

Problem Set 3 - Production Economies

Solutions

December 1, 2007

5. Imagine a world consisting of two countries, referred to as the home and foreign countries. If x denotes a variable for the home country, we use an asterisk to designate the corresponding variable x^* for the foreign country. The countries produce two commodities $h = 1, 2$ using a common technology subject to constant returns to scale. There are two factors of production, capital and labor. Each country has one unit of labor and the capital stocks are k and k^* , respectively. The output per worker of commodity h is represented by a neoclassical production function

$$f_h(k_h) = \frac{1}{1 - \alpha_h} (k_h)^{\alpha_h},$$

where $0 < \alpha_h < 1$. In each country, consumers' preferences are represented by a Cobb-Douglas utility function

$$u(c_1, c_2) = \ln c_1 + \ln c_2,$$

for any $(c_1, c_2) \geq 0$, where c_h denotes consumption of commodity h . We assume that factors of production are owned by consumers who supply them inelastically. Suppose that the home country has a larger capital stock than the foreign country, $k > k^*$, and that f_1 is more productive than f_2 , in the sense that $\alpha_1 > \alpha_2$.

- (i) Define a competitive equilibrium with free trade for this model.
- (ii) Show that the equilibrium is equivalent to the solution of a single maximization problem.
- (iii) Solve the maximization problem to determine the allocation of capital and labor and the output of commodities in each country.
- (iv) Determine the commodity prices (p_1, p_2) and (p_1^*, p_2^*) and the factor prices (r, w) and (r^*, w^*) in each country.
- (v) Which country exports each commodity? Relate this result to the Heckscher-Ohlin theory.

Solution:

i) Let e^* denote the exchange rate in the foreign country, that is, let e^* be the value of one unit of foreign currency in terms of the domestic currency. Free-trade implies that

$$(p_1, p_2) = e^* (p_1^*, p_2^*).$$

Nominal prices do not really matter in this problem. Hence, we can normalize e^* to 1 so that $p_h = p_h^*$.

The problem of the consumer in home country is:

$$\begin{aligned} \max & \ln c_1 + \ln c_2 \\ \text{s.t.} & p_1 c_1 + p_2 c_2 \leq rk + w \\ & c_1, c_2 \geq 0. \end{aligned}$$

The problem of the consumer in foreign country is:

$$\begin{aligned} \max & \ln c_1^* + \ln c_2^* \\ \text{s.t.} & p_1 c_1^* + p_2 c_2^* \leq r^* k^* + w^* \\ & c_1^*, c_2^* \geq 0. \end{aligned}$$

The problem of the domestic firm h is:

$$\begin{aligned} \max & p_h f_h(k_h) l_h - r k_h l_h - w l_h \\ \text{s.t.} & k_h, l_h \geq 0. \end{aligned}$$

Similarly the problem of the foreign firm h is:

$$\begin{aligned} \max & p_h f_h(k_h^*) l_h^* - r^* k_h^* l_h^* - w^* l_h^* \\ \text{s.t.} & k_h^*, l_h^* \geq 0. \end{aligned}$$

Since the factors of production are not mobile, an allocation $(c_h, c_h^*, k_h, l_h, k_h^*, l_h^*)_{h=1,2}$ is attainable if:

$$\begin{aligned} k_1 l_1 + k_2 l_2 &= k, \\ k_1^* l_1^* + k_2^* l_2^* &= k^*, \\ l_1 + l_2 &= l_1^* + l_2^* = 1, \\ c_1 + c_1^* &= f_1(k_1) l_1 + f_1(k_1^*) l_1^*, \end{aligned}$$

and

$$c_2 + c_2^* = f_2(k_2) l_2 + f_2(k_2^*) l_2^*.$$

Definition: An equilibrium is an attainable allocation $(c_h, c_h^*, k_h, l_h, k_h^*, l_h^*)_{h=1,2}$ and prices $(p_1, p_2, r, w, r^*, w^*)$ such that given the prices, the allocation solves households' and firms' problems.

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ii-iii) The two Welfare Theorems are helpful when approaching this problem: Under some regularity assumptions, the equilibrium and efficient allocations will coincide. The standard Welfare Theorems apply for closed economies, and thus for this problem, we need to extend the Theorems to open economies.

We shall start with writing down a Negishi problem where consumers in the two countries have weights $\eta, 1 - \eta > 0$.

$$\begin{aligned}
& \max \eta (\ln c_1 + \ln c_2) + (1 - \eta) (\ln c_1^* + \ln c_2^*) && \text{(Problem0)} \\
& s.t. \ c_1 + c_1^* = f_1(k_1)l_1 + f_1(k_1^*)l_1^* \\
& \quad c_2 + c_2^* = f_2(k_2)l_2 + f_2(k_2^*)l_2^* \\
& \quad k_1l_1 + k_2l_2 = k \\
& \quad k_1^*l_1^* + k_2^*l_2^* = k^* \\
& \quad l_1 + l_2 = 1 \\
& \quad l_1^* + l_2^* = 1 \\
& \quad c_h, c_h^*, k_h, k_h^*, l_h, l_h^* \geq 0, \ h = 1, 2.
\end{aligned}$$

Note that first order conditions with respect to c_h, c_h^* imply $\frac{c_1}{c_1^*} = \frac{c_2}{c_2^*} = \frac{\eta}{1-\eta}$. This in turn implies that $c_h = \eta(c_h + c_h^*)$ and $c_h^* = (1 - \eta)(c_h + c_h^*)$. If we define $\bar{c}_h := c_h + c_h^*$, we can substitute the previous equations into the Negishi problem's objective function to see that

$$\begin{aligned}
& \max \eta (\ln \eta \bar{c}_1 + \ln \eta \bar{c}_2) + (1 - \eta) (\ln (1 - \eta) \bar{c}_1 + \ln (1 - \eta) \bar{c}_2) = \\
& \max \eta (\ln \bar{c}_1 + \ln \bar{c}_2) + (1 - \eta) (\ln \bar{c}_1 + \ln \bar{c}_2) = \\
& \max \ln \bar{c}_1 + \ln \bar{c}_2.
\end{aligned}$$

We've thus reduced the problem into one of a representative consumer:

$$\begin{aligned}
& \max \ln \bar{c}_1 + \ln \bar{c}_2 && \text{(Problem1)} \\
& s.t. \ \bar{c}_1 = f_1(k_1)l_1 + f_1(k_1^*)l_1^* \\
& \quad \bar{c}_2 = f_2(k_2)l_2 + f_2(k_2^*)l_2^* \\
& \quad k_1l_1 + k_2l_2 = k \\
& \quad k_1^*l_1^* + k_2^*l_2^* = k^* \\
& \quad l_1 + l_2 = 1 \\
& \quad l_1^* + l_2^* = 1 \\
& \quad k_h, k_h^*, l_h, l_h^* \geq 0, \ h = 1, 2.
\end{aligned}$$

Note that the two problems above will lead to the same efficient factor allocation. Since both consumers have identical, homothetic utility, in equilibrium the representative consumer will demand the same amount as the two consumers. Consequently, it should not be surprising that the two problems are equivalent.

I leave it to you to check that in Problem1 first order conditions w.r.t. k_h, k_h^* imply

$$\left(\frac{k_1}{k_1^*}\right)^{\alpha_1-1} = \frac{f_1'(k_1)}{f_1'(k_1^*)} = \frac{f_2'(k_2)}{f_2'(k_2^*)} = \left(\frac{k_2}{k_2^*}\right)^{\alpha_2-1}. \quad (1)$$

Similarly, using (1) one can simplify first order conditions w.r.t. l_h, l_h^* to show that

$$\left(\frac{k_1}{k_1^*}\right)^{\alpha_1} = \frac{f_1(k_1) - f_1'(k_1)k_1}{f_1(k_1^*) - f_1'(k_1^*)k_1^*} = \frac{f_2(k_2) - f_2'(k_2)k_2}{f_2(k_2^*) - f_2'(k_2^*)k_2^*} = \left(\frac{k_2}{k_2^*}\right)^{\alpha_2}. \quad (2)$$

Now dividing (2) by (1), we have

$$\frac{k_1}{k_1^*} = \frac{k_2}{k_2^*}.$$

If we substitute this back into (1) and use the condition $\alpha_1 > \alpha_2$, we see that $k_h = k_h^*$. Hence, if we define $\bar{k}_h := k_h = k_h^*$, Problem1 can be further reduced to¹

$$\begin{aligned} & \max \ln \bar{c}_1 + \ln \bar{c}_2 && \text{(Problem1a)} \\ & s.t. \bar{c}_1 = f_1(\bar{k}_1) (l_1 + l_1^*) \\ & \bar{c}_2 = f_2(\bar{k}_2) (l_2 + l_2^*) \\ & \bar{k}_1 l_1 + \bar{k}_2 l_2 = k \\ & \bar{k}_1 l_1^* + \bar{k}_2 l_2^* = k^* \\ & l_1 + l_2 = 1 \\ & l_1^* + l_2^* = 1 \\ & \bar{k}_h, l_h, l_h^* \geq 0, \quad h = 1, 2. \end{aligned}$$

Now if we define $\bar{l}_h := l_h + l_h^*$, by summing the 3rd and 4th constraints in Problem1a, we have $\bar{k}_1 \bar{l}_1 + \bar{k}_2 \bar{l}_2 = k + k^*$, and by summing 5th and 6th constraints, we have $\bar{l}_1 + \bar{l}_2 = 2$. Hence, the constraints of Problem1a are more restrictive than those of the following problem, which is easier to solve:

$$\begin{aligned} & \max \ln \bar{c}_1 + \ln \bar{c}_2 && \text{(Problem2)} \\ & s.t. \bar{c}_1 = f_1(\bar{k}_1) \bar{l}_1 \\ & \bar{c}_2 = f_2(\bar{k}_2) \bar{l}_2 \\ & \bar{k}_1 \bar{l}_1 + \bar{k}_2 \bar{l}_2 = k + k^* \\ & \bar{l}_1 + \bar{l}_2 = 2 \\ & \bar{k}_h, \bar{l}_h \geq 0, \quad h = 1, 2. \end{aligned}$$

In Problem2, we've aggregated the two economies into a one big economy which has all the resources. As we already mentioned, this economy is more efficient because its resource constraints are "looser" than in Problem1a. Our next task is to solve Problem2, and then we will find a set of conditions where the solution to Problem2 coincides with the solution of Problem1a (and then the equilibrium allocation).

If we define $\lambda := \bar{l}_1$, Problem2 is equivalent to:

$$\begin{aligned} & \max \ln (\lambda f_1(\bar{k}_1)) + \ln ((2 - \lambda) f_2(\bar{k}_2)) && \text{(Problem2a)} \\ & s.t. \lambda \bar{k}_1 + (2 - \lambda) \bar{k}_2 = k + k^* \\ & \bar{k}_1, \bar{k}_2 \geq 0, \quad 0 \leq \lambda \leq 2. \end{aligned}$$

¹Note that, in general, constraints of Problem1a are more restrictive than those of Problem1: we impose the additional restriction $k_h = k_h^*$ ($= \bar{k}_h$). Hence, the fact that the solution of Problem1 is feasible in Problem1a implies that the same allocation solves this latter problem as well.

Assuming the solution is interior, the Lagrangian for this problem is

$$\mathcal{L} = \ln(\lambda f_1(\bar{k}_1)) + \ln((2-\lambda)f_2(\bar{k}_2)) + \mu(k + k^* - \lambda\bar{k}_1 - (2-\lambda)\bar{k}_2).$$

Recalling $\bar{c}_1 = \lambda f_1(\bar{k}_1)$ and $\bar{c}_2 = (2-\lambda)f_2(\bar{k}_2)$, the first order conditions with respect to \bar{k}_1, \bar{k}_2 give

$$\frac{1}{\bar{c}_1} f_1'(\bar{k}_1) = \mu = \frac{1}{\bar{c}_2} f_2'(\bar{k}_2). \quad (3)$$

Similarly, foc's w.r.t. λ can be written as

$$\frac{1}{\bar{c}_1} f_1(\bar{k}_1) - \mu\bar{k}_1 = \frac{1}{\bar{c}_2} f_2(\bar{k}_2) - \mu\bar{k}_2. \quad (4)$$

By substituting (3) into this last equation we obtain

$$\frac{1}{\bar{c}_1} (f_1(\bar{k}_1) - f_1'(\bar{k}_1)\bar{k}_1) = \frac{1}{\bar{c}_2} (f_2(\bar{k}_2) - f_2'(\bar{k}_2)\bar{k}_2). \quad (5)$$

By substituting $f_h := (1 - \alpha_h)^{-1} (k_h)^{\alpha_h}$ in (3) and (5) we obtain

$$\frac{1}{\bar{c}_1} \frac{\alpha_1}{1 - \alpha_1} \bar{k}_1^{-\alpha_1 - 1} = \frac{1}{\bar{c}_2} \frac{\alpha_2}{1 - \alpha_2} \bar{k}_2^{-\alpha_2 - 1} \quad (6)$$

$$\frac{1}{\bar{c}_1} \bar{k}_1^{-\alpha_1} = \frac{1}{\bar{c}_2} \bar{k}_2^{-\alpha_2}. \quad (7)$$

Dividing (7) by (6) gives $\frac{\alpha_2}{1 - \alpha_2} \bar{k}_1 = \frac{\alpha_1}{1 - \alpha_1} \bar{k}_2$, which is equivalent to

$$\bar{k}_2 = \gamma \bar{k}_1 \quad (8)$$

where $\gamma := \frac{1 - \alpha_1}{\alpha_1} \frac{\alpha_2}{1 - \alpha_2}$. Note that $\gamma < 1$ so that $\bar{k}_2 < \bar{k}_1$.

Now notice that (7) can be rewritten as:

$$\frac{1}{\frac{1}{1 - \alpha_1} \bar{k}_1^{-\alpha_1} \lambda} \bar{k}_1^{-\alpha_1} = \frac{1}{\frac{1}{1 - \alpha_2} \bar{k}_2^{-\alpha_2} (2 - \lambda)} \bar{k}_2^{-\alpha_2},$$

which implies that

$$\frac{1 - \alpha_1}{\lambda} = \frac{1 - \alpha_2}{(2 - \lambda)}.$$

This gives efficient labor allocation as a function of primitives:

$$\lambda = \frac{2(1 - \alpha_1)}{2 - \alpha_2 - \alpha_1}. \quad (9)$$

Using (8) in the resource constraint of Problem2a we get:

$$\lambda\bar{k}_1 + (2 - \lambda)\gamma\bar{k}_1 = k + k^*$$

Solving this equation for \bar{k}_1 we find

$$\bar{k}_1 = \frac{k + k^*}{\lambda + \gamma(2 - \lambda)} \quad (10)$$

so that

$$\bar{k}_2 = \gamma \bar{k}_1 = \frac{\gamma(k + k^*)}{\lambda + \gamma(2 - \lambda)}. \quad (11)$$

Equations (9)-(11) complete the description of the efficient factor allocation for Problem2a.

Before we proceed, recall that Problem0-Problem1a (and similarly Problem2-Problem2a) are equivalent in the sense that they lead to same efficient factor allocation. Now to complete the answer of the problem we need to show the following:

A) Under suitable conditions, the solution to Problem1a coincides with the solution to Problem2.

B) The solution of Problem1 is the equilibrium allocation of factors.

Equivalence of Problem1a and Problem2.

Since the constraint set of Problem1a is a subset of the constraint set of Problem2, all we need to do is to determine conditions under which the solution of Problem2 will be feasible in Problem1a: the allocation which maximizes the common objective function on a larger constraint set will also maximize the function on the smaller constraint set.

Let us denote the solution of Problem2 with $(\bar{k}_1, \bar{k}_2, \lambda)$ and recall that $\lambda := (l_1 + l_1^*)$. A bit caution here: when solving Problem2, we explicitly found λ , but we did not determine the particular values of l_1 and l_1^* . Indeed, our purpose is just to determine whether we can decompose λ into particular values of $l_1, l_1^* \in (0, 1)$ such that the 3rd and 4th constraints of Problem1a are satisfied.

Bearing in mind that $l_2 = (1 - l_1)$, the 3rd constraint of Problem1a can be written as

$$l_1 = \frac{k - \bar{k}_2}{\bar{k}_1 - \bar{k}_2}. \quad (12)$$

Similarly, the 4th constraint of Problem1a is equivalent to

$$l_1^* = \frac{k^* - \bar{k}_2}{\bar{k}_1 - \bar{k}_2}. \quad (13)$$

Note that under (12) and (13),

$$l_1 + l_1^* = \frac{k + k^* - 2\bar{k}_2}{\bar{k}_1 - \bar{k}_2} = \lambda.$$

Therefore, we only need to find the conditions where, under (12) and (13), the points l_1, l_1^* belong to $(0, 1)$.² That is, we want

$$0 < \frac{k - \bar{k}_2}{\bar{k}_1 - \bar{k}_2} < 1$$

²We work with the open interval $(0, 1)$ since we've assumed an interior solution.

and

$$0 < \frac{k^* - \bar{k}_2}{k_1 - \bar{k}_2} < 1.$$

Using the fact that $k^* < k$, these two conditions are equivalent to

$$\bar{k}_2 < k^* < k < \bar{k}_1. \quad (14)$$

Hence, if k^* and k are sufficiently close to each other so that (14) holds, we can conclude that Problem1a and Problem2 are equivalent.

Equivalence between the solution of Problem1 and equilibrium factor allocation.

Suppose that the equilibrium factor allocation does not solve Problem1. Recall that we have a representative consumer here. Since equilibrium requires attainability, equilibrium consumption of the representative consumer is feasible in Problem1. Hence, the hypothesis that the equilibrium allocation does not solve Problem1 simply means that the consumption bundle (c_1, c_2) implied by the solution of Problem1 is not in the budget set of the representative agent. In other words, we must have

$$p_1 c_1 + p_2 c_2 > rk + w + r^* k^* + w^*, \quad (15)$$

where $(p_1, p_2, r, w, r^*, w^*)$ are the equilibrium prices. Next, if we substitute $c_h = f_h(k_h)l_h + f_h(k_h^*)l_h^*$ into (15), we obtain:

$$p_1 (f_1(k_1)l_1 + f_1(k_1^*)l_1^*) + p_2 (f_2(k_2)l_2 + f_2(k_2^*)l_2^*) > rk + w + r^* k^* + w^*. \quad (16)$$

Since in equilibrium firms are maximizing profits, then their equilibrium profits must be less than 0 due to CRS production. Consequently,

$$\begin{aligned} p_h f_h(k_h)l_h - rk_h l_h - wl_h &\leq 0, \quad h = 1, 2, \\ p_h f_h(k_h^*)l_h^* - r^* k_h^* l_h^* - w^* l_h^* &\leq 0, \quad h = 1, 2. \end{aligned}$$

By summing over h , we then get:

$$\begin{aligned} \sum_{h=1}^2 p_h f_h(k_h)l_h &\leq \sum_{h=1}^2 rk_h l_h + wl_h, \\ \sum_{h=1}^2 p_h f_h(k_h^*)l_h^* &\leq \sum_{h=1}^2 r^* k_h^* l_h^* + w^* l_h^*. \end{aligned}$$

Note that by attainability of efficient allocation,

$$\begin{aligned} \sum_{h=1}^2 rk_h l_h + wl_h &= rk + w, \\ \sum_{h=1}^2 r^* k_h^* l_h^* + w^* l_h^* &= r^* k^* + w^*. \end{aligned}$$

Combining the four equations, we have

$$\begin{aligned} \sum_{h=1}^2 p_h f_h(k_h) l_h &\leq rk + w, \\ \sum_{h=1}^2 p_h f_h(k_h^*) l_h^* &\leq r^* k^* + w^*. \end{aligned}$$

Adding these two inequalities together yields the contradiction to (16). Consequently, the factor allocation that solves Problem1 is part of an equilibrium. Thus, we only need to find prices that make such an allocation optimal for both the firms and the consumers.

Let us denote the solution of Problem1 with $(k_h, k_h^*, l_h, l_h^*)_{h=1,2}$. From (1) and (2) we know that $k_h = k_h^*$. Again, define $\bar{k}_h := k_h = k_h^*$ and assume that (14) holds so that Problem1 and Problem2a are equivalent. Let $\lambda := l_1 + l_1^*$, $\bar{c}_1 := \lambda f_1(\bar{k}_1)$ and $\bar{c}_2 := (2 - \lambda) f_2(\bar{k}_2)$. Then the allocation at our hand must satisfy equations (3) and (4). Let μ be the associated multiplier. Define prices as

$$\begin{aligned} p_h &: = \frac{1}{\bar{c}_h}, \quad h = 1, 2, \\ r &: = r^* := \mu, \\ w &: = w^* := \frac{1}{\bar{c}_1} f_1(\bar{k}_1) - \mu \bar{k}_1 = \frac{1}{\bar{c}_2} f_2(\bar{k}_2) - \mu \bar{k}_2. \end{aligned}$$

With prices defined as above, (3) and (4) reduce to the following conditions, which are the foc's of the firms' problems:

$$\begin{aligned} p_h f_h'(\bar{k}_h) &= r = r^*, \quad h = 1, 2, \\ p_h f_h(\bar{k}_h) - r \bar{k}_h &= w, \quad h = 1, 2, \\ p_h f_h(\bar{k}_h) - r^* \bar{k}_h &= w^*, \quad h = 1, 2. \end{aligned} \tag{17}$$

These equations show that the given allocation solves the firms' problems at the given prices.

Next, we must show that \bar{c}_h is the optimal consumption demand, given prices. Note that at the given prices, consumers demand

$$\hat{c}_1 = \frac{rk + w}{2p_1} \quad \text{and} \quad \hat{c}_1^* = \frac{r^* k^* + w^*}{2p_1}.$$

Thus

$$\begin{aligned}
\widehat{c}_1 + \widehat{c}_1^* &= \frac{rk + w + r^*k^* + w^*}{2p_1} & (18) \\
&= \frac{r(k + k^*) + 2w}{2p_1} \\
&= \frac{r((l_1 + l_1^*)\bar{k}_1 + (l_2 + l_2^*)\bar{k}_2) + 2w}{2p_1} \\
&= \frac{r(\lambda\bar{k}_1 + (2 - \lambda)\bar{k}_2) + 2w}{2p_1} \\
&= \frac{\lambda(r\bar{k}_1 + w) + (2 - \lambda)(r\bar{k}_2 + w)}{2p_1} \\
&= \frac{\lambda p_1 f_1(\bar{k}_1) + (2 - \lambda) p_2 f_2(\bar{k}_2)}{2p_1},
\end{aligned}$$

where the last equality follows from (17). Now observe that (5) implies

$$p_1(1 - \alpha_1)f_1(\bar{k}_1) = p_1(\bar{k}_1)^{\alpha_1} = p_2(\bar{k}_2)^{\alpha_2} = p_2(1 - \alpha_2)f_2(\bar{k}_2).$$

Multiplying both sides of the expression above by $(2 - \lambda)$ and using the formula for λ in (9) (which must be true since our allocation solves Problem2a), we get

$$\lambda p_1 f_1(\bar{k}_1) = (2 - \lambda) p_2 f_2(\bar{k}_2).$$

Substituting this into (18), we obtain

$$\widehat{c}_1 + \widehat{c}_1^* = \frac{2\lambda p_1 f_1(\bar{k}_1)}{2p_1} = \lambda f_1(\bar{k}_1) = \bar{c}_1.$$

Similarly, we can also show that the aggregate demand for good 2 equals to \bar{c}_2 .

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iv) When solving this part of the question, we assume that Problem1 and Problem2a are equivalent so that we can use solution of Problem2a to find the prices.

Let $(\bar{k}_1, \bar{k}_2, \lambda)$ be the solution of Problem2a as determined by equations (9), (10) and (11). Since equilibrium allocation solves Problem1, which is assumed to be the same as Problem2a, we can immediately recover equilibrium factor allocations as

$$\widehat{k}_h = \widehat{k}_h^* = \bar{k}_h, \quad h = 1, 2. \quad (19)$$

$$\begin{aligned}
\widehat{l}_1 &= \frac{k - \bar{k}_2}{\bar{k}_1 - \bar{k}_2}, \\
\widehat{l}_1^* &= \frac{k^* - \bar{k}_2}{\bar{k}_1 - \bar{k}_2},
\end{aligned}$$

where in the last two equations we use formulas (12),(13). For future use, note that $k > k^*$ and $\bar{k}_1 > \bar{k}_2$ imply

$$\widehat{l}_1 > \widehat{l}_1^*. \quad (20)$$

From (19) and the foc's of the firms' problems, we have

$$p_h f'_h(\bar{k}_h) = r = r^*, \quad h = 1, 2. \quad (21)$$

So, if we can find commodity prices, this equation will give us the interest rates. As usual, we can only pin down the ratio of commodity prices. From (21), it immediately follows that

$$\frac{p_2}{p_1} = \frac{f'_1(\bar{k}_1)}{f'_2(\bar{k}_2)}.$$

If we use the normalization $p_1 = 1$, from this equation we find that $p_2 = \frac{f'_1(\bar{k}_1)}{f'_2(\bar{k}_2)}$. Similarly, from (21) we find that $r = r^* = f'_1(\bar{k}_1)$.

Now, equation (19), and foc's of the firms' problems imply:

$$p_h f_h(\bar{k}_h) - p_h f'_h(\bar{k}_h) \bar{k}_h = w = w^*.$$

Substituting $p_1 = 1$, shows that wages are equal to $f_1(\bar{k}_1) - f'_1(\bar{k}_1) \bar{k}_1 = (\bar{k}_1)^{\alpha_1}$.

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v) According to Hecksher-Ohlin theory, the country which has a higher (per-capita) capital must be exporting the commodity produced by the technology which is more capital intensive. We have seen that in equilibrium, $\widehat{k}_1 = \widehat{k}_1^* = \bar{k}_1 > \bar{k}_2 = \widehat{k}_2^* = \bar{k}_2$. That is, technology 1 is more capital intensive. Hence, we must be able to show that home country exports commodity 1.

Now, recall that the equilibrium demands for the first good is given by

$$\widehat{c}_1 = \frac{rk + w}{2p_1} \quad \text{and} \quad \widehat{c}_1^* = \frac{r^*k^* + w^*}{2p_1}. \quad (22)$$

Since equilibrium profits in home country are 0, we must have

$$p_1 f_1(\bar{k}_1) \widehat{l}_1 + p_2 f_2(\bar{k}_2) \widehat{l}_2 = rk + w. \quad (23)$$

Similarly,

$$p_1 f_1(\bar{k}_1) \widehat{l}_1^* + p_2 f_2(\bar{k}_2) \widehat{l}_2^* = r^*k^* + w^*. \quad (24)$$

Combining (22),(23),(24) gives:

$$\widehat{c}_1 = \frac{p_1 f_1(\bar{k}_1) \widehat{l}_1 + p_2 f_2(\bar{k}_2) \widehat{l}_2}{2p_1} \quad \text{and} \quad \widehat{c}_1^* = \frac{p_1 f_1(\bar{k}_1) \widehat{l}_1^* + p_2 f_2(\bar{k}_2) \widehat{l}_2^*}{2p_1}.$$

Hence,

$$\begin{aligned}
f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1 &= f_1(\bar{k}_1)\hat{l}_1 - \frac{p_1 f_1(\bar{k}_1)\hat{l}_1 + p_2 f_2(\bar{k}_2)\hat{l}_2}{2p_1} \\
&= \frac{f_1(\bar{k}_1)\hat{l}_1}{2} - \frac{p_2 f_2(\bar{k}_2)(1 - \hat{l}_1)}{2p_1} \\
&= \hat{l}_1 \left(\frac{f_1(\bar{k}_1)}{2} + \frac{p_2 f_2(\bar{k}_2)}{2p_1} \right) - \frac{p_2 f_2(\bar{k}_2)}{2p_1}.
\end{aligned}$$

Similarly,

$$f_1(\bar{k}_1)\hat{l}_1^* - \hat{c}_1^* = \hat{l}_1^* \left(\frac{f_1(\bar{k}_1)}{2} + \frac{p_2 f_2(\bar{k}_2)}{2p_1} \right) - \frac{p_2 f_2(\bar{k}_2)}{2p_1}.$$

We already now that $\hat{l}_1 > \hat{l}_1^*$ from (20). Hence, these last two equations immediately imply

$$f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1 > f_1(\bar{k}_1)\hat{l}_1^* - \hat{c}_1^*. \quad (25)$$

Now, we want to show that home country exports good 1. That is, we want

$$f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1 > 0. \quad (26)$$

By attainability, we have

$$f_1(\bar{k}_1)\hat{l}_1^* - \hat{c}_1^* = -(f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1). \quad (27)$$

Combining, (25) and (27) yields

$$\begin{aligned}
f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1 &> -(f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1) \Leftrightarrow \\
2(f_1(\bar{k}_1)\hat{l}_1 - \hat{c}_1) &> 0,
\end{aligned}$$

thus proving (26). Using an analogous argument, we can show that that foreign country exports the labor intensive good 2. ■