(\varepsilon)$ units of consumption when old, where

$$v(1-\varepsilon, 1+\phi(\varepsilon)) = v(1,1).$$

It is easy to show that if $u$ is a $C^1$ function then $\phi(\varepsilon)$ is a well defined, increasing $C^1$ function for $0 \leq \varepsilon < 1$ and

$$\phi'(0) = \frac{\frac{\partial v}{\partial x_1}(1,1)}{\frac{\partial v}{\partial x_2}(1,1)}.$$

We assume that $\phi'(0) = 1$ as in the linear example above. The curvature of the utility function implies that $\phi''(\varepsilon) > 0$, so for any $\varepsilon_0 > 0$ there is a number $a(\varepsilon_0) > 0$ such that

$$\phi(\varepsilon) > (1 + \alpha(\varepsilon_0))\varepsilon$$

for all $\varepsilon \geq \varepsilon_0$. The same analysis applies at any date $t$. If the young agent at $t$ gives the old agent $\varepsilon \geq \varepsilon_0$ then in order to be no worse off the young agent will have to receive at least $\phi(\varepsilon) > (1 + \alpha(\varepsilon_0))\varepsilon$ when old. Thus, if we begin by transferring $\varepsilon_0 > 0$ to the old generation at $t = 0$, we must transfer $\varepsilon_t$ at date $t$ where the sequence of transfers $\{\varepsilon_t\}$ satisfies

$$\varepsilon_{t+1} \geq \phi(\varepsilon_t)$$

at each date $t$. But that means

$$\varepsilon_t \geq (1 + a(\varepsilon_0))^{t-1} \varepsilon_0,$$

which implies $\varepsilon_t \to \infty$, so the transfers eventually become infeasible.

A little reflection will show that the same argument applies to the case $\phi'(0) > 1$ without the curvature assumption. This corresponds to the case $p \cdot \sum_{t=0}^{\infty} e_i < \infty$ in the theorem.

In a model with production, Cass used the curvature of the production function in a similar way to obtain a complete characterization of the efficiency of a production path.

## 3.3 Land

[To be completed]