Markov Equilibria in Dynamic Matching and Bargaining Games[1]

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Rubinstein and Wolinsky (1990) show that a simple homogeneous market with exogenous matching has a continuum of (non-competitive) perfect equilibria; however, the unique Markov perfect equilibrium of this model is competitive. By contrast, in the more general case of heterogeneous markets, even the Markov property is not enough to guarantee the perfectly competitive outcome. We define a market game that allows for heterogeneous values on both sides of the market and exhibit a number of examples of (non-competitive) Markov-perfect equilibria, with and without discounting. Unlike the homogeneous case, these

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equilibria allow for inefficient trades and for trade at non-uniform prices. The non-competitive equilibrium may be unique.

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1. INTRODUCTION

The theory of competitive equilibrium provides an elegant and simple account of how markets work. By contrast, the strategic analysis of markets tends to be complex and intractable. In particular, dynamic market games have many equilibria, in which a variety of different kinds of behavior are sustained by threats and counter-threats. In a seminal paper, Rubinstein and Wolinsky (1990), henceforth RW, analyze a market for a single indivisible good. There is a finite number of buyers and sellers who are matched in pairs and bargain over the terms of trade. The trading process can be formalized as a simple, extensive-form game. RW show that this dynamic matching and bargaining game[2] possesses a large set of perfect-equilibrium outcomes,[3] a result reminiscent of the Folk Theorem for repeated games.[4] Most of these equilibria do not correspond to the perfectly competitive outcome.

RW also consider conditions under which perfect-equilibrium outcomes are competitive. For example, in a game with *exogenous, random matching*, it is shown that any perfect equilibrium in which equilibrium strategies are Markovian (anonymous) is competitive. In a similar vein, Sabourian (2001), henceforth S, has investigated a refinement based on lexicographic minimization of the complexity costs of implementing strategies. In the context of RW's model, he shows that perfect equilibria satisfy this refinement only if they are competitive.

These results suggest that perfectly competitive behavior may obtain where agents are required to use simple strategies. Unfortunately, both RW and S restrict their attention to a very simple environment, a market

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[2] In the sequel, when we refer to market games, we have in mind games of this general type.

[3] We use the term perfect equilibrium to embrace both subgame perfect equilibrium and sequential or perfect Bayesian equilibrium, as appropriate. In markets where a single pair of agents is matched at any time, it is sufficient to use the concept of subgame perfect equilibrium. Sequential or perfect Bayesian equilibrium is needed when simultaneous matching and bargaining are allowed. In the following discussion, where the technical differences are not important, we use the term perfect equilibrium to cover both cases.

[4] An important feature of RW is that it analyzes a market with a finite number of agents. The preceding literature (Rubinstein and Wolinsky (1985), Gale (1986a,b,c, 1987), Binmore and Herrera (1988a,b), McLennan and Sonnenschein (1991), Osborne and Rubinstein (1990)) assumes a non-atomic continuum of agents, each of whom has a negligible effect on equilibrium.

for a single indivisible good consisting of B identical buyers and S identical sellers, each of whom wants to trade at most one unit of the good. In a heterogenous market, where buyers (or sellers) have a range of valuations of the good, things turn out to be more complicated. In particular, the refinements proposed by RW are insufficient to guarantee competitive behavior, even with exogenous, sequential matching.[5]

The rest of the paper is organized as follows. In Section 2, we explore the differences between homogeneous and heterogeneous markets and explain why the methods used in RW may not suffice to characterize competitive behavior in the richer, heterogeneous environment. The market game is defined formally in Section 3. In Sections 4 to 6, we present a series of examples that illustrate the types of non-competitive equilibrium behavior that can easily arise in heterogeneous markets.

Section 4 analyzes a four-person game with sequential matching and no discounting. We construct a continuum of asymmetric Markov-perfect equilibria of the game and show that the outcomes are non-competitive. In fact, trade occurs at non-uniform prices that do not belong to the competitive interval. We modify the example to allow for inefficient trade and show that inefficient trade does occur in equilibrium.

In Section 5 we introduce a discounting into a symmetric four-person example. We show that there exists a unique, symmetric, Markov-perfect equilibrium, when the discount factor is close to one, and that trade occurs at non-uniform prices that do not belong to the competitive interval. Again, we argue that this example can be extended to allow for inefficient trade.

The model used in Sections 4 and 5 assume sequential matching: at most one pair can bargain and trade at any time. In Section 6, we consider markets with simultaneous trade and show that this is another robust source of non-competitive behavior.

2. DYNAMIC MATCHING AND BARGAINING GAMES

2.1. Homogeneous markets

RW study the following market game. There are S sellers and $B > S$ buyers. Each seller has one unit of an indivisible good and each buyer wants to buy at most one unit of the good. A seller's valuation of the good is 0 and a buyer's valuation is 1. Time is divided into discrete periods or dates indexed $t = 1, 2, ...$ At each date, the agents are randomly matched in pairs consisting of one seller and one buyer (each feasible configuration of pairwise matches has equal probability). One member of the pair is randomly chosen to be the proposer and the other is the responder. Each member has probability 1/2 of being chosen as proposer. The proposer offers to trade at a price $p \in [0, 1]$. The responder accepts or rejects the

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[5] As is well known, endogenous matching gives rise to both indeterminacy of equilibrium and non-competitive equilibrium outcomes, as we discuss in Section **??** below.

offer. Unmatched buyers are forced to remain inactive throughout the period.

If agreement is reached, the two agents trade at the agreed price p and leave the market. The buyer receives a payoff $1 - p$ and the seller receives a payoff p. There is no discounting.

Agents have complete information about the past play of the game, but, at the moment when they choose their actions, they do not know the identity of the other matches or the actions simultaneously chosen by other agents. Simultaneous moves require the use of perfect Bayesian equilibrium as the solution concept.

The central result RW obtain for this model is the following: *For every price p^* between* 0 *and* 1 *and for every one to one function β from the set of sellers to the set of buyers there exists a perfect equilibrium in which seller s sells his unit to buyer $\beta(s)$ for a price of p^*.*

The strategies required to support these equilibria are quite complicated, in the sense that there is no limit to the number of potential deviations and each additional deviation from the equilibrium path requires a tailor-made response that discourages the deviation. RW also show that *if each player adopts a Markov strategy, the unique perfect-equilibrium outcome is such that the good is sold for a price* $p = 1$.

Two points are worth noting. First, RW actually use a weaker assumption than Markov strategies, which they call *anonymity.* Markov strategies are functions of a minimal set of payoff-relevant variables at each date. They cannot be conditioned on variables that do not directly affect the future payoffs of the game. Anonymous strategies, by contrast, are allowed to depend on the date t as well. Secondly, the Markov assumption (or anonymity) has the immediate effect of preventing the agents from punishing a deviator, because deviations are not remembered after they occur.

Two further variations of this basic model are considered in RW. One is to introduce a common discount factor $0 < \delta < 1$ and study the perfect equilibrium for the case of a single seller and B buyers. There is a unique perfect equilibrium in which trade takes place immediately; the prices at which trade takes place depend on whether the buyer or seller was chosen to propose. However, as δ converges to 1, the equilibrium price converges to 1. This can be interpreted as demonstrating that the multiple equilibria found by RW are not robust. However, in a further variant of the basic model, there is a single seller who can select, at the beginning of each period, the buyer with whom he wants to bargain in that period. In this model, with endogenous matching, the indeterminacy of equilibrium returns in a strong form. This suggests that endogenous matching is also a source of indeterminacy and (hence) non-competitive behavior.

It is important to note that the equilibria constructed by RW in the case of endogenous matching are non-Markovian: they rely on punishment strategies that remember deviations from the equilibrium path. However, it is not difficult to construct examples of non-competitive Markov-perfect equilibria with endogenous matching.[6]

In the sequel, we restrict attention to purely exogenous matching.

2.2. Heterogeneous markets

These are important results, but unfortunately they only apply to the case of a homogenous market consisting of B identical buyers and S identical sellers. A market consisting of *heterogeneous* buyers and sellers is quite different. For the purposes of the present paper, we define a heterogeneous market as follows. As in the homogeneous case, there is a single indivisible good that is exchanged for money and each agent wants to trade at most one unit of the good. Without loss of generality, we can assume that there are equal numbers of buyers and sellers.[7] Buyers are indexed by $i = 1, ..., n$ and sellers are indexed by $j = 1, ..., n$. Buyer i's valuation of the good is denoted by $v_i \geq 0$ and seller j's valuation is denoted by $w_j \geq 0$. We assume that buyers and sellers can be ordered so that $v_1 > v_2 > ... > v_n$ and $w_1 < w_2 < ... < w_n$. We also assume that the valuations satisfy the (generic) assumption that $v_i \neq w_j$ for $i, j = 1, ..., n$.

These valuations define demand and supply curves that determine the competitive, market-clearing price(s) in the usual way. The marginal traders $i = j = m$ are defined by the conditions $v_m > w_m$ and $v_{m+1} < w_{m+1}$. The interval of perfectly competitive prices for the market is

$$\max\{v_{m+1}, w_m\} \leq p \leq \min\{v_m, w_{m+1}\}.$$

Exchange between any two inframarginal traders $i, j \leq m$ is efficient whereas exchange between an inframarginal trader and an extramarginal trader is inefficient. Exchange between two extramarginal traders is not individually rational.

Compared with the heterogeneous market, we can see that the homogeneous market is special in several respects.

- *Efficient trade*: In a heterogeneous market, inefficient trade can be individually rational. In a homogeneous market, by contrast, any trade is both individually rational and efficient.

- *Division of surplus*: In the homogeneous market, except for the special case $B = S$, the competitive equilibrium price is either 0 or 1

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[6] The exogenous random matching assumption in Rubinstein and Wolinsky (1985) and (1990) requires an agent who is matched with a new partner to abandon his original partner and bargain with the new one. It is well known, however, that if agents are allowed to stay with their original partners, there is an equilibrium which is equivalent to two-person bargaining. We are indebted to a referee for reminding us of this possibility.

[7] Sellers with extremely high valuations and buyers with extremely low valuations cannot trade in any case.

and all of the surplus goes to one side of the market.[8] In a heterogeneous market, there will typically be agents receiving positive payoffs on both sides of the market. The characterization of equilibrium outcomes becomes much more complicated as a result.

- *Invariance of the competitive prices*: In the homogeneous market, the set of competitive prices remains constant, independently of the set of agents remaining in the market. For example, if $B > S$ then no matter how many pairs of agents have traded, the number of remaining buyers is greater than the number of remaining sellers and the competitive price remains equal to 1. In the heterogeneous market, this need not be so. For example, if the competitive interval is $[w_m, v_m]$ and the marginal buyer and seller trade first, the competitive interval becomes
$$[\max\{v_{m+1}, w_{m-1}\}, \min\{v_{m-1}, w_{m+1}\}],$$
which is strictly larger given the assumption that $v_{m-1} > v_m > v_{m+1}$ and $w_{m-1} < w_m < w_{m+1}$. Similarly, if an inframarginal buyer $i \leq m$ trades with an extramarginal seller $j > m$, or an extramarginal buyer $i > m$ trades with an inframarginal seller $j \leq m$, then the competitive interval changes. In some cases, the new competitive interval may not even intersect the old one.

For all these reasons, and a number of others, the analysis of a heterogeneous market is more complicated, conceptually, analytically, and substantively, than the analysis of a homogeneous market.

3. THE MARKET GAME

A heterogeneous market is defined, as in Section 2, by the valuations $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$). The trading game is defined by the following rules:

- At each date, a pair of agents consisting of one buyer and one seller is chosen at random from the agents remaining in the market. One member of the pair is chosen at random to be the proposer; the remaining agent becomes the responder.
- The agent chosen to be proposer offers a price p. The responder must accept or reject this offer. If the offer is accepted, the good is traded at the agreed price and both agents leave the market. If the proposal is rejected, there is no trade and all agents begin the next period with the same endowments.

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[8] The analysis in Sabourian (2001) depends crucially on the fact that one side of the market receives a zero payoff in a competitive equilibrium.

An agent's information at the beginning of date t consists of the matches, proposals and responses observed in all previous periods. During the period, all the agents observe the set of agents remaining in the market, the choice of proposer and responder, the price offered by the proposer, and the response. An agent's strategy maps all the available information into a choice of action at each date.

Given the market parameters $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$, the game is defined as follows. Play occurs at a countable sequence of dates $t = 1, 2, ...$ At each date t, the set of players remaining in the game is denoted by N. The set N is *balanced*, that is, it contains an equal number of buyers and sellers. An ordered pair $\langle k, \ell \rangle$ is randomly selected from the set N, where the first agent k is the proposer and the second agent ℓ is the responder. We assume that:

- The pair $\langle k, \ell \rangle$ consists of a buyer and a seller;
- Each remaining buyer has an equal probability of being chosen and each remaining seller has an equal probability of being chosen;
- The buyer and seller chosen have equal probability of being chosen as proposer and responder.

We adopt this particular matching rule for simplicity. The matching probabilities are not important as long as each agent remaining in the game has a positive probability of being chosen and the matching probabilities are stationary, that is, they depend on the set of agents remaining in the market, but not on the date. The assumption that only one buyer and one seller are matched at each date is crucial, however, as the example in Section **??** shows.

We study the *Markov perfect equilibria* (MPE) of this game. At each date, players observe the set of remaining players N and the matched pair $\langle k, \ell \rangle$, then the proposer makes his proposal, and the responder, having observed the proposal, accepts or rejects the proposal. The Markov assumption requires that proposals depend only on the set of remaining players and the matched pair at that date and that the responses depend only on the set of remaining players, the matched pair, and the proposal at that date.

4. RANDOM SEQUENTIAL MATCHING WITH NO DISCOUNTING

In this section we maintain the rules of the game defined in the preceding section and show the existence of a continuum of non-competitive Markov perfect equilibria for a market with sequential random matching and no discounting. The idea behind the example is the following. We assume there are four agents, two buyers and two sellers, and that all four agents are inframarginal. Once the first pair has traded, the remaining pair constitutes a homogeneous market in which any price that lies between their valuations is a Markov perfect equilibrium (MPE) of the subgame. By choosing the prices in the two-person subgames appropriately, we can determine the payoffs and prices in the four-person game before any trade has occurred. The trick is to ensure that there exists a MPE corresponding to a particular choice of prices in the two-person subgames. Given that a continuum of MPE exists, it is easy to choose these prices so that the resulting equilibria will be non-competitive.

Suppose $n = 2$ and $v_1 > v_2 > w_2 > w_1$. Thus $m = n = 2$ and there is no possibility of inefficient trade. We construct a MPE of the game as follows. Let p_{ij} denote the price at which buyer i and seller j will trade if they are the last pair of agents left in the market, where $w_j \leq p_{ij} \leq v_i$ for $i, j = 1, 2$. In addition, we assume that

$$p_{22} < p_{11} \tag{1}$$

$$p_{21} = \frac{1}{2}(p_{11} + p_{22}). \tag{2}$$

These conditions guarantee that v_2 pays a low price in the subgame where he is matched with w_2 and w_1 receives a high price in the two-person subgame where he is matched with v_1. When v_2 and w_1 are matched in the two-person subgame, they trade at the average of these two prices. This assumption leads v_2 and w_1 to anticipate high payoffs in the two-person subgames, with the result that they do not want to trade with each other in the four-person subgame.

Let the numbers $(v_1^*, v_2^*, w_1^*, w_2^*)$ be defined implicitly as follows:

$$\begin{aligned} v_1 - v_1^* &= \frac{1}{2}(p_{11} + p_{21}) = w_1 + w_1^*, \\ v_2 - v_2^* &= \frac{1}{2}(p_{22} + p_{21}) = w_2 + w_2^*. \end{aligned}$$

Our strategy for proving the existence of a continuum of non-competitive equilibria is to treat the numbers v_i^* and w_j^* as if they represent the reservation utilities of the agents and then show that these are precisely the payoffs that agents achieve in equilibrium. Direct calculation shows that

$$v_1^* + w_1^* = v_1 - w_1 \tag{3a}$$

$$v_2^* + w_2^* = v_2 - w_2 \tag{3b}$$

$$v_1^* + w_2^* = v_1 - w_2 - \frac{1}{2}(p_{11} - p_{22}) < v_1 - w_2 \tag{3c}$$

$$v_2^* + w_1^* = v_2 - w_1 - \frac{1}{2}(p_{22} - p_{11}) > v_2 - w_1. \tag{3d}$$

So, in the early stages of the game (before any trade has occurred), each of three pairs of agents $(i,j) \neq (2,1)$ can trade the good.[9]

When a pair of agents $(i,j) \neq (2,1)$ is matched, if i is the proposer he offers a price $p = w_j^* + w_j$, which is accepted by j, and if j is the proposer he offers a price $p = v_i - v_i^*$, which is also accepted. When the pair $(i,j) = (2,1)$ is matched, no trade occurs. If the pairs $(1,1)$ or $(2,2)$ are the first to trade, the price does not depend on the identity of the proposer. If the pair $(1,2)$ is the first to trade, the price does depend on the identity of the proposer, but the mean price is clearly

$$\begin{aligned} \bar{p} &= \frac{1}{2}\left(v_1 - v_1^* + w_2 + w_2^*\right) \\ &= \frac{1}{4}\left(2p_{21} + p_{11} + p_{22}\right) \\ &= p_{21}. \end{aligned}$$

Each of the three cases is equally likely. The data are summarized in the following table.

Probability	First trade	Mean Price	Second Trade	Price
1/3	$(1,1)$	$v_1 - v_1^* = w_1 + w_1^*$	$(2,2)$	p_{22}
1/3	$(2,2)$	$v_2 - v_2^* = w_2 + w_2^*$	$(1,1)$	p_{11}
1/3	$(1,2)$	p_{21}	$(2,1)$	p_{21}

After the first trade occurs, there are three possible subgames that can occur on the equilibrium path, depending on which of three pairs traded first. If $(1,1)$ trade first, the subgame consists of buyer 2 and seller 2 who exchange the good at a price of p_{22}. If $(2,2)$ trade first, the subgame consists of buyer 1 and seller 1 who exchange the good at a price of p_{11}. If $(1,2)$ trade first, the subgame consists of buyer 2 and seller 1 who exchange the good at a price of p_{21}.

It is clear that these outcomes can be supported as MPE of the subgames: any individually rational trades are MPE outcomes for the two-person subgames. By inspection of the equations and inequalities in (3), the strategies in the first stages of the game, before the first trade occurs, are best responses, given the assumed payoffs from the continuation game. It remains to show that the assumed payoffs are achieved by these strategies. Direct calculation yields

$$\begin{aligned} v_1^* &= \frac{1}{3}v_1^* + \frac{1}{3}(v_1 - p_{11}) + \frac{1}{3}(v_1 - p_{21}) \\ &= v_1 - \frac{1}{2}\left(p_{11} + p_{21}\right), \end{aligned}$$

where the three terms in the first line correspond to first trades by $(1,1)$, $(2,2)$

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[9] Trade between buyer 2 and seller 1 is not individually rational because, by construction, $v_2^* + w_1^* > v_2 - w_1$. Thus, we only have to consider the three subgames in which the first pair to trade is $(i,j) = (1,1), (1,2), (2,2)$.

and $(1,2)$, respectively. Similarly,

$$\begin{aligned} v_2^* &= \frac{1}{3}\left(v_2 - p_{22}\right) + \frac{1}{3}v_2^* + \frac{1}{3}\left(v_2 - p_{21}\right) \\ &= v_2 - \frac{1}{2}\left(p_{22} + p_{21}\right), \end{aligned}$$

$$\begin{aligned} w_1^* &= \frac{1}{3}w_1^* + \frac{1}{3}\left(p_{11} - w_1\right) + \frac{1}{3}\left(p_{21} - w_1\right) \\ &= \frac{1}{2}\left(p_{11} + p_{21}\right) - w_1, \end{aligned}$$

and

$$\begin{aligned} w_2^* &= \frac{1}{3}\left(p_{22} - w_2\right) + \frac{1}{3}w_2^* + \frac{1}{3}(p_{21} - w_2) \\ &= \frac{1}{2}\left(p_{22} + p_{21}\right) - w_2, \end{aligned}$$

as required.

This completes the description of the equilibrium path. It remains to define the strategies that will support this outcome. In the four-person game, before any trade has occurred, the proposer offers a price that gives the responder his equilibrium payoff (when this is possible) and the responder accepts. If a proposer deviates from the equilibrium offer, the responder rejects any offer that gives him less than his equilibrium payoff and accepts any offer that gives him more than his equilibrium payoff. The strategies for the two-person subgames are defined similarly.

These strategies are clearly independent of the history of play. Thus we have a Markov perfect equilibrium for every value of the parameters $(p_{11}, p_{12}, p_{21}, p_{22})$ satisfying (1) and (2). Since the parameters p_{11} and p_{22} can be chosen arbitrarily within some small intervals, there exists a continuum of Markov-perfect equilibria.

The following proposition summarizes the preceding discussion.

Proposition 1. *Let* $w_1 < w_2 < v_2 < v_1$ *define a market and let* p_{ij} *denote the price at which buyer* i *and seller* j *trade if they are the last pair of agents left in the market, where* $w_j \leq p_{ij} \leq v_i$ *for any* $i, j = 1, 2$. *Assume that the parameters* $\{p_{ij}\}$ *satisfy*

$$\begin{aligned} p_{22} &< p_{11}, \\ p_{21} &= \frac{1}{2}(p_{11} + p_{22}). \end{aligned}$$

Then there exists a continuum of MPE, each parameterized by $\{p_{ij}\}$, *with equilibrium payoffs* $(v_1^*, v_2^*, w_1^*, w_2^*)$ *defined by*

$$\begin{aligned} v_1 - v_1^* &= \frac{1}{2}\left(p_{11} + p_{21}\right) = w_1 + w_1^*, \\ v_2 - v_2^* &= \frac{1}{2}\left(p_{22} + p_{21}\right) = w_2 + w_2^*. \end{aligned}$$

In each such equilibrium, if v_i and w_j are the first pair of agents to trade, they trade at the price $v_i - v_i^* = w_j + w_j^*$ if $i = j = 1, 2$, at the price $v_1 - v_1^*$ if w_2 proposes to v_1, and $w_2 + w_2^*$ if w_2 proposes to v_1. The agents v_2 and w_1 cannot be the first pair to trade.

Proposition 1 shows that trade occurs at non-uniform prices and that some of these prices do not belong to the competitive interval (for some choices of p_{11} and p_{22}, seller 1 and buyer 1 trade at a price higher than v_2). In fact, it is also easy to check that the equilibrium payoffs v_1^*, v_2^*, w_1^*, and w_2^* do not correspond to competitive payoffs.

The essential intuition behind the technicalities of Proposition 1 is the following: as soon as one pair of agents has traded, the two-person subgame defined by the remaining agents will have an interval of competitive prices that is quite different from the competitive interval in the preceding four-person game. Therefore, the prices at which trade occurs in a two-person subgame can be very different from the competitive prices in a four-person market. It is the anticipation of extreme prices in the various two-person continuation games that supports trade at non-competitive prices in the four-person game.

One of the special features of the example described in Proposition 1 is that all agents are assumed to be inframarginal and, hence, all trade is efficient. To illustrate the robustness of the non-competitive outcome and also allow for inefficient trade, consider the following (symmetric) example. Let $v_1 = 1$, $v_2 = v$, $w_1 = 0$ and $w_2 = 1 - v$. We assume that $0 < v < 1/2$, so that v_2 and w_2 are extramarginal. Trade between v_1 and w_2 or between v_2 and w_1 is individually rational but inefficient. The following proposition shows that there exists an equilibrium in which trade occurs with positive probability between any of the three individually rational pairs. The proof, which we omit, uses standard arguments.

PROPOSITION 2. *Let $v_1 = 1$, $v_2 = v$, $w_1 = 0$ and $w_2 = 1 - v$ define a market. Then there exists an equilibrium in which the equilibrium payoffs are*

$$\begin{aligned} v_1^* &= w_1^* = \frac{1}{3}v + \frac{5}{24} \\ v_2^* &= w_2^* = \frac{1}{3}v - \frac{1}{24}. \end{aligned}$$

The fact that all payoffs are positive implies that all agents trade with positive probability. If trade were efficient, only v_1 and w_1 could trade and the sum of equilibrium payoffs would be one. Adding up the equilibrium payoffs, we see that

$$v_1^* + w_1^* + v_2^* + w_2^* = \frac{4}{3}v + \frac{1}{3} < 1,$$

since $v < 1/2$, so there is indeed a loss of surplus.

The construction of the equilibrium uses familiar arguments. We assume that in the two-person subgames, which occur after one pair has traded, the remaining agents split the surplus equally. This is clearly a MPE for the subgame. Using these continuation payoffs, we can solve for the unique MPE payoffs of the four-person subgame. These payoffs are consistent with all four agents trading with positive probability.

It should be clear that the existence of non-competitive equilibria does not rely on the number of agents or the specific valuations assumed here. The essential part of the construction can be embedded in any model with four or more inframarginal agents. Extramarginal agents can be added without disturbing the equilibrium as long as their valuations are outside the range where trade takes place.

A much deeper question is what happens as we let $n \to \infty$. Even if we start with a large number of agents, the market will eventually become small. As long as there is a large number of equilibria in the small subgames, we can probably bootstrap these to generate multiple non-competitive equilibria for arbitrarily large n.

5. DISCOUNTING

In market games with exogenous matching, introducing a small amount of discounting is sometimes enough to ensure the uniqueness of perfect equilibrium. As mentioned in the introduction, under certain conditions, the unique perfect equilibrium with discounting converges to the competitive outcome as the discount rate converges to 1. The next example shows that this strategy does not work for markets with heterogeneous buyers and sellers.

5.1. Symmetric equilibrium

Consider a symmetric market in which there are two buyers and two sellers and all agents are inframarginal. The valuations of the agents are denoted by $w_1 < w_2 < v_2 < v_1$, where without loss of generality we can assume that

$$\begin{aligned} w_1 &= 0 \\ w_2 &= w \\ v_2 &= v \\ v_1 &= v + w \end{aligned}$$

for some $0 < w < v$. Payoffs are discounted using a common discount factor $0 < \delta < 1$. The next result characterizes the symmetric MPE payoffs for this market, under the assumption that δ is not too small.

Proposition 3. *Let* $(w_1, w_2, v_1, v_2) = (0, w, v + w, v)$ *define a symmetric market with common discount factor* δ*. For some* $\delta^* < 1$ *and all*

$\delta^* < \delta < 1$, there is a unique symmetric MPE of the game with discounting in which the equilibrium payoffs are defined implicitly by the equations:

$$v_1^* = w_1^* = \frac{(2+\delta)v + w - \delta v_2^*}{8-3\delta} \tag{4}$$

$$v_2^* = w_2^* = \frac{(1+2\delta)v - \delta w - \delta v_1^*}{8-3\delta}. \tag{5}$$

After the first pair of agents has traded, the remaining two agents are in a two-person bargaining game and we know from the bargaining literature (e.g., Rubinstein (1982)) that with equal discount factors they will split the surplus. More precisely, the division of the surplus will depend on who is the proposer but ex ante they have equal chances of being proposer and hence receive equal payoffs. If w_1 and v_1 (resp. w_2 and v_2) are left to trade in the two-person subgame, they each receive $(v+w)/2$ (resp. $(v-w)/2$). If w_1 and v_2 (resp. w_2 and v_1) are left to trade in the two-person subgame, they will each receive $v/2$ (resp. $v/2$). As $\delta \to 1$, the prices at which trade occurs become independent of the proposer. If w_1 and v_1 (resp. w_2 and v_2) are left to trade in the two-person subgame, the price at which they trade will be approximately $(v+w)/2$; by contrast, if w_1 and v_2 (resp. w_2 and v_1) are left to trade in the two-person subgame, the price at which they trade will be approximately $v/2$ (resp. $w + v/2$). The competitive interval is $[w_2, v_2] = [w, v]$, which need not contain $v/2$ or $w + v/2$.

In a symmetric MPE, v_1 and w_1 receive the same payoff and v_2 and w_2 receive the same payoff. The numbers v_1^*, v_2^*, w_1^*, and w_2^* are implicitly defined by (4) and (5). Our strategy, as before, is to treat these numbers as if they represent the reservation utilities of the agents, define individual strategies accordingly, and then show that these are precisely the payoffs achieved in equilibrium. Substituting v_2^* into the first equation and simplifying yields

$$v_1^* = \frac{(8-3\delta)\left[(2+\delta)v + w\right] - \delta(1+2\delta)v + \delta^2 w}{(8-3\delta)^2 - \delta^2}.$$

Note that

$$\lim_{\delta\to 1} v_1^* = \lim_{\delta\to 1} w_1^* = \frac{1}{2}v + \frac{1}{4}w$$

and

$$\lim_{\delta\to 1} v_2^* = \lim_{\delta\to 1} w_2^* = \frac{1}{2}v - \frac{1}{4}w.$$

Thus,

$$\lim_{\delta\to 1} \delta\left(v_1^* + w_1^*\right) = v + \frac{1}{2}w < v + w \tag{6}$$

and

$$\lim_{\delta\to 1} \delta\left(v_2^* + w_2^*\right) = v - \frac{1}{2}w > v - w. \tag{7}$$

Thus, by (6) and (7), for δ close to 1, as long as there are four agents remaining, v_1 and w_1 must trade whenever matched and v_2 and w_2 cannot trade whenever matched. Summing the equations that define v_1^* and v_2^* we get

$$\delta\left(v_1^*+v_2^*\right)=\frac{\delta}{8-3\delta}\left[(2+\delta)v+w+(1+2\delta)v-\delta w\right]-\frac{\delta^2(v_1^*+v_2^*)}{8-3\delta}.$$

Thus,

$$\begin{aligned}(8\delta-3\delta^2+\delta^2)\left(v_1^*+v_2^*\right) &= \delta\left((3+\delta)v+(1-\delta)w\right)\\ \delta(8-2\delta)\left(v_1^*+v_2^*\right) &= \delta\left((3+\delta)v+(1-\delta)w\right)\\ \delta\left(v_1^*+v_2^*\right) &= \frac{\delta}{(8-2\delta)}\left((3+\delta)v+(1-\delta)w\right)\\ &< \frac{\delta(4+\delta)v}{(8-2\delta)}<v.\end{aligned}$$

Since $v_1^*=w_1^*$ and $v_2^*=w_2^*$ this establishes that, as long as there are four agents remaining, whenever v_1 and w_2 meet or v_2 and w_1 meet, they must trade.

The Markov strategies for the game with four agents remaining are as follows. If $(i,j)=(2,2)$ are matched, no individually rational trade is possible, so the proposer makes an offer that the responder must reject. Otherwise, if $(i,j)\neq(2,2)$ and i is chosen as the proposer, he proposes a price

$$p_{ij}=w_j+\delta w_j^*,$$

and w_j accepts any price greater than or equal to p_{ij} and rejects any other price. If j is chosen as the proposer, then he proposes a price

$$p_{ji}=v_i-\delta v_i^*$$

and v_i accepts any price less than or equal to p_{ji} and rejects any other price. The two-person subgames have unique MPE.

It remains to show that these strategies constitute a symmetric MPE. In the game with four agents remaining, there are four equally probable matches: (v_1,w_1), (v_1,w_2), (v_2,w_1), (v_2,w_2). The fourth results in no trade. The payoff to v_1 must satisfy

$$v_1^*=\frac{1}{4}\left(\frac{1}{2}\delta v_1^*+\frac{1}{2}\left(v+w-\delta w_1^*\right)\right)+\frac{1}{4}\left(\frac{1}{2}\delta v_1^*+\frac{1}{2}\left(v-\delta w_2^*\right)\right)+\frac{1}{4}\delta\frac{1}{2}v+\frac{1}{4}\delta v_1^*$$

and the payoff to v_2 must satisfy

$$v_2^*=\frac{1}{4}\left(\frac{1}{2}\delta v_2^*+\frac{1}{2}\left(v-\delta w_1^*\right)\right)+\frac{1}{4}\delta\frac{1}{2}(v-w)+\frac{1}{4}\delta\frac{1}{2}v+\frac{1}{4}\delta v_2^*.$$

Using symmetry, $v_1^* = w_1^*$ and $v_2^* = w_2^*$, these equations can be rewritten successively as

$$\begin{aligned}\left(1-\frac{3}{8}\delta\right)v_1^* &= \frac{1}{8}(v+w)+\frac{1}{8}v+\frac{1}{8}\delta v-\frac{1}{8}\delta v_2^*,\\ \left(1-\frac{3}{8}\delta\right)v_2^* &= \frac{1}{8}v+\frac{\delta}{8}(v-w)+\frac{1}{8}\delta v-\frac{1}{8}\delta v_1^*,\end{aligned}$$

or

$$\begin{aligned}(8-3\delta)\,v_1^* &= (2+\delta)v+w-\delta v_2^*,\\ (8-3\delta)\,v_2^* &= (1+2\delta)v-\delta w-\delta v_1^*,\end{aligned}$$

which gives the definitions above.

This completes the construction of a symmetric, Markov-perfect equilibrium. To see that it is unique, one only has to note that any symmetric, Markov-perfect equilibrium will have payoffs $w_1^{**}, w_2^{**}, v_1^{**}$, and v_2^{**} that will uniquely determine the individual strategies. It can be further shown that, in any symmetric MPE, the pair (i,j) must trade in the initial four-person subgames if and only if $(i,j) \neq (2,2)$. Then the preceding calculations show that the payoffs are unique and equal to w_1^*, w_2^*, v_1^*, and v_2^*.

5.2. Non-competitive prices

The equilibrium described in Proposition 3 is non-competitive because (a) trade takes place at non-uniform prices and (b) these prices may not belong to the competitive interval. This is true even as $\delta \to 1$ (recall that we have maintained the assumption that δ is not too far from 1). Let $p_{k\ell}^{(n)}$ denote the limiting value, as $\delta \to 1$, of the price at which k and ℓ trade when there are n agents left in the market, k is the proposer, and ℓ is the responder. Solving the equations above, we see that the equilibrium payoffs in the limit as $\delta \to 1$ are

$$\begin{aligned}v_1^* &= w_1^* = \frac{1}{2}v+\frac{1}{4}w,\\ v_2^* &= w_2^* = \frac{1}{2}v-\frac{1}{4}w.\end{aligned}$$

Note that

$$\begin{aligned}v_1^*+w_1^* &= v+\frac{1}{2}w < v+w,\\ v_2^*+w_2^* &= v-\frac{1}{2}w > v-w,\\ v_2^*+w_1^* &= v_1^*+w_2^* = v.\end{aligned}$$

When there are four agents in the market, $(i,j)=(2,2)$ cannot trade, $(i,j)=(1,1)$ trade at prices depending on the identity of the proposer, and

$(i,j) = (1,2), (2,1)$ trade at unique price, independently of the identity of the proposer. When there are only two agents left, they trade at a price that splits the surplus equally, independently of the identity of the proposer. Thus, we have the following corollary.

Corollary 1. *In the equilibrium described in Proposition 3, the equilibrium prices have the following limiting values as $\delta \to 1$:*

$$p^{(4)}_{k\ell} = \begin{cases} \frac{1}{2}v + \frac{1}{4}w & \textit{if } \ell \textit{ is either seller 1 or buyer 2,} \\ \frac{1}{2}v + \frac{3}{4}w & \textit{if } \ell \textit{ is either seller 2 or buyer 1,} \end{cases}$$

$$p^{(2)}_{k\ell} = \begin{cases} \frac{1}{2}(v+w) & \textit{if } (k,\ell) = (1,1),(2,2), \\ \frac{1}{2}v & \textit{if } \{k,\ell\} = \{i,j\}, (i,j) = (2,1), \\ \frac{1}{2}v + w & \textit{if } \{k,\ell\} = \{i,j\}, (i,j) = (1,2). \end{cases}$$

This example could also be extended to allow for inefficient trades in the spirit of Proposition 2. In fact, since we assume in the proof of Proposition 2 that agents split the surplus equally in the two-person subgames, we conjecture that the equilibrium payoffs described in Proposition 2 are in fact the limit as $\delta \to 1$ of the MPE payoffs for the corresponding games with $\delta < 1$.

The example can also be generalized to allow for asymmetry and a larger number of buyers and sellers: the restrictive assumptions used here are for illustrative purposes only. Again, the idea behind the construction does not depend essentially on the number of agents or the precise valuations. We can add extramarginal agents without disturbing the equilibrium, as long as their valuations lie outside the region where trade takes place. The determinateness of equilibrium in the two-person subgames implies that some trade takes place at non-competitive prices in any MPE.

A deeper question is what happens when $n \to \infty$. On the one hand, the market must eventually become small and then the arguments above will continue to apply. On the other hand, it might be that most agents trade at the competitive price, even though the market becomes non-competitive when the number of remaining players is small enough. Since the equilibrium is unique for small n, we cannot bootstrap multiple equilibrium for small n to provide non-competitive equilibrium for large n. This question is beyond the scope of this paper and would require very different techniques to analyze.

5.3. Uniqueness

Proposition 3 characterizes the payoffs from a symmetric MPE but we can easily show that these are the unique MPE payoffs of the game with symmetric valuations (as long as δ is not too far from 1). We begin by noting that any two-person subgame is isomorphic to a standard Rubinstein-type bargaining game. It is well known that the MPE payoffs of such a game are uniquely determined and symmetric.

Next, consider a fixed but arbitrary MPE of the game and let v_i^* (resp. w_j^*) denote the equilibrium payoff of buyer i (resp. seller j) in this MPE. These represent the expected utilities of the agents in any subgame before trade has occurred. Similarly, let v_{ij}^* (resp. w_{ji}^*) denote the equilibrium payoff of buyer i (resp. seller j) in the two-person subgame where he is matched with seller j (resp. buyer i).

In the four-person subgame, before any trade has taken place, it must always be possible for v_1 and w_1 to trade because feasibility implies that $v_1^* + w_1^* \leq v + w$ so $\delta\left(v_1^* + w_1^*\right) < v + w$.

If v_1 and w_1 were the only agents who could trade in the four-person subgame, then v_2 and w_2 would have to wait until the two-person subgame to trade and this would imply $v_2^* < v_{22}^*$ and $w_2^* < w_{22}^*$, where the inequalities are strict because of discounting. But this implies that $\delta(v_2^* + w_2^*) < v - w$, contradicting the assumption that they are unable to trade in the four-person subgame. Intuitively, if they are going to trade eventually, they might as well trade immediately.

Similarly, if the only pairs that can trade in the four-person subgame are (v_1, w_1) and (v_2, w_2) then effectively the two pairs are engaged in Rubinstein bargaining, the outcomes will be the same as in the respective two-person subgames (except that the discounting will be greater) and this implies that (v_1, w_2) and (v_2, w_1) can trade.

So we know that in the MPE under consideration, the pair (v_1, w_1) and at least one of the pairs (v_1, w_2) and (v_2, w_1) can trade.

Now suppose that w_1 cannot trade with v_2 in the four person subgame. Then v_1 effectively controls trade in the four-person subgame, because no trade can take place unless he is a party to it. This implies that, for any $\varepsilon > 0$ and for δ sufficiently close to 1, $w_1^* < w_{12}^* + \varepsilon$. As a responder, w_2 must accept any offer that gives him more than $\frac{1}{2}(\delta w_2^* + \delta w_{12}^*)$. This implies that, for δ close to 1, v_1 can achieve almost $(v+w) - \frac{1}{2}(\delta w_2^* + \delta w_{12}^*)$ by waiting until he has a chance to propose to w_2. This implies that $w_2^* < \frac{1}{2}(\delta w_2^* + \delta w_{12}^*) + \varepsilon$ or, $w_2^* < w_{12}^* + \varepsilon$, in which case it is clear that w_2 and v_2 can trade in the four-person subgame.

In a similar way, we get a contradiction by assuming that v_1 and w_2 cannot trade in the four person subgame, and this leads to the conclusion that each of the pairs (v_1, w_1), (v_1, w_2) and (v_2, w_1) can trade in this subgame for δ sufficiently close to 1. Then it can be shown that v_2 and w_2 cannot trade in this subgame. The worst thing that can happen to v_2 and w_2 is that they end up in a two-person subgame with each other, in which case they split the surplus $v - w$, so, ignoring the effect of discounting, $(v - w)/2$ is a lower bound for their equilibrium payoff. However, if they end up in a two-person subgame with w_1 or v_1, as the case may be, they do much better and end up with $v/2$, again ignoring the small effect of discounting. Taking into account this possibility, the sum of their payoffs $v_2^* + w_2^*$ is more than $v - w$, so it is never rational for them to trade with each other while there are still four agents in the market. Thus, the trading possibilities are the same as in the symmetric equilibrium characterized by Proposition 3.

The equilibrium payoffs must satisfy the following conditions:

$$v_1^* = \frac{1}{2}\left(\frac{1}{4}(v + w - \delta w_1^*) + \frac{1}{4}(v - \delta w_2^*) + \frac{1}{2}\delta v_1^*\right) + \frac{1}{4}\delta v_{12}^* + \frac{1}{4}\delta v_1^*$$

$$v_2^* = \frac{1}{4}\left(\frac{1}{2}(v - \delta w_1^*) + \frac{1}{2}\delta v_2^*\right) + \frac{1}{4}\delta v_{22}^* + \frac{1}{4}\delta v_2^*$$

$$w_1^* = \frac{1}{2}\left(\frac{1}{4}(v + w - \delta v_1^*) + \frac{1}{4}(v - \delta v_2^*) + \frac{1}{2}\delta w_1^*\right) + \frac{1}{4}\delta w_{12}^* + \frac{1}{4}\delta w_1^*$$

$$w_2^* = \frac{1}{4}\left(\frac{1}{2}(v - \delta v_1^*) + \frac{1}{2}\delta w_2^*\right) + \frac{1}{4}\delta w_{22}^* + \frac{1}{4}\delta w_2^*.$$

It is straightforward to show that these equations have a unique solution for v_1^*, v_2^*, w_1^*, and w_2^*. So there is a unique set of equilibrium payoffs, which must agree with the same as the payoffs of the symmetric equilibrium.

COROLLARY 2. *The equilibrium payoffs described in Proposition 3 are the unique MPE payoffs.*

6. EXOGENOUS, SIMULTANEOUS MOVES

The bulk of this paper is devoted to games with sequential matching, i.e., games in which exactly one pair of agents is allowed to bargain and trade at a time. In this section we briefly discusse the class of games with simultaneous matching and show that new sets of non-competitive MPE arise in these models too for reaons that are quite different from the ones discussed in Sections 4 and 5.

Assume that $m = 1$ and $n = 2$ and that

$$w_1 < v_2 < w_2 < v_1.$$

Matching is random and every buyer-seller pair has equal probability and each member of the pair has an equal probability of being chosen as the proposer.

Consider the following strategies: when the pair (i, j) such that $v_i > w_j$ is formed, the proposer offers to trade at the price $p_{ij} \in [w_j, v_i]$; the responder accepts any price that is at least as good as p_{ij} and rejects any other offer. We claim that these strategies constitute a MPE for the simultaneous matching game. The strategies are clearly stationary: they depend only on the matched pair of agents. The strategies clearly form a subgame perfect equilibrium of the subgames in which only a single buyer and seller are left. Furthermore, when the pair (i, j) forms, they are either the only agents left in the market or they expect the other two agents, who are currently matched, to trade immediately and leave the market. Thus, if i and j do not trade in the current period, they expect to trade at the price p_{ij} in the continuation game. Then clearly it is optimal for the agents to follow the specified strategy in the current period.

This shows that the exogenous simultaneous move game has a continuum of stationary (Markov) non-competitive equilibria.

Random matching is not necessary for this example. The same construction works with deterministic matching as long as agents are matched simultaneously. In this case, the structure of the game is not stationary, but the strategies are stationary since they depend only on the current match. The definition of a MPE should allow strategies to depend on time because the matching rule depends on time, but a stationary strategy does not have to depend on time.

We also note that the strategies in these examples are stationary as a function of N.

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