

# Problem Set 0

## Optimization Theory (A Brief Review)

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### Solutions

1. There are two commodities with quantities  $x_1 \geq 0$  and  $x_2 \geq 0$  and prices  $p_1 \geq 0$  and  $p_2 \geq 0$ . The consumer's income is  $y > 0$ . The utility function is denoted by  $u(x_1, x_2)$  and is defined by

$$u(x_1, x_2) = a_1x_1 + a_2x_2 - b_1x_1^2 - b_2x_2^2$$

for any  $(x_1, x_2) \geq 0$  where  $a_1, a_2, b_1, b_2$  are all positive constants. The consumer chooses an element of her consumption set  $x \in \mathbf{R}_+^2$  to maximize her utility  $u(x)$  subject to a budget constraint  $p \cdot x \leq y$ .

1. Prove that the Constraint Qualification is satisfied at any point  $x$  in the budget set  $\{x \in \mathbf{R}_+^2 \mid p \cdot x \leq y\}$ .

2. Suppose that  $x^*$  is a solution to the consumer's problem for some  $p \geq 0$  and  $y > 0$ . What first order conditions must be satisfied by  $x^*$ ?

3. Calculate the value of the consumer's demand function  $f(p, y)$  for any  $p \geq 0$  and  $y > 0$ .

Let's first start with a few observations about the problem. The first thing we notice is that since  $p_1$  and  $p_2$  can be zero, the set  $B(p_1, p_2, y) = \{(x_1, x_2) \in \mathbf{R}_+^2 \mid p \cdot x \leq y\}$  is not necessarily compact, so at this point we can't be sure that the problem has a solution. Another thing we can observe is that our utility function has a bliss point. In fact, we have a separable utility function which allows us here to find  $x_1$  and  $x_2$  values  $\bar{x}_1$  and  $\bar{x}_2$  such that the consumer will never want to consume more of  $x_i$  than  $\bar{x}_i$ . It's easy to see that

$$\bar{x}_1 = \frac{a_1}{2b_1} \text{ and } \bar{x}_2 = \frac{a_2}{2b_2}.$$

The existence of the upper bounds above imply that when trying to maximize  $u(x_1, x_2)$  we can concentrate on the set

$$\tilde{B}(p_1, p_2, y) = \{(x_1, x_2) \in \mathbf{R}_+^2 \mid p \cdot x \leq y \text{ and } (x_1, x_2) \leq (\bar{x}_1, \bar{x}_2)\}.$$
<sup>1</sup>

But now  $\tilde{B}(p_1, p_2, y)$  is a compact set and since our function  $u(\cdot)$  is continuous we know that our problem indeed has a solution. Moreover, since  $u(\cdot)$  is a strictly concave function and  $\tilde{B}(p_1, p_2, y)$  is a convex set, we know that this solution is unique.<sup>2</sup> All these imply that we have a well defined demand function  $f(p_1, p_2, y)$  associated with the problem above. Finally, observe that  $\tilde{B}(p_1, p_2, y) = \tilde{B}\left(\frac{p_1}{y}, \frac{p_2}{y}, 1\right)$ , so without loss of generality we can assume that  $y = 1$  when solving the consumer's problem.

1. Our constraint function  $g(x)$  in this case is given by

$$g(x) = \begin{bmatrix} -x_1 \\ -x_2 \\ p \cdot x - 1 \end{bmatrix}.$$

We also have that

$$\nabla g_1(x) = [ -1 \quad 0 ], \quad \nabla g_2(x) = [ 0 \quad -1 ], \quad \nabla g_3(x) = [ p_1 \quad p_2 ].$$

Since  $\nabla g_1(x)$  and  $\nabla g_2(x)$  are linearly independent, if the third restriction is not binding at  $x$  then the constraint qualification is satisfied at  $x$ . If the third restriction is binding at  $x$  this implies that there exists  $i \in \{1, 2\}$  such that  $p_i > 0$  and  $x_i > 0$ , which implies that the  $i_{th}$  restriction is not binding. Moreover, in this case it's easy to see that  $\nabla g_3(x)$  and  $\nabla g_j(x)$ ,  $j \neq i$ , are linearly independent, so the constraint qualification is satisfied.

2. If  $x^* \in B(p_1, p_2, 1)$  is a solution to the consumer's problem, there must exist non-negative scalars  $\lambda_1, \lambda_2, \lambda_3$  such that the following conditions are true

$$\begin{aligned} -a_1 + 2b_1x_1^* &= \lambda_1 - \lambda_3p_1 \\ -a_2 + 2b_2x_2^* &= \lambda_2 - \lambda_3p_2 \\ \lambda_1(-x_1^*) &= 0 \\ \lambda_2(-x_2^*) &= 0 \\ \lambda_3(p \cdot x^* - 1) &= 0. \end{aligned}$$

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<sup>1</sup>The formal argument goes like this: For any  $x \in B(p, y)$  such that it's not true that  $x \leq \bar{x}$ , there exists  $\tilde{x} \in \tilde{B}(p, y)$  such that  $u(\tilde{x}) > u(x)$ . So if  $u(x^*) \geq u(x) \quad \forall x \in \tilde{B}(p, y)$ , then  $u(x^*) \geq u(x) \quad \forall x \in B(p, y)$ .

<sup>2</sup>To see this, suppose that  $x^*$  and  $\hat{x}$  are distinct points which solve the consumer's problem. Since  $\tilde{B}(p, y)$  is convex we know that  $(0.5x^* + 0.5\hat{x}) \in \tilde{B}(p, y)$ . But the strict concavity of  $u(\cdot)$  implies that  $u(0.5x^* + 0.5\hat{x}) > u(x^*)$ , which contradicts the fact that  $x^*$  solves the consumer's problem.

The conditions above can be simplified to

$$a_1 - 2b_1x_1^* \leq \lambda_3 p_1, \text{ with equality if } x_1^* > 0 \quad (1)$$

$$a_2 - 2b_2x_2^* \leq \lambda_3 p_2, \text{ with equality if } x_2^* > 0 \quad (2)$$

$$\lambda_3 (p \cdot x^* - 1) = 0 \quad (3)$$

3. To find the consumer's demand function, we'll consider 4 different cases:

Case 1:  $p \cdot \bar{x} \leq 1$

In this case the consumer can buy  $\bar{x}$ , so clearly the best choice is exactly  $\bar{x}$ , which means that  $f(p, 1) = \bar{x}$ .

Case 2:  $f_1(p, 1) = x_1^* = 0$

If the consumer is not consuming the first good, (1) implies that  $\lambda_3 > 0$  and  $p_1 > 0$ .<sup>3</sup> Now, using (3), the fact that  $\lambda_3 > 0$  implies that  $p \cdot x^* = 1$ .<sup>4</sup> But this condition implies that

$$x_2^* = \frac{1}{p_2}.$$

But now, (2) tells us that

$$\lambda_3 = \frac{1}{p_2} \left( a_2 - \frac{2b_2}{p_2} \right).$$

Finally using the condition above in (1) we get the relation

$$\frac{a_1}{\left( a_2 - \frac{2b_2}{p_2} \right)} \leq \frac{p_1}{p_2} \quad (4)$$

Note that we did things in a kind of reverse order here, in the sense that we started with a condition on the demand function and found a condition on the prices that would give us such a demand. But it's not really a problem here, because we can easily check that if we impose the condition (4), then  $x_1^* = 0$  and  $x_2^* = \frac{1}{p_2}$  satisfy the first order conditions for the consumer's problem. Since we have a concave utility function and a convex budget set, we know that this is sufficient to identify  $\left( 0, \frac{1}{p_2} \right)$  as the solution for the consumer's problem in this case. So we can rewrite case 2 as

$$\text{Case 2: } \frac{a_1}{\left( a_2 - \frac{2b_2}{p_2} \right)} \leq \frac{p_1}{p_2}$$

As argued above in this case we have  $f(p, 1) = \left( 0, \frac{1}{p_2} \right)$ .

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<sup>3</sup>Notice that  $p_1 > 0$  is intuitive. Why wouldn't the consumer consume a good that is free?

<sup>4</sup>This is also intuitive. Why wouldn't the consumer buy good 1 if he still had money left?

Case 3:  $\frac{a_2}{\left(a_1 - \frac{2b_1}{p_1}\right)} \leq \frac{p_2}{p_1}$

Exactly the same reasoning as above shows that in this case we have  $f(p, 1) = \left(\frac{1}{p_1}, 0\right)$ .

Case 4: None of the conditions above are satisfied.

In this case we know that  $x_1^* > 0$  and  $x_2^* > 0$ . Moreover since the consumer is not yet satiated it must be the case that he is exhausting his budget. So we have the following system of equations

$$\begin{aligned} a_1 - 2b_1x_1^* &= \lambda_3 p_1 \\ a_2 - 2b_2x_2^* &= \lambda_3 p_2 \\ p_1x_1^* + p_2x_2^* &= 1. \end{aligned}$$

Solving the system above we find that

$$f(p, 1) = \left( \frac{p_2^2 a_1 - p_1 p_2 a_2 + 2p_1 b_2}{2(p_2^2 b_1 + p_1^2 b_2)}, \frac{p_1^2 a_2 - p_1 p_2 a_1 + 2p_2 b_1}{2(p_2^2 b_1 + p_1^2 b_2)} \right).$$

In summary, our demand function is given by

$$f(p, 1) = \begin{cases} \bar{x} & \text{if } p \cdot \bar{x} \leq 1 \\ \left(0, \frac{1}{p_2}\right) & \text{if } \frac{a_1}{\left(a_2 - \frac{2b_2}{p_2}\right)} \leq \frac{p_1}{p_2} \text{ and } p \cdot \bar{x} > 1 \\ \left(\frac{1}{p_1}, 0\right) & \text{if } \frac{a_2}{\left(a_1 - \frac{2b_1}{p_1}\right)} \leq \frac{p_2}{p_1} \text{ and } p \cdot \bar{x} > 1 \\ \left(\frac{p_2^2 a_1 - p_1 p_2 a_2 + 2p_1 b_2}{2(p_2^2 b_1 + p_1^2 b_2)}, \frac{p_1^2 a_2 - p_1 p_2 a_1 + 2p_2 b_1}{2(p_2^2 b_1 + p_1^2 b_2)}\right) & \text{otherwise.} \end{cases}$$