# Strategic Foundations of General Equilibrium

## 1 Strategic foundations of general equilibrium

A strategic foundation of general equilibrium requires of three elements:

- A description of an economy.
- An extensive-form game.
- A solution concept that implies a competitive outcome under certain conditions.

There are several reasons why strategic foundations of general equilibrium are desirable:

- Game theory has changed the definition of what counts as a reasonable model of equilibrium:
  - all endogenous variables should be chosen/determined by players;
  - each strategy profile should determine a unique, feasible outcome.
- To answer the question "When is competitive equilbrium appropriate?"

## 2 The core of an exchange economy

Edgeworth's *Mathematical Psychics*. The contract curve and the core of an economy.

Let $\mathcal{E} = \{(X_i, u_i, e_i)\}_{i=1}^m$ be an exchange economy. A **coalition** is any non-empty set of agents $S \subseteq \{1, ..., m\}$. Let $x$ be an attainable allocation for $\mathcal{E}$. A coalition $S$ can **improve** on $x$ if and only if there exists an attainable allocation $x'$ such that

$$u_i(x_i') > u_i(x_i) \text{ for all } i \in S$$

and

$$\sum_{i \in S} x_i' = \sum_{i \in S} e_i.$$

The **core** of the exchange economy $\mathcal{E}$, denoted $C(\mathcal{E})$, is defined to be the set of attainable allocations that cannot be improved on by any coalition.

An attainable allocation $x$ is called a **Walras allocation** for $\mathcal{E}$ if there is a price vector $p \neq 0$ such that $(x, p)$ is a Walrasian equilibrium. Let $W(\mathcal{E})$ denote the set of Walras allocations for $\mathcal{E}$.

**Proposition 1** $W(\mathcal{E}) \subseteq C(\mathcal{E})$.

For a given exchange economy $\mathcal{E}$, we define an $r$-fold replica of $\mathcal{E}$, denoted by $\mathcal{E}^r$, to be an exchange economy consisting of $rm$ agents, indexed $\{(1,1), ..., (m,1), ..., (1,r), ..., (m,r)\}$, where agent $(i,k)$ is the $k$-th agent of type $i$, that is, an agent with characteristics $(X_i, u_i, e_i)$. Attainable allocations and the core of an $r$-fold replica are defined in the usual way. A **symmetric** allocation for $\mathcal{E}^r$ has the property that $x_{ik} = x_{ik'}$, for every $i$, $k$, and $k'$. We abuse notation by writing a symmetric allocation as $x = (x_1, ..., x_i, ..., x_m)$. Suppose that $u_i$ is $C^1$ for every $i = 1, ..., m$.

**Proposition 2** *Suppose that $x$ is an attainable, symmetric equilibrium for $\mathcal{E}$, that $x \in C(\mathcal{E}^r)$ for $r = 1, 2, ...$, and that $x_i$ belongs to the interior of $X_i$ for $i = 1, ..., m$. Then $x \in W(\mathcal{E})$.*

The allocation $x$ must be Pareto-efficient, so there exists a price vector $p \neq 0$ such that $\lambda_i p = \nabla u_i(x_i)$ for every $i = 1, ..., m$. Then we claim that $(x, p)$ is a Walrasian equilibrium. If not, then $p \cdot x_i > p \cdot e_i$ for some $i$. Define an attainable allocation $x^r$ for $\mathcal{E}^r$ as follows:

$$\begin{aligned} x^r_{(i,1)} &= e_i, \\ x^r_{(i,k)} &= x_i + \frac{1}{rm-1}(x_i - e_i), \; k = 2, ..., r, \\ x^r_{(j,k)} &= x_j + \frac{1}{rm-1}(x_i - e_i), \; j \neq i, \; k = 1, ..., r. \end{aligned}$$

It is easy to see that $x^r$ is attainable for $\mathcal{E}^r$ and that

$$u_j\left(x^r_{(j,k)}\right) = u_j(x_j) + \frac{1}{rm-1} \nabla u_j(x_j) \cdot (x_i - e_i) + o\left(\frac{1}{nm-1}\right).$$

The for $r$ sufficiently large, the coalition $S$ consisting of everyone other than agent $(i,1)$ is improving.

Another way of expressing this result, assuming that all allocations are symmetric, is

$$\cap_{r=1}^{\infty} C(\mathcal{E}^r) = W(\mathcal{E}).$$

Other solution concepts: Shapley value, bargaining set.

The Nash Program.

# 3 Market games

Nash-Cournot. A typical firm maximizes

$$P(nq^* + q)q - C(q)$$

and the first-order condition is

$$P'(nq^* + q)q + P(nq^* + q) = C'(q).$$

In a symmetric equilibrium,

$$P'\left(\left(n+1\right)q^{*}\right)q^{*}+P\left(\left(n+1\right)q^{*}\right)=C'\left(q^{*}\right).$$

Suppose there are $r\left(n+1\right)$ firms. Then a typical firm maximizes

$$P\left(\frac{\left(r\left(n+1\right)-1\right)q^{*}+q}{r}\right)q-C\left(q\right)$$

and the first-order condition is

$$P'\left(\frac{\left(r\left(n+1\right)-1\right)q^{*}+q}{r}\right)\frac{q}{r}+P\left(\frac{\left(r\left(n+1\right)-1\right)q^{*}+q}{r}\right)=C'\left(q\right).$$

In a symmetric equilibrium, we have

$$P'\left(\left(n+1\right)q^{r}\right)\frac{q^{r}}{r}+P\left(\left(n+1\right)q^{r}\right)=C'\left(q^{*}\right).$$

As $r\to\infty$, $q^{r}\to q^{0}$. Then

$$P(\left(n+1\right)q^{0})=C'\left(q^{0}\right).$$

What follows is a precis of Shapley and Shubik (1977). There are $I$ agents, indexed $i=1,...,I$ and $\ell+1$ goods, indexed $h=1,...,\ell+1$. Each agent has an endowment $e_{i}\in\mathbf{R}_{+}^{\ell}$ and a utility function $u_{i}:\mathbf{R}_{+}^{\ell}\to\mathbf{R}$. This includes fiat money as a special case. As a simplifying assumption, we assume that agent $i$ supplies all of the goods $h\neq\ell+1$ in his endowment.

There are $\ell$ separate trading posts, one for each good $h=1,...,\ell$. Each agent submits a vector of bids

$$b_{i}=\left(b_{i1},...,b_{i\ell}\right)\in\mathbf{R}_{+}^{\ell},$$

where $b_{ih}$ is the amount of good $\ell+1$ ("money") offered in exchange for good $h$. The cash in advance constraint requires

$$\sum_{h=1}^{\ell}b_{ih}\leq e_{i\ell+1}.$$

Define the price $p_{h}$ by putting

$$p_{h}=\frac{\sum_{i=1}^{I}b_{ih}}{\sum_{i=1}^{I}e_{ih}}$$

for $h=1,...,\ell$ and define the trading rule by

$$x_{ih}=\begin{cases}\frac{b_{ih}}{p_{h}} & \text{if } p_{h}\neq0,\\ 0 & \text{if } p_{h}=0.\end{cases}$$

Define the payoff function $\pi_{i}\left(b\right)=u_{i}\left(x_{i1},...,x_{ih},...,x_{i\ell+1}\right)$.

**Theorem 3** *If $u_i$ is continuous, concave and non-decreasing and for each good $h = 1, ..., \ell$ there exist two agents with positive endowments of $h$ whose utility is increasing in $h$, then a Nash equilibrium $b^*$ exists.*

**Definition 4** *A replica economy $\mathcal{E}^r$ consists of $r$ agents of each type $i = 1, ..., I$.*

**Theorem 5** *Assume that for infinitely many $r$, there is a symmetric, interior Nash equilibrium $\hat{b}^r = \left(\hat{b}_1^r, ..., \hat{b}_I^r\right)$ for $\mathcal{E}^r$ and let $p^r$ denote the corresponding price vector. Let $p^*$ be the limit point of $p^r$ and define $p_{\ell+1}^* = 1$. Then $\left(p^*, p_{\ell+1}^*\right)$ is a competitive equilibrium price vector.*

# 4 Dynamic matching and bargaining games

This treatment, from Osborne and Rubinstein (1990), is based on a working paper by Gale (1986). The model is described as follows.

*Dates.* Time is divided into a countable number of dates $t = 1, 2, ....$

*Goods.* There are $\ell$ goods, indexed $h = 1, ..., \ell$, and the consumption set for every agent is $\mathbf{R}_+^\ell$.

*Agents.* There are $K$ types of agents, $k = 1, ..., K$. An agent of type $k$ has a utility function

$$u_k : \mathbf{R}_k^\ell \cup \{D\} \to \mathbf{R} \cup \{-\infty\}$$

where $u_k(D) = -\infty$ is the utility associated with infinite delay. The fraction of agents of type $k$ is denoted by $n_k > 0$, where $\sum_{k=1}^K n_k = 1$.

*Assumption 1.* For each $k$, there exists a function $\phi_k : \mathbf{R}_k^\ell \to \mathbf{R}$ that is increasing, strictly concave on the interior of $\mathbf{R}_k^\ell$, and such that $\phi_k(x) = 0$, for any $x \in \partial \mathbf{R}_k^\ell$. Let

$$X_k = \left\{x \in \mathbf{R}_k^\ell : \phi_k(x) \geq \bar{\phi}_k\right\}$$

and assume that

$$u_k(x) = \begin{cases} \phi_k(x) & \text{if } x \in X_k, \\ -\infty & \text{if } x \notin X_k. \end{cases}$$

We also assume that $e_k \in X_k$.

*Assumption 2.* There is a unique tangent to the indifference curve through any point $x \in X_k$. Let $S_k(p)$ be the expansion path at $p$. Then for any $z \in \mathbf{R}^\ell$*such* that $p \cdot z > 0$, there exists a number $L \in \mathbf{Z}$ such that $u_k(c + z/L) > u_k(c)$ for any $c \in S_k(p)$.

*Matching.* An agent is matched in any period $t$ with a fixed probability $0 < \alpha < 1$. The probability of being matched with an agent whose characteristics are $c \in C$ is proportional to the measure of agents with characteristic $C$. Once a pair of agents is matched, each has a probability 1/2 of being chosen as the proposer and the other is a responder.

*Exit.* After rejecting an offer, the responder decides whether to exit.

Now we define an equilibrium for the model. An agent's information set consists of the 4-tuple $(t, x, k, y)$ at the beginning of period $t$, where $x$ is his consumption bundle, $k$ is his opponent's type, and $y$ is his opponent's bundle.

If he is chosen as the proposer, he offers to trade $z$; if he is the responder, he receives an offer $z$ and responds by accepting the offer, $Y$, rejecting the offer and staying in the game, $NS$, or rejecting the offer and leaving the game, $NX$.

We analyze a symmetric equilibrium in which every agent of type $k$ uses the same strategy.

In any period, the state is represented by a finite array $(k_i, c_i, v_i)_{i=1,...,I}$.

An agent is characterized by an ordered pair $(k, c)$.

A strategy profile is denoted $\sigma = (\sigma_1, ..., \sigma_k, ..., \sigma_K)$.

We assume the "law of large numbers" holds, that is, there is no aggregate uncertainty and the number of agents of a particular type who are matched is always equal to the ex ante probability. Then for any strategy profile $\sigma$ there is a unique state of the market $\rho(\sigma, t)$ at date $t$. Let $\rho(\sigma, t) = (k_i, c_i, v_i)_{i=1,...,I}$. Then

(a) the set of agents $(k_j, c_j)$ matched with $(k_h, c_h)$ and chosen as proposer has measure

$$\frac{\frac{1}{2}\alpha\nu_j\nu_h}{\sum_{i=1}^{I}\nu_i}.$$

(b) if $\sigma$ leads to an offer $z$ that is accepted, then a measure $\frac{1}{2}\alpha\nu_j\nu_h / \sum_{i=1}^{I}\nu_i$ is transferred from $(k_j, c_j)$ to $(k_j, c_j + z)$ and an equal measure is transferred from $(k_h, c_h)$ to $(k_h, c_h - z)$.

(c) if $\sigma$ instructs $(k_h, c_h)$ to reject and exit, $(k_h, c_h)$ is reduced by $\frac{1}{2}\alpha\nu_j\nu_h / \sum_{i=1}^{I}\nu_i$.

(d) otherwise, the measure remains the same.

**Definition 6** *A* ***market equilibrium*** *consists of a strategy profile $\sigma^*$ such that for any $(z, c, c', k, t)$, the agent's behavior is optimal given that the agent holds $c$ in $t$ and proposes or responds to $z$ from an agent of type $k$ and holds $c'$, given the other strategies and beliefs that of the market is $\rho(\sigma^*, t)$.*

An allocation is an array $x = (x_1, ..., x_K) \in \mathbf{R}_+^{\ell K}$ and it is attainable if

$$\sum_{k=1}^{K} x_k = \sum_{k=1}^{K} e_k.$$

An allocation is **competitive** (Walrasian) if there exists a price vector $p \neq 0$ such that $x_k$ maximizes $u_k$ over the set $\{x \in X_k : p \cdot x_k \leq p \cdot e_k\}$.

**Proposition 7** *For any market equilibrium $\sigma$ there is a Walras allocation $x = (x_1, ..., x_K)$ such that every agent of type $k$ leaves the market with $x_k$.*