# Solution - Problem Set 2
# Economies with Uncertainty

*1. Consider an economy with two goods and two states of nature. There are four contingent commodities (since goods are distinguished by the state in which they are delivered). The states are indexed by $s = 1, 2$ and the contingent commodities are indexed by $h = 1, 2$ (for state $s = 1$) and $h = 3, 4$ (for state $s = 2$). There are two agents $i = 1, 2$ with endowments $e_1 = (1, 2, 3, 4)$ and $e_2 = (4, 3, 2, 1)$, respectively. The agents both regard the two states as equally likely and they have identical von Neumann-Morgenstern utility functions*

$$u(c_1, c_2) = \frac{1}{3}\ln c_1 + \frac{2}{3}\ln c_2.$$

*Define a competitive equilibrium assuming that there are complete markets for contingent commodities. Find the equilibrium price vector and the equilibrium allocation. Next define an equilibrium with Arrow securities for the economy, assuming there are markets for Arrow securities before the state is known and spot markets for goods after the true state is revealed. Find the equilibrium prices for commodities and securities and the equilibrium allocation of commodities and securities.*

**Solution for the Arrow-Debreu Economy**

An *Arrow-Debreu equilbirium* is an attainable allocation $(x_1^*, x_2^*) \in \mathbb{R}_+^4 \times \mathbb{R}_+^4$ and a vector of prices $p^* \neq 0$ such that for $i = 1, 2$, $x_i^*$ solves

$$\max_{x_i \in \mathbb{R}_+^4} \frac{1}{2}\left[\frac{1}{3}\ln x_{i,1} + \frac{2}{3}\ln x_{i,2}\right] + \frac{1}{2}\left[\frac{1}{3}\ln x_{i,3} + \frac{2}{3}\ln x_{i,4}\right] \quad \text{subject to } p^* x_i \leq p^* e_i.$$

First, let's find the equilibrium allocation. The consumers' demand is Cobb-Douglas (in four commodities), and thus demand for both agents is

$$x_{i,h}^*(p, pe_i) = \frac{pe_i}{6p_h} \text{ for } h = 1, 3 \text{ and}$$

$$x_{i,h}^*(p, pe_i) = \frac{pe_i}{3p_h} \text{ for } h = 2, 4.$$

Note that $e_{1,h} + e_{2,h} = 5$ for all commodities. Since the allocation is attainable in equilibrium, it follows that

$$\sum_{i=1}^{2} x^*_{i,h} = \frac{5\,(p_1 + p_2 + p_3 + p_4)}{6p_h} = 5 \text{ for } h = 1, 3 \text{ and}$$
$$\sum_{i=1}^{2} x^*_{i,h} = \frac{5\,(p_1 + p_2 + p_3 + p_4)}{3p_h} = 5 \text{ for } h = 2, 4.$$

Clearly, $p^*_1 = p^*_3 = p^1$ and $p^*_2 = p^*_4 = p^2$. Substituting these conditions into the market clearing equations above, it follows $p^2 = 2p^1$, and therefore the price vector $p^* = (1, 2, 1, 2)$ and allocation $(x^*_1, x^*_2) = \left(\left(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}\right), \left(\frac{7}{3}, \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)\right)$ constitutes a competitive equilibrium.

**Solution for the Arrow Securities Economy**

For the remainder of the problem, $p_{s,h}$ will represent the spot price for good $h$ in state $s$, while $p_s$ will represent the vector of spot prices in state $s$.

An *equilibrium with Arrow securities* is an attainable allocation of commodities $(x^*_1, x^*_2) \in \mathbb{R}^4_+ \times \mathbb{R}^4_+$, an attainable allocation of securities $(z^*_1, z^*_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, a commodity price vector $p^* \neq 0$ and an asset price vector $q^* \neq 0$ such that for $i = 1, 2$, $(x^*_i, z^*_i)$ solves

$$\max_{(x_i, z_i) \in \mathbb{R}^4_+ \times \mathbb{R}^2} \sum_{s=1}^{2} \frac{1}{2} u(x_{i,s}) \quad \text{subject to } q^* z_i \leq 0 \text{ and } p^*_s x_{i,s} \leq p^*_s e_{i,s} + z_{i,s} \text{ for } s = 1, 2.$$

From the Arrow-Debreu equilibrium, we know that

$$(x^*_1, x^*_2) = \left(\left(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}\right), \left(\frac{7}{3}, \frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)\right).$$

To calculate the equilibrium security allocation, note that consumer 1's spot market budget constraints are

$$\begin{aligned} \frac{8}{3}\left[p^*_{1,1} + p^*_{1,2}\right] &= p^*_{1,1} + 2p^*_{1,2} + z_{1,1} \text{ in state 1 and} \\ \frac{8}{3}\left[p^*_{2,1} + p^*_{2,2}\right] &= 3p^*_{2,1} + 4p^*_{2,2} + z_{1,2} \text{ in state 2.} \end{aligned}$$

Therefore, the equilibrium security allocation must satisfy

$$z_1^* = \begin{pmatrix} \frac{5}{3}p_{1,1}^* + \frac{2}{3}p_{1,2}^* \\ -\frac{1}{3}p_{2,1}^* - \frac{4}{3}p_{2,2}^* \end{pmatrix} = -z_2^*,$$

where the second equality follows from attainability of $(z_1^*, z_2^*)$. In state $s$, therefore, consumer 1 will have wealth $\frac{8}{3}\left[p_{s,1}^* + p_{s,2}^*\right]$, and thus in each spot market, consumer 1's demand for the first commodity will be

$$x_{1,1}^*\left(p_s^*, \frac{8}{3}\left[p_{s,1}^* + p_{s,2}^*\right]\right) = \frac{\frac{8}{3}\left[p_{s,1}^* + p_{s,2}^*\right]}{3p_{s,1}^*} \implies p_{s,2}^* = 2p_{s,1}^* \text{ for } s = 1, 2.$$

Moreover, since demand is identical in both states, then $p_{1,h}^* = p_{2,h}^*$ for $h = 1, 2$, ie the spot price for good $h$ is the same in both states.[1] Thus $p^* = (1, 2, 1, 2)$ as before, and the corresponding security allocation is

$$z_1^* = \begin{pmatrix} 3 \\ -3 \end{pmatrix} = -z_2^*,$$

with security price vector $q^* = (1, 1)$.

*2. Consider an economy with two goods $h = 1, 2$, two assets $k = 1, 2$, two dates $t = 0, 1$, and two states of nature $s = 1, 2$. The two states have equal probability at date 0; the true state is revealed at date 1. The two assets are traded at the first date. One asset is a promise to deliver one unit of good 1 in each state at date 1; the other asset is a promise to deliver one unit of good 2 in each state at date 1. The two goods are traded on spot markets at date 1. Let $q \in \mathbf{R}_+^2$ denote the equilibrium asset prices at date 0 and let $p(s) \in \mathbf{R}_+^2$ denote the equilibrium goods prices at date 1 in state $s = 1, 2$. Each consumer $i$ chooses a portfolio $z_i \in \mathbf{R}^2$ of assets at date 0 and a consumption bundle $x_i \in \mathbf{R}_+^4$ at date 1, subject to the budget constraints*

$$q \cdot z_i \leq 0,$$

$$p(s) \cdot (x_i(s) - e_i(s)) \leq p(s) \cdot z_i, \text{ for } s = 1, 2.$$

[1] Since $\frac{\frac{8}{3}\left[p_{1,1}^* + p_{1,2}^*\right]}{3p_{1,1}^*} = \frac{\frac{8}{3}\left[p_{2,1}^* + p_{2,2}^*\right]}{3p_{2,1}^*}$ and $p_{s,2}^* = 2p_{s,1}^*$ for $s = 1, 2$, it follows that $p_{1,1}^* = p_{2,1}^*$ and $p_{1,2}^* = p_{2,2}^*$, ie good $h$ has the same price in both states.

*Suppose that the spot price vectors $\{p(s)\}$ are linearly independent. Show that the budget constraints are equivalent to a single budget constraint of the form*

$$\sum_{s=1}^{2} \hat{p}(s) \cdot (x_i(s) - e_i(s)) \leq 0,$$

*for some price vector $\hat{p} \in \mathbf{R}_{+}^{4}$.*

**Solution 1:**

Our task is to find a price vector $\hat{p}$ such that the consumption allocation in the Radner equilibrium at prices $(p, q)$ coincide with the consumption allocation in the A-D equilibrium at prices $\hat{p}$. In other words, we are looking for a $\hat{p}$ such that for any $x_i(1), x_i(2) \in \mathbf{R}_{+}^{2}$,

$$\sum_{s=1}^{2} \hat{p}(s) \cdot (x_i(s) - e_i(s)) \leq 0 \quad \Leftrightarrow$$

$$\exists z_i \in \mathbf{R}^2 : qz_i \leq 0 \quad \text{and} \quad \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ p(2) \cdot (x_i(2) - e_i(2)) \end{bmatrix} \leq \begin{bmatrix} p(1) \cdot z_i \\ p(2) \cdot z_i \end{bmatrix}.$$

First, let's discuss the intuition of the problem. Since we already know how to transform Arrow security equilibrium prices into A-D equilibrium prices, we shall first focus on the relationship between a Radner equilibrium and Arrow security equilibrium. In both environments, the consumer wants to redistribute her wealth across different states to maximize her expected utility. In a Radner environment, this redistribution is indirectly determined by the chosen asset portfolio and corresponding spot prices in each state. Here, spot prices may or may not allow the consumer to achieve her desired wealth distribution. In contrast, an Arrow securities economy allows the consumer to directly achieve a wealth distribution by purchasing the securities. Under the asset structure defined in this question, linear independence of prices $p(1)$ and $p(2)$ allows the consumer to implement any wealth distribution across states. Since markets are essentially complete, the Radner equilibrium will be equivalent to the A-S equilibrium, and consqeuently equivalent to the A-D equilibrium.

Now we'll answer the question formally. Let us first find a hypothetical Arrow security price vector $\pi := (\pi_1, \pi_2)$ which will then be used to find $\widehat{p}$.

First, note that an asset portfolio $z_i$ induces the wealth level $p(s) \cdot z_i$ in state $s$. If we define

$$P := \begin{bmatrix} p(1) \\ p(2) \end{bmatrix},$$

then the asset portfolio $z_i$ induces the following wealth distribution across states:

$$\begin{bmatrix} p(1)z_i \\ p(2)z_i \end{bmatrix} = Pz_i.$$

To obtain the same wealth distribution in an Arrow-security market, a consumer would have to buy $p(1)z_i$ units of security 1 and $p(2)z_i$ units of security 2. At our hypothetical security price vector $\pi$, this would cost:

$$\pi_1 p(1)z_i + \pi_2 p(2)z_i = \pi P z_i.$$

It is natural to expect that a given wealth distribution across states must have the same cost at date 0 in the A-S and Radner markets, and thus:

$$\pi P z_i = q z_i \quad \text{for all } z_i \in \mathbf{R}^2. \tag{1}$$

In (1), the right hand side of the equation is the cost of obtaining the wealth distribution $Pz_i$ in the Radner market. Clearly, (1) is equivalent to:

$$\pi P = q. \tag{2}$$

Since prices are linearly independent, then $P$ is invertible, and thus (2) is equivalent to:

$$\pi = qP^{-1}. \tag{3}$$

Equation (3) yields the Arrow security price vector that we were looking for. Therefore $(x_i, Pz_i)_{i=1,2}$ $(p, \pi)$ is an A-S equilibrium.[2] Then from the lecture notes, the corresponding A-D prices are

$$\widehat{p}(s) := \pi_s p(s) \quad \text{for } s = 1, 2. \tag{4}$$

To complete the proof, we must show that $\widehat{p}$ defined by (4) induces the same budget set. We claim, and later prove, that

$$\pi \geq 0. \tag{5}$$

[2] Verify this as an exercise - the proof should not be difficult after you've read the remainder of this solution.

For now it will suffice to note that (5) is a no arbitrage condition under the assumption that preferences are monotonic. Assuming (5), we can proceed as follows: Take any $x_i(1), x_i(2) \in \mathbf{R}^2_+$, and portfolio $z_i \in \mathbf{R}^2$ such that

$$qz_i \leq 0 \quad \text{and} \quad p(s) \cdot (x_i(s) - e_i(s)) \leq p(s)z_i \text{ for } s = 1, 2.$$

Then we have

$$\begin{aligned}\sum_{s=1}^{2} \hat{p}(s) \cdot (x_i(s) - e_i(s)) &= \sum_{s=1}^{2} \pi_s p(s) \cdot (x_i(s) - e_i(s)) \\ &\leq \sum_{s=1}^{2} \pi_s p(s) z_i = \pi P z_i = q z_i \leq 0,\end{aligned}$$

where the first inequality uses the fact that $\pi \geq 0$.

Conversely assume now that $\sum_{s=1}^{2} \hat{p}(s) \cdot (x_i(s) - e_i(s)) \leq 0$. In order to support the bundle $(x_i(1), x_i(2))$ in the Radner market, the consumer needs wealth $p(s) \cdot (x_i(s) - e_i(s))$ in state $s$. But since $P$ has full rank, we can a find an asset portfolio $z_i$ which induces this wealth distribution at date 1:

$$Pz_i = \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ p(2) \cdot (x_i(2) - e_i(2)) \end{bmatrix} \implies z_i = P^{-1} \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ p(2) \cdot (x_i(2) - e_i(2)) \end{bmatrix} \tag{6}$$

Now note that:

$$\begin{aligned} qz_i &= \pi P z_i = \pi \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ p(2) \cdot (x_i(2) - e_i(2)) \end{bmatrix} \\ &= \sum_{s=1}^{2} \pi_s p(s) \cdot (x_i(s) - e_i(s)) = \sum_{s=1}^{2} \hat{p}(s) \cdot (x_i(s) - e_i(s)) \leq 0. \end{aligned}$$

Hence, the portfolio $z_i$ defined by (6) is affordable at date 0 in the Radner environment, and therefore the bundle $(x_i(1), x_i(2))$ is affordable at date 1. This completes the proof of equivalence under the assumption (5).

Finally, we must prove that $\pi \geq 0$ as we claimed. Without loss of generality, suppose to the contrary that $\pi_1 < 0$. Let $\overline{z_i}$ be the (Radner) equilibrium asset demand of consumer $i$. Since $P$ has full rank, there exists an asset portfolio $z'$ such that

$$Pz' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This combined with the definition of $\pi$ implies

$$qz' = \pi P z' = \pi_1 < 0.$$

This in turn implies that

$$q(\overline{z_i} + z') < q\overline{z_i} \leq 0.$$

Hence, the agent $i$ could afford the asset portfolio $\overline{z_i} + z'$ at date 0. This would give her the wealth distribution $P(\overline{z_i} + z')$ at date 1. Now note that

$$P(\overline{z_i} + z') = P\overline{z_i} + Pz' = P\overline{z_i} + \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

Hence, compared to $\overline{z_i}$, the portfolio $\overline{z_i} + z'$ gives more return at state 1 and the same return at state 2. Assuming that $i$'s utility is monotonic in state 1, this contradicts optimality of $\overline{z_i}$. In fact, by buying more and more of $z'$, agents can increase their state 1 consumption arbitrarily without reducing their state 2 consumption. ■

**Solution 2:**

Define the set of attainable date 1 wealth distributions as

$$W := \left\{ Pz : z \in \mathbf{R}^2, qz \leq 0 \right\},$$

where, as in Solution 1, we define

$$P := \left[ \begin{array}{c} p(1) \\ p(2) \end{array} \right].$$

Note that $W \cap \mathbf{R}^2_{++}$ is empty, as agents could increase their wealths in both states arbitrarily, which would contradict existence of a Radner equilibrium. Since $W$ is a convex set which does not intersect the interior of $\mathbf{R}^2_+$, then we can separate these two sets. In other words, there exists a vector $\pi \neq 0$ such that

$$\sup_{w \in W} \pi w \leq \inf_{y \in \mathbf{R}^2_+} \pi y. \tag{7}$$

Note that $\inf_{y \in \mathbf{R}^2_+} \pi y = 0$ - otherwise, there would be a vector $y' \in \mathbf{R}^2_+$ such that $\pi y' < 0$. But since $ny'$ is in $\mathbf{R}^2_+$ for any $n$, then $\inf_{y \in \mathbf{R}^2_+} \pi y = -\infty$, which would contradict (7). Therefore,

$$\sup_{w \in W} \pi w \leq 0 = \inf_{y \in \mathbf{R}^2_+} \pi y. \tag{8}$$

An immediate implication of (8) is that $\pi_1 = \pi \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pi_2 = \pi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0$. Since $\pi \neq 0$, then

$$\pi > 0. \tag{9}$$

Next, let's show that

$$\Psi := \left\{ w \in \mathbf{R}^2 : \pi w \leq 0 \right\} = W. \tag{10}$$

From (8) we already know that $W \subset \Psi$. To prove the converse, suppose by contradiction that there exists a $w \in \Psi \setminus W$. Since $P$ has full rank, then there exists a $z \in \mathbf{R}^2$ such that

$$Pz = w.$$

Then, the hypothesis $w \notin W$ is equivalent to $qz > 0$. But then $q(-z) < 0$, and hence, the point $-w = P(-z)$ belongs to $W$. Since $W \subset \Psi$, it follows that $\pi(-w) \leq 0$, and since $w \in \Psi$, then $\pi w = 0$. Note, however, that $q(-z) < 0$ implies that there exists a $z' >> -z$, sufficiently close to $-z$, such that $qz' < 0$. Then

$$\begin{aligned} \pi P z' &= \pi P (z' - z) + \pi P z = \pi P (z' - z) + \pi w \\ &= \pi P (z' - z) = \pi \begin{bmatrix} p(1)(z' - z) \\ p(2)(z' - z) \end{bmatrix}. \end{aligned}$$

If preferences are monotonic, then $p(1), p(2) > 0$. Since $(z' - z) >> 0$ and $\pi > 0$, it follows that

$$\pi P z' > 0,$$

which contradicts $qz' < 0$, and thus $W = \Psi$.

As usual, define

$$\widehat{p}(s) := \pi_s p(s) \quad \text{for } s = 1, 2.$$

Take any $x_i(1), x_i(2) \in \mathbf{R}^2_+$. First assume there is a $z_i \in \mathbf{R}^2$ such that

$$qz_i \leq 0 \quad \text{and} \quad p(s) \cdot (x_i(s) - e_i(s)) \leq p(s) z_i \text{ for } s = 1, 2.$$

Then we have

$$\begin{aligned} \sum_{s=1}^{2} \hat{p}(s) \cdot (x_i(s) - e_i(s)) &= \sum_{s=1}^{2} \pi_s p(s) \cdot (x_i(s) - e_i(s)) \\ &\leq \sum_{s=1}^{2} \pi_s p(s) z_i \quad = \pi P z_i \leq 0, \end{aligned}$$

where the first inequality follows from (9) and the second one follows from (10) and the definition of the set $W$.

Conversely assume now that $\sum_{s=1}^{2}\hat{p}(s)\cdot(x_i(s)-e_i(s))\leq 0$. In order to support the bundle $(x_i(1), x_i(2))$ in the Radner market, we need a wealth of $w_s := p(s)\cdot(x_i(s)-e_i(s))$ in state $s$. But since $P$ has full rank, we can find an asset portfolio $z_i$ which induces this wealth distribution at date 1:

$$Pz_i = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =: w \implies z_i = P^{-1}w.$$

Now note that

$$\begin{aligned} \pi P z_i &= \pi w = \sum_{s=1}^{2} \pi_s p(s)\cdot(x_i(s)-e_i(s)) \\ &= \sum_{s=1}^{2} \hat{p}(s)\cdot(x_i(s)-e_i(s)) \leq 0. \end{aligned}$$

Hence, the wealth distribution $w$ belongs to the set $\Psi$, which equals to $W$. In other words, the bundle $(x_i(1), x_i(2))$ is affordable at date 1 in the Radner market, and we're done. ■

*3. Consider an example of the economy described in the preceding question in which there are two consumers $i = 1, 2$ with von Neumann-Morgenstern utility functions*

$$V_1(c_1, c_2) = 2\ln c_1 + \ln c_2$$

*and*

$$V_2(c_1, c_2) = \ln c_1 + 2\ln c_2$$

*and endowments*

$$e_1(s) = \begin{cases} (1,1) & s = 1 \\ (2,2) & s = 2 \end{cases}$$

*and*

$$e_2(s) = \begin{cases} (2,2) & s = 1 \\ (1,1) & s = 2 \end{cases}.$$

*Define a competitive equilibrium for this economy. Characterize the set of equilibria for this economy and determine whether markets are "complete" or "incomplete" for each equilibrium. Explain what you mean by "complete" in this context.*

**Solution:**

A *competitive Radner equilibrium* is an attainable allocation $(x_1^*, x_2^*, z_1^*, z_2^*)$ and prices $p$ and $q$ such that, for $i = 1, 2$, $(x_i^*, z_i^*)$ solves

$$\max_{(x_i, z_i) \in \mathbf{R}_+^4 \times \mathbf{R}^2} \frac{1}{2} V_i (x_{i1}(1), x_{i2}(1)) + \frac{1}{2} V_i (x_{i1}(2), x_{i2}(2))$$
$$s.t.\ q \cdot z_i \leq 0,$$
$$p(s) \cdot (x_i(s) - e_i(s)) \leq p(s) \cdot z_i, \text{ for } s = 1, 2.$$

Note that *there does not exist a Radner equilibrium.* To prove this, we will consider two cases: First, we will assume that $p(1)$ and $p(2)$ are linearly independent and derive a contradiction. Second, we will assume that $p(1)$ and $p(2)$ are linearly dependent, and again derive a contradiction.

$p(1)$ **and** $p(2)$ **linearly independent**

Suppose $p(1)$ and $p(2)$ are linearly independent and that they are part of a Radner equilibrium $(x_1^*, z_1^*, x_2^*, z_2^*, p, q)$. From Question 2, we know that there exists $\hat{p}$ that makes the Radner equilibrium budget constraints equivalent to the Arrow-Debreu budget constraints. This implies that $(\hat{p}, x_1^*, x_2^*)$ is an A-D equilibrium. To see this, notice that $(x_1^*, x_2^*)$ is obviously attainable in the A-D economy. Moreover, from Question 2, we know that $(x_1^*, x_2^*)$ satisfies the A-D budget constraints. Finally, since we know that for any $x_i$ that satisfies consumer $i$'s A-D budget constraint there exists a $z_i$ such that $(x_i, z_i)$ satisfies the Radner equilibrium budget constraint, it must be the case that $V_i(x_i^*) \geq V_i(x_i)$ for any such budget feasible $x_i$.

Now let us identify the set of Arrow-Debreu equilibria. Let

$$a_1 = \hat{p} \cdot e_1 \text{ and } a_2 = \hat{p} \cdot e_2$$

be consumers' wealth in the A-D economy. Since both consumers have standard Cobb-Douglas preferences we know that their demand functions will be:

$$x_{11}(1) = \frac{a_1}{3\hat{p}_1(1)},\ x_{12}(1) = \frac{a_1}{6\hat{p}_2(1)},\ x_{11}(2) = \frac{a_1}{3\hat{p}_1(2)},\ x_{12}(2) = \frac{a_1}{6\hat{p}_2(2)}$$

for consumer 1 and

$$x_{21}(1) = \frac{a_2}{6\hat{p}_1(1)},\ x_{22}(1) = \frac{a_2}{3\hat{p}_2(1)},\ x_{21}(2) = \frac{a_2}{6\hat{p}_1(2)},\ x_{22}(2) = \frac{a_2}{3\hat{p}_2(2)}$$

for consumer 2. Imposing market clearing, we have

$$\hat{p}_1(1) = \frac{1}{3}\left(\frac{a_1}{3} + \frac{a_2}{6}\right) = \hat{p}_1(2)$$

and

$$\hat{p}_2(1) = \frac{1}{3}\left(\frac{a_1}{6} + \frac{a_2}{3}\right) = \hat{p}_2(2).$$

Consequently, the A-D equilibrium prices satisfy $\hat{p}(1) = \hat{p}(2)$. Given the definition of $\hat{p}$, this simply means $\pi_1 p(1) - \pi_2 p(2) = 0$. But since $\pi$ is non-zero, this contradicts the linear independence of $p(1)$ and $p(2)$. Hence, there is no Radner equilibrium with $p(1)$, $p(2)$ linearly independent.

**$p(1)$ and $p(2)$ linearly dependent**

Next, assume that the Radner equilibrium prices $p(1)$ and $p(2)$ are linearly dependent. We next sketch the proof of the following fact

$$q = \mu p(1) \quad \text{for some } \mu > 0. \tag{11}$$

By monotonicity of preferences, to prove (11) it suffices to show that $q$ belongs to the set $S := \{\alpha p(1) : \alpha \in \mathbf{R}\}$. If $q$ is not in $S$, then by applying the separating hyperplane theorem we can find a portfolio $z$ such that $qz < 0 = pz$ for all $p \in S$. In particular,

$$\begin{bmatrix} q \\ p(1) \\ p(2) \end{bmatrix} z = \begin{bmatrix} a < 0 \\ 0 \\ 0 \end{bmatrix}.$$

Given the monotonicity of preferences, we know that $p(1), p(2) > 0$. Now, we can find another portfolio $z' >> z$, sufficiently close to $z$, so that

$$qz' < 0. \tag{12}$$

Since $p(1), p(2) > 0$ and since $z' >> z$, we must also have

$$\begin{aligned} p(1)z' &> p(1)z = 0, \\ p(2)z' &> p(2)z = 0. \end{aligned} \tag{13}$$

(12) and (13) yields the contradiction, since $z'$ is an affordable portfolio that induces positive wealth in both states, thus creating an arbitrage opportunity.

Since $p(1)$ and $p(2)$ are collinear, by (11) $q$ is also collinear with $p(2)$; that is, there is a $\beta > 0$ such that $\beta p(2) = q = \mu p(1)$. Then for any $z_i \in \mathbf{R}^2$,

$$qz_i \leq 0 \Leftrightarrow p(s) z_i \leq 0, \text{ for } s = 1, 2.$$

Clearly, consumers will always choose $z_i$ such that $p(s) z_i = 0$ for $s = 1, 2$: since there is no opportunity of inducing a positive return to a state, the best consumers can do is to avoid losses. By the additivity of expected utility function, it is then clear that consumers must in fact be maximizing at each state independent of the other state. Formally we can find consumer $i$'s optimal consumption choice by solving the following problem for each $s = 1, 2$:

$$\begin{aligned} &\max_{x_i(s) \in \mathbf{R}^2_+} V_i(x_i(s)) \\ &s.t.\ p(s) \cdot (x_i(s) - e_i(s)) \leq 0. \end{aligned}$$

Again, since the consumers have standard Cobb-Douglas preferences we know that their demands will be:

$$\begin{aligned} x_{11}(1) &= \frac{2}{3}\frac{(p_1(1) + p_2(1))}{p_1(1)}, \\ x_{12}(1) &= \frac{1}{3}\frac{(p_1(1) + p_2(1))}{p_2(1)}, \\ x_{11}(2) &= \frac{4}{3}\frac{(p_1(2) + p_2(2))}{p_1(2)}, \\ x_{12}(2) &= \frac{2}{3}\frac{(p_1(2) + p_2(2))}{p_2(2)}, \end{aligned}$$

for consumer 1 and

$$\begin{aligned} x_{21}(1) &= \frac{2}{3}\frac{(p_1(1) + p_2(1))}{p_1(1)}, \\ x_{12}(1) &= \frac{4}{3}\frac{(p_1(1) + p_2(1))}{p_2(1)}, \\ x_{11}(2) &= \frac{1}{3}\frac{(p_1(2) + p_2(2))}{p_1(2)}, \\ x_{12}(2) &= \frac{2}{3}\frac{(p_1(2) + p_2(2))}{p_2(2)}, \end{aligned}$$

for consumer 2. Clearing the goods' markets we get:

$$\frac{p_2(1)}{p_1(1)} = \frac{5}{4}$$

and

$$\frac{p_2(2)}{p_1(2)} = \frac{4}{5}$$

which contradicts the assumption that the two price vectors are linearly dependent, and thus there does not exist a Radner equilibrium. ■

*4. Consider an economy with production in which there is a single good, two dates $t = 0, 1$, and two states of nature $s = 1, 2$. There is a single firm that chooses a production plan $y = (y_1, y_2)$ at date 0, where $y_s$ denotes the output of the good at date 1 in state $s = 1, 2$. A production plan $y$ is feasible if it satisfies the constraints*

$$y \geq 0 \text{ and } y_1 + y_2 \leq 1.$$

*There are two consumers, $i = 1, 2$, whose preferences are given by the utility functions*

$$U_1(c_1, c_2) = c_0 + 2 \ln c_1 + \ln c_2$$

*and*

$$U_2(c_1, c_2) = c_0 + \ln c_1 + 2 \ln c_2$$

*where $c_0$ denotes consumption at date 0 and $c_s$ denotes consumption in state $s = 1, 2$ at date 1. Each agent has an endowment of $w$ units of the good at date 0 and none at date 1 and owns a share $\bar{\theta}_i = 1/2$ of the firm. There are no markets for contingent commodities at date 0. The only way that goods can be reallocated at date 1 is by trading shares in the firm at date 0. At date 1 each agent $i$ consumes his share of the profits $\theta_i y_s$ where $y = (y_1, y_2)$ is the firm's production plan. Taking the firm's production plan $y$ as given, the consumers trade as if they were in a pure exchange economy in which the only commodities are shares and consumption at date 1. Describe the preferences and endowments of the pure exchange economy and then define a competitive equilibrium for the pure exchange economy. What necessary conditions must be satisfied by the firm's choice of production plan if the equilibrium is to be efficient? What objective function would you suggest for the firm to ensure an efficient choice of production plan?*

**Solution:**

Since consumers cannot trade contingent commodities at date 1, then $c_{is} = \theta_i y_s$. We can thus represent the consumers' preferences as

$$U_1(c_{10}, \theta_1) = c_{10} + 2\ln\theta_1 y_1 + \ln\theta_1 y_2$$

and

$$U_2(c_{20}, \theta_2) = c_{20} + \ln\theta_2 y_1 + 2\ln\theta_2 y_2.$$

Using the utility functions above, we can write consumer 1's problem as

$$\max_{(c_{10},\theta_1)\in\mathbf{R}_+\times[0,1]} c_{10} + 2\ln\theta_1 y_1 + \ln\theta_1 y_2$$
$$s.t.\ c_{10} + p\theta_1 \leq w + \frac{p}{2},$$

where the price of the good at $t = 0$ is normalized to 1 and $p$ is the price of a share of the firm. Similarly consumer 2's problem is

$$\max_{(c_{20},\theta_2)\in\mathbf{R}_+\times[0,1]} c_{20} + \ln\theta_2 y_1 + 2\ln\theta_2 y_2$$
$$s.t.\ c_{20} + p\theta_2 \leq w + \frac{p}{2}.$$

An allocation will be attainable in this economy if

$$c_{10} + c_{20} = 2w \text{ and } \theta_1 + \theta_2 = 1.$$

Therefore a competitive equilibrium for this economy is an attainable allocation $(c_{10}, c_{20}, \theta_1, \theta_2)$ and a share price $p$ such that given $p$ and the firm's production plan, $(c_{i,0}, \theta_i)$ solves consumer $i$'s problem.

Notice that an allocation solves the consumers' problem if and only if it solves[3]

$$\max_{(c_{10},\theta_1)\in\mathbf{R}_+\times[0,1]} c_{10} + 3\ln\theta_1$$
$$s.t.\ c_{10} + p\theta_1 \leq w + \frac{p}{2},$$

and

$$\max_{(c_{20},\theta_2)\in\mathbf{R}_+\times[0,1]^2_+} c_{20} + 3\ln\theta_2$$
$$s.t.\ c_{20} + p\theta_2 \leq w + \frac{p}{2}.$$

[3] Here we're using the fact that $\ln\theta y = \ln\theta + \ln y$.

In other words, both consumers solve the same problem. Moreover, we can easily see that the problem has a unique solution - to see this, note that by substituting out $c_{10}$ by inserting the budget constraint, we have

$$\max_{0 \leq \theta_1 \leq \min\{w/p+1/2,1\}} w + \frac{p}{2} - p\theta_1 + 3 \ln \theta_1.$$

But now the problem above is simply a maximization of a strictly concave function over a convex set, which has a unique solution. Given that both consumers make the same choice we know that the equilibrium for our economy must have the following allocation:

$$c_{10} = c_{20} = w \text{ and } \theta_1 = \theta_2 = 1/2.$$

It is easy to see that $p = 6$ supports such an equilibrium.

Now we have to find necessary conditions that the firm's production plan must satisfy in order for this equilibrium to be constrained efficient. First, we must define what we mean by constrained efficiency. We can show that in the unconstrained economy, which is not restricted to exchanging shares, this equilibrium will never be efficient. In this problem, therefore, we are talking about efficiency under the restrictions imposed by the exchange economy. That is, an allocation $(c_{10}, c_{20}, \theta_1, \theta_2, y_1, y_2)$ is (constrained) efficient if it is attainable and there is no other attainable allocation that Pareto dominates $(c_{10}, c_{20}, \theta_1, \theta_2, y_1, y_2)$.

Independently of $(y_1, y_2)$, our equilibrium will satisfy $c_{10} = c_{20} = w$ and $\theta_1 = \theta_2 = 1/2$ as discussed above. Since consumers have quasi-linear utilities and both consumers are consuming a positive amount at $t = 0$, we can prove that a necessary condition for the equilibrium to be efficient is that $(y_1, y_2)$ solves:

$$\max_{(y_1,y_2)\in\mathbf{R}^2_+} 2\ln\left(\frac{1}{2}y_1\right) + \ln\left(\frac{1}{2}y_2\right) + \ln\left(\frac{1}{2}y_1\right) + 2\ln\left(\frac{1}{2}y_2\right)$$
$$s.t.\ y_1 + y_2 = 1.$$

The solution to the problem above is $y_1 = y_2 = 1/2$. This condition is equivalent to saying that the equilibrium can be efficient only if the firm is producing the same quantity in both states. To prove this, suppose that $y_1^* \neq y_2^*$. We need to show that there exists an attainable allocation $(c_{10}, c_{20}, \theta_1, \theta_2, y_1, y_2)$ that Pareto dominates $(w, w, 1/2, 1/2, y_1^*, y_2^*)$. Suppose, without loss of generality, that $y_1^* > y_2^*$. Let $\lambda \in (0,1)$ be such that

$$2\ln y_1^* + \ln y_2^* - \left(2\ln\left(\lambda\frac{1}{2} + (1-\lambda)\,y_1^*\right) + \ln\left(\lambda\frac{1}{2} + (1-\lambda)\,y_2^*\right)\right) = \varepsilon < w.$$

Now let us look at the following allocation: $\left(w+\varepsilon, w-\varepsilon, 1/2, 1/2, y_1^\lambda, y_2^\lambda\right)$, where $y_1^\lambda \equiv \lambda\frac{1}{2} + (1-\lambda)\,y_1^*$ and $y_2^\lambda \equiv \lambda\frac{1}{2} + (1-\lambda)\,y_2^*$. By construction we have

$$\begin{aligned}
& U_1\left(w+\varepsilon, 1/2, y_1^\lambda, y_2^\lambda\right) = \\
& w + \varepsilon + 2\ln y_1^\lambda + \ln y_2^\lambda + 3\ln\frac{1}{2} = \\
& w + \varepsilon + 2\ln y_1^* + \ln y_2^* - \varepsilon + 3\ln\frac{1}{2} = \\
& w + 2\ln y_1^* + \ln y_2^* + 3\ln\frac{1}{2} = U_1\left(w, 1/2, y_1^*, y_2^*\right).
\end{aligned}$$

Now note that

$$\begin{aligned}
& U_1\left(w+\varepsilon, 1/2, y_1^\lambda, y_2^\lambda\right) + U_2\left(w-\varepsilon, 1/2, y_1^\lambda, y_2^\lambda\right) = \\
& w + 6\ln\frac{1}{2} + 3\left(\ln y_1^\lambda + \ln y_2^\lambda\right) > \\
& w + 6\ln\frac{1}{2} + 3\left(\lambda\left(\ln\frac{1}{2} + \ln\frac{1}{2}\right) + (1-\lambda)\left(\ln y_1^* + \ln y_2^*\right)\right) > \\
& w + 6\ln\frac{1}{2} + 3\left(\ln y_1^* + \ln y_2^*\right) = \\
& U_1\left(w, 1/2, y_1^*, y_2^*\right) + U_2\left(w, 1/2, y_1^*, y_2^*\right),
\end{aligned}$$

which implies that

$$U_2\left(w-\varepsilon, 1/2, y_1^\lambda, y_2^\lambda\right) > U_2\left(w, 1/2, y_1^*, y_2^*\right).$$

This shows that $(w, w, 1/2, 1/2, y_1^*, y_2^*)$ is not strongly Pareto efficient, but we can easily see that it is not weakly efficient either. For that we only need to transfer a small amount of the good at time 0 from consumer 2 to consumer 1. We still need to show that $(w, w, 1/2, 1/2, 1/2, 1/2)$ is indeed efficient. It is easy to see that $(w, w, 1/2, 1/2, 1/2, 1/2)$ solves the following problem:

$$\begin{aligned}
& \max_{c_{10}, c_{20}, \theta_1, \theta_2, y_1, y_2} U_1\left(c_{10}, \theta_1, y_1, y_2\right) + U_2\left(c_{20}, \theta_2, y_1, y_2\right) \\
& s.t.\ c_{10} + c_{20} \leq w,\ c_{10}, c_{20} \geq 0 \\
& \theta_1 + \theta_2 = 1,\ \theta_1, \theta_2 \geq 0 \\
& y_1 + y_2 \leq 1,\ y_1, y_2 \geq 0.
\end{aligned}$$

But this implies that $(w, w, 1/2, 1/2, 1/2, 1/2)$ has to be efficient, because any attainable allocation that Pareto dominates $(w, w, 1/2, 1/2, 1/2, 1/2)$ would be the solution for the problem above instead of $(w, w, 1/2, 1/2, 1/2, 1/2)$ itself. If the firm's objective is

$$f(y_1, y_2) = \ln y_1 + \ln y_2,$$

then the firm would produce $(1/2, 1/2)$. ■

*7. Consider a dynamic economy with an infinite horizon. Time is divided into a countable number of dates indexed by $t = 0, 1, \ldots$. There are two agents $i = 1, 2$. Uncertainty is represented by a sequence of i.i.d. r.v.s $\{s_t\}_{t=1}^{\infty}$. At each date $t$ the random variable $s_t$ takes the values 1 and 2 with equal probability. There is a single good at each date and the endowments of the two agents are functions of the random variable at each date:*

$$e_1(s_t) = \begin{cases} 1 & \text{if } s_t = 1 \\ 3 & \text{if } s_t = 2 \end{cases} \quad \text{and} \quad e_2(s_t) = \begin{cases} 2 & \text{if } s_t = 1 \\ 4 & \text{if } s_t = 2 \end{cases}.$$

*Each agent maximizes the expected value of the utility function*

$$\sum_{t=0}^{\infty} \beta^t \ln c_t,$$

*where $c_t$ denotes consumption at date $t$ and $0 < \beta < 1$ is a parameter. The initial state of the economy is $s_0 = 1$. Suppose there is a complete set of contingent commodity markets at date 0. Calculate the equilibrium prices of contingent commodities for this Arrow-Debreu economy. Derive a formula for the wealth of each agent $i = 1, 2$. Derive the equilibrium demand for contingent commodities for each agent $i = 1, 2$ in terms of his relative wealth.*

*Now suppose that there is a market for two Arrow securities at each date. An Arrow security at date $t$ is defined to be a security that pays one unit of the good at date $t + 1$ if state $s_t + 1$ occurs and nothing otherwise. What are the prices of the Arrow securities at date $t$? What are the equilibrium demands for Arrow securities at date $t$?*

**Solution:**

Since the agents have identical, homothetic utility functions, we can calculate prices $p_t(s_t)$ using the representative consumer's maximization problem. Note that the aggregate endowment is

$$e(s_t) = \begin{cases} 3 & \text{if } s_t = 1 \\ 7 & \text{if } s_t = 2 \end{cases}$$

and the representative consumer solves

$$\max_{\{c_t(s_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s_t=1}^{2} \pi_t(s_t|s^{t-1}) \ln c_t(s_t)$$
$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s_t=1}^{2} p_t(s_t) c_t(s_t) \leq \sum_{t=0}^{\infty} \sum_{s_t=1}^{2} p_t(s_t) e_t(s_t),$$

where $\pi_t(s_t|s^{t-1})$ is the probability of state $s_t$ occuring at time $t$, given history $s^{t-1}$. Under the iid assumption, it follows that $\pi_t(s_t|s^{t-1}) = \pi(s_t) = \frac{1}{2}$. The problem thus reduces to

$$\max_{\{c_t(s_t)\}} \ln c_0 + \frac{1}{2} \sum_{t=1}^{\infty} \beta^t \sum_{s_t=1}^{2} \ln c_t(s_t)$$
$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s_t=1}^{2} p_t(s_t) c_t(s_t) \leq \sum_{t=0}^{\infty} \sum_{s_t=1}^{2} p_t(s_t) e_t(s_t).$$

Normalize $p_0 = 1$, and let $\lambda$ denote the Lagrange multiplier on the representative agent's budget constraint. Taking first order conditions with respect to $c_0$ and $c_t(s_t)$, we have

$$\frac{1}{c_0} = \lambda \qquad \text{and} \qquad \frac{\beta^t}{2} \frac{1}{c_t(s_t)} = \lambda p_t(s_t) \qquad \Rightarrow \qquad p_t(s_t) = \frac{\beta^t}{2} \frac{c_0}{c_t(s_t)}.$$

In equilibrium, markets must clear, and thus $c_t(s_t) = e(s_t)$. It follows, then, that

$$p_t(1) = \frac{\beta^t}{2} \qquad \text{and} \qquad p_t(2) = \frac{3\beta^t}{14}.$$

Given these prices, the agents' time 0 wealth is:

$$w^1 = 1 + \sum_{t=1}^{\infty}\sum_{s_t=1}^{2} p_t(s_t)e_t^1(s_t) = 1 + 1\sum_{t=1}^{\infty}\frac{\beta^t}{2} + 3\sum_{t=1}^{\infty}\frac{3\beta^t}{14} = 1 + \frac{8}{7}\frac{\beta}{1-\beta}$$

$$w^2 = 2 + \sum_{t=1}^{\infty}\sum_{s_t=1}^{2} p_t(s_t)e_t^2(s_t) = 2 + 2\sum_{t=1}^{\infty}\frac{\beta^t}{2} + 4\sum_{t=1}^{\infty}\frac{3\beta^t}{14} = 2 + \frac{13}{7}\frac{\beta}{1-\beta}.$$

From the first order conditions for agent $i$'s maximization problem, note that

$$c_t^i(s_t) = \frac{\beta^t}{2}\frac{c_0^i}{p_t(s_t)}$$

for $i = 1, 2$. Plugging this into the budget constraint, we have

$$\sum_{t=0}^{\infty}\beta^t c_0^i = w^i \qquad \Rightarrow \qquad c_0^i = (1-\beta)w^i.$$

From here it's straight-forward to calculate demand:

$$\begin{aligned} c^1(1) &= \frac{7+\beta}{7} & \qquad c^1(2) &= \frac{7+\beta}{3} \\ c^2(1) &= \frac{14-\beta}{7} & \qquad c^2(2) &= \frac{14-\beta}{3}. \end{aligned}$$

Now let's consider the Arrow securities economy. There are four prices to calculate: $q(s'|s)$ for $s, s' = 1, 2$, where $q(s'|s)$ is the price of a security that delivers 1 unit of the good if state $s'$ occurs, given that the current state is $s$. From the standard Euler equation,[4] we have

$$q(s'|s) = \frac{\beta}{2}\frac{c(s)}{c(s')},$$

[4] From the representative agent's first order conditions at $s_t$ and $s_{t+1}|s_t$, we have $\frac{\beta^t}{2}\frac{1}{c(s_t)} = \lambda q_t(s_t)$ and $\frac{\beta^{t+1}}{(2)^2}\frac{1}{c(s_{t+1})} = \lambda q_{t+1}(s_{t+1}|s_t)$. By normalizing the time $t$ price to 1, we have that $q(s_{t+1}|s_t) = \frac{\beta}{2}\frac{c(s_t)}{c(s_{t+1})}$.

where $c(s)$ is the representative consumer's consumption in state $s$. By invoking market clearing and substituting the respective endowments in for $c(s)$ and $c(s')$, we have

$$\begin{aligned} q(1|1) &= \frac{\beta}{2} & q(2|1) &= \frac{3\beta}{14} \\ q(1|2) &= \frac{7\beta}{6} & q(2|2) &= \frac{\beta}{2}. \end{aligned}$$

Intuitively, the price $q(2|1)$ is relatively small because in the bad state $(s = 1)$ the agents have small endowments, and therefore the cost of a security in a future good state $(s' = 2)$ must be low in order to reduce the incentive for agents to borrow against their possibility large future endowment. Conversely, $q(1|2)$ is relatively large because in the good state, the agents would ideally want to purchase securities to insure against a possible future bad state $s' = 1$.

Next, let's compute the corresponding security demands. First, note that $z^i(s'|1) = z^i(s'|2) = z^i(s')$ for $i = 1, 2$ and $s' = 1, 2$ since the states are iid. In other words, the agents' $s'$ security demand at time $t + 1$ is independent of the state at time $t$. To compute these demands for agent $i$, we must solve the following system of equations:

$$\begin{aligned} c^i(1) + q(1|1)z^i(1) + q(2|1)z^i(2) &= e^i(1) + z^i(1) \\ c^i(2) + q(1|2)z^i(1) + q(2|2)z^i(2) &= e^i(2) + z^i(2). \end{aligned}$$

For agent 1, these equations are

$$\begin{aligned} \frac{7+\beta}{7} + \frac{\beta}{2}z^1(1) + \frac{3\beta}{14}z^1(2) &= 1 + z^1(1) \\ \frac{7+\beta}{3} + \frac{7\beta}{6}z^1(1) + \frac{\beta}{2}z^1(2) &= 3 + z^1(2). \end{aligned}$$

After some (tedious) algebra, we can solve for $z^1(1)$ in each of these two budget constraints:

$$
\begin{aligned}
z^1(1) &= \frac{2}{7}\frac{\beta}{2-\beta}\left[1+\frac{3}{2}z^1(2)\right] \\
z^1(1) &= \frac{2}{7}\frac{2-\beta}{\beta}\left[1+\frac{3}{2}z^1(2)\right].
\end{aligned}
$$

Note, however, that for $\beta \in (0,1)$, we have $\frac{2-\beta}{\beta} > \frac{\beta}{2-\beta}$, and thus it must be that

$$
1+\frac{3}{2}z^1(2) = 0 \qquad \Rightarrow \qquad z^1(2) = -\frac{2}{3}.
$$

Consequently, $z^1(1) = 0$. By the market clearing condition for the securities, we have

$$
\begin{aligned}
z^1(1) &= 0 \qquad\qquad z^1(2) = -\frac{2}{3} \\
z^2(1) &= 0 \qquad\qquad z^2(2) = \frac{2}{3}.
\end{aligned}
$$