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Vintage Capital and Inequality

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If machines are indivisible, a vintage capital model must give rise to income inequality. If new machines are always better than old ones and if society cannot provide everyone with a new machine all of the time, inequality will result. I explore this mechanism in detail. If technology resides in machines and if a firm or worker must use just one technology at a time, a variety of machines will be in use, and workers' productivities will differ. This is because not everyone can be given the latest vintage machine all of the time. Inequality thus originates in the limited capacity of the capital goods sector. If machine quality and skill are complements, a worker who is paired with the best machine will acquire more skill, and inequality persists indefinitely. Moreover, if the used equipment market or the process of labor turnover function without frictions, a perfect positive assignment between the quality of labor and of capital can be maintained by a process of continual reassignment. This serves to enhance the degree of equilibrium inequality. Paradoxically, in this type of model, free migration of labor across borders raises cross-country inequality instead of lowering it as it does in some other models. *Journal of Economic Literature* Classification Number: O31.

1. INTRODUCTION

At some point in history, per capita outputs, physical capital stocks, and human capital stocks were about the same everywhere. Today, there is huge inequality in per capita outputs. Endowments of natural resources just cannot explain who is rich and who is poor. The common approach is

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to assume some exogenous inequality in some initial condition or policy. By its exogenous nature, this approach cannot explain the origins of income inequality. The model I develop here assumes none of this exogeneity. Income disparity is not a consequence of different initial conditions, but the result of different investment choices made by economies.

The main point of the paper is that when machines are indivisible, a vintage capital model has a natural nonconvexity. New machines are better than old ones, and machines are scarce goods. Therefore, it is infeasible, or at least wasteful and suboptimal, to replace all of the existing 486 and Pentium computers because we now have the Pentium II. Then, who should use the new Pentium IIs? The answer depends on technology and market structure. It is plausible and empirically well founded to suppose that new technologies and skills are complements, and so the new machines will be used by the most skilled workers. Therefore, they will increase inequality. In fact, small differences in skills will translate into larger differences in productivities. This is due to the nonconvexity. Had the economy been convex, instead of producing some much better machines, we would have improved all of the existing machines by a small increment.

The theoretical novelty is in how the model gets inequality among individuals to persist and in how it avoids the technological leapfrogging that is a feature of other vintage capital models such as Parente [30]. This is done by assuming frictionless reassignment of machines and workers—through a used-equipment market or through the movement of labor among (or even within) plants. The steady-state equilibrium takes the following form: Workers accumulate human captial at a constant rate, which determines the growth rate of the economy. New vintages also improve at a constant rate. Therefore, the distribution of both machine quality and worker skills is invariant over time. When a new vintage arrives, the highest-skilled individual abandons his machine and switches to the best one, the second best gets the machine just abandoned by the most skilled, and so on, until the lowest-skilled worker, who simply scraps his machine. But it is also a fact of life that people do work with a machine for extended periods of time, and if better machines are invented at each instant (as they are in my continuous time model), the model implies that workers switch technologies more often than they in fact do.

1.1. Explaining Inequality

Why do two people of the same age earn different wages per hour? A labor economist might list three reasons.

(1) *Endowments*: One worker's IQ exceeds the other's or his parents have nurtured and educated him better.

(2) *Luck*: The better paid worker was in the right place at the right time.

(3) *Compensating differentials*: The two workers were the same to begin with but chose different careers, each with the same discounted lifetime income, but with a different age−earnings profile.

If asked why the incomes per head of two *countries* differ, a development economist would offer roughly the same three answers.

(1) *Endowments*: One country "started out" with more physical and human capital, and more natural resources (as in Lucas [28, Sect. 4]).

(2) *Luck*: A country happens to have a government whose policies hinder development and growth (as in Parente and Prescott [29]) or it is affected by adverse shocks such as wars and famines or sectoral shocks (as in Acemoglu and Zilibotti [2]).

(3) *Compensating differentials*: A high per capita income of a developed country is a reward for sacrifices its citizens made in the past.

The first two answers are less pertinent for development, where they involve large groups of people, than for returns to labor, where they deal with individuals. First, IQ differs less (if at all) among groups than it does among individuals, and second, not only does risk matter far less to societies than to individuals, but, in addition, the effects of some aggregate shocks, particularly those technological in origin, are shared by many countries. So answers 1 and 2 immediately raise the question *why* a country started out with more capital or why it has a good government.

We do not want to abandon questions like these entirely to historians, sociologists, or political scientists. Yet the bulk of development theory implicitly does just that, for it tells us that if two countries start with the same tastes, production possibilities, initial conditions, and government policies, their subsequent development will be identical. Such theory explains the persistence of inequality, but not its origin or its magnitude. Lucas [28, Sect. 4], Romer [34], or Parente and Prescott [29] can explain income inequality only if they assume inequality in something *else*. This remark applies to a degree even to the "club-convergence" models of Azariadis and Drazen [4] and Galor and Zeira [17] in which countries start out close together, but on different sides of an unstable steady-state equilibrium, although this literature is motivated by concerns similar to those I raise here, and I will return to it shortly.

In contrast, answer 3 presumes no exogenous variation in anything. To save space I will refer to such arguments as "type-three explanations." I shall now review several models of this type, and in so doing, I shall interpret them as models of world equilibrium.

1.2. Type-Three Explanations for the Development Puzzle

My model is a type-three explanation: inequality must occur in the long run no matter where countries start out. The long-run extent of inequality is determined uniquely and it does not depend on initial conditions. It arises when the basic elements of two well-known models—the vintage capital model and the assignment model—are both pertinent. In the *vintage capital model*, the economy cannot replace all its old capital at each date, and different technologies must coexist. In the *assignment model*, members of two heterogeneous populations are paired with each other in fixed proportions. I combine these two models and then assume that agents can invest in improving their skills and in building better machines. A balanced growth path of this economy must entail a *nondegenerate* distribution of skill and of machine quality around trend, simply because society cannot give everyone a frontier machine all the time; those that do get the best machines will want to acquire more skill. The best equipment is assigned to workers with the greatest human capital skills, and this enhances the superiority of their skills even further. So continuing income inequality is explained by heterogeneity in the quality of plant and equipment, which in turn originates in the finite capacity of the capital goods sector.

Other type-three explanations of inequality typically assume (just as I do) that people can engage in just one activity at a time or use just one technology at a time. In one class of models, different activities involve different rates of productivity growth. In Lucas [28, Sect. 5] and Boldrin and Scheinkman [9], activities are defined by the goods they produce—computers and potatoes, say. Computer producers improve their productivity faster than do potato growers, and a development gap arises between them. In Chari and Hopenhayn [11], activities are defined by their technological vintage. Some old vintages survive alongside new ones, sustained by people that have become skilled in them, and each technological vintage offers a different age−earnings profile.

Arrow [3] and Parente [29] offer another type-three explanation: Fixed costs of adopting technology cause a producer's net revenues to fall right after an adoption. Faced with a temporary drop in income, the adopter borrows in order to smooth his consumption. For the rate of interest to be constant, people must stagger their adoption decisions, so that a range of technological vintages is in use at each date.[1] Now because of a savings indeterminacy that stems from an assumed linear utility function, Arrow

[1] In my (1982) equilibrium version [25] of Baumol's [7] model of cash management, heterogeneity is the only equilibrium outcome for essentially the same reasons: a constant rate of interest can obtain only if people stagger their cash acquisitions.

has to specify the degree of inequality exogenously.[2] More fundamentally, Arrow's and Parente's models have two unrealistic implications. First, in Arrow's model, there is no machine-specific learning by doing, and so a technological switch is a ‘‘productivity miracle’’ that is then followed by sustained ‘‘disaster’’ relative to the leading technology, followed by another miracle, then another prolonged disaster, and so on. In Parente's model, it is the other way around: because he assumes machine-specific learning by doing, a technological switch is a ‘‘productivity disaster’’ which is then followed by a ‘‘miracle,’’ followed by another disaster, then another miracle, and so on; even at the micro level the data do not show such endless leapfrogging at any level of aggregation. Second, *if one were to start Arrow's or Parente's economy off on its balanced growth equilibrium*, cross section wealth differentials would be smaller than income differentials, and the data do not confirm this either. And third, initial conditions matter even in the long run. By contrast, in my model:

(a) There is no turnover in the distribution of productivities and wages.

(b) *If* one were to start my economy off on *its* balanced growth equilibrium, wealth differentials (with wealth interpreted as the present value of labor income) and consumption differentials would be the same as the wage differentials.

(c) The variances in labor incomes would not depend on initial conditions, and the distributions of both variables would be log-uniform, which means that they would be skewed to the right, as evidence shows.

But an agent's steady-state wealth need not equal the present value of his steady-state income. The agent may have accumulated assets or liabilities along the transition path. If we start everyone off equal and if everyone in equilibrium has the same options, then although agents' income trajectories may differ, their wealth and consumption will be the same at all dates. And that, of course, is wildly at odds with reality. It is inappropriate, therefore, to compare models' implications for wealth and consumption levels without considering the full dynamics.

In a very different model, Lucas [28, pp. 31–34] carries out the full dynamics and they reveal that even with infinitely lived agents, initial homogeneity need not imply equal lifetime wealth. Lucas assumes that a country will specialize in producing a good—‘‘low tech’’ or ‘‘high tech’’—and that learning is external to an agent though fully internal to a country. With two initially identical countries, his model has two asymmetric equilibria. Each is a mirror image of the other and is indexed by the

[2] As an indirect consequence of the assumption imbedded in Eq. (38) of his paper.

identity of the country that specializes in the high-tech good and becomes the eventual leader. It is better to reside in the country that an equilibrium designates as the leader, but migration is ruled out and hence the residents of the leading country enjoy higher wealth and lifetime consumption. If countries are initially different in Lucas' model, then equilibrium is unique and each country's development path is determined by its initial comparative advantage.

I can think of two ways to generate wealth inequality in my model: imperfect capital markets and leisure in the utility function. I shall discuss them briefly in Section 5, but I will not carry out the full dynamics. Neither alternative requires that there be multiple equilibria across which levels of lifetime utility for the residents of different countries reverse themselves. In a Lucas equilibrium, the follower wishes that the *other* equilibrium had been "selected." This is true even at the outset when the two countries are the same and when the identity of the leader is determined through some unspecified mechanism. Perhaps this state of affairs would suffer from a fragility that I cannot articulate.[3] In any event, because his equilibrium assigns different lifetime utilities to the two countries, Lucas' development gap is more than a simple compensating differential.

Now let us return to the "club convergence" models of Azariadis, Drazen, Galor, and Zeira. To generate long-run inequality, these models require that there be some initial inequality, albeit minimal. Suppose one were to insist that the initial conditions of different individuals or countries all be within ε of one another. These papers offer no special reason why any of these nearly identical countries should initially find themselves on a different side of an unstable steady-state equilibrium than the rest. In contrast, Lucas' model has a "self-organized criticality" even if countries are initially homogeneous: if computers and potatoes are both essential in consumption, countries will specialize in different goods, and inequality will emerge. Similarly, in my model, incomes cannot remain equal.

1.3. Plan of the Paper

The model is easiest to describe "backwards." Section 2 analyzes the equilibrium that arises at any date if the distributions of quality of physical and human capital are given. This determines the equilibrium prices of each type of labor and capital at each date. Section 3 develops the dynamics—optimal saving, accumulation of human capital, and innovation—and shows that a unique balanced growth path exists. Section 4 discusses

[3] Incidentally, this preference (on the part of the player that equilibrium selection has designated as the follower) for being in the other equilibrium results from a comparison of lifetime incomes, and it would not disappear if we were to perturb the initial conditions away from equality.

some properties of the equilibrium, stressing the implications for inequality. Section 5 discusses the key assumptions and relates the model to the literature. The Appendix contains technical details.

2. THE STATIC ASSIGNMENT MODEL

I begin with the problem of assigning machines to workers, taking the distribution of quality of each factor as given. Such a problem was first considered by Tinbergen [40] and Sattinger [36]. A firm uses machines and workers in fixed proportions, normalized to one machine and one worker. Each machine is owned by a firm. Let k be the quality of a machine, and s the skill of a worker.

The Firm's Decision Problem

With the quality of its machine k given, the firm optimizes over s and its profit is

$$\pi(k) = \max_{s}\{F(k,s) - w(s)\},$$

where F is a linearly homogeneous production function with $F_{12} > 0$, and $w(\cdot)$ is a price function for labor skill that the firm takes as given. The assignment $s = \phi(k)$ that solves this problem satisfies

$$F_2(k,\phi(k)) - w'(\phi(k)) = 0. \tag{1}$$

With ϕ so determined, the rental value of capital k and the profit of the firm that owns it is

$$\pi(k) = F(k,\phi(k)) - w(\phi(k)). \tag{2}$$

The Scrapping Decision

By the envelope theorem, $\pi'(k) = F_1(k,\phi(k)) > 0$, so that the worst machines are the least profitable. The population of workers is fixed, whereas better machines are constantly introduced and the worst machines are constantly scrapped. So, old machines are in excess supply even at a zero price—they are unprofitable because labor is too expensive. Wages are endogenous, however, and the force that pins *them* down uniquely is free entry of firms. What matters at this point is that any firm can use a machine from the scrap heap, combine it with labor, and produce output. Normalize the size of the population to unity, so that the measure of employed machines is also 1. Let $s_{\min}$ denote the lowest quality of labor.

Free entry bids up the wage of the lowest-skill workers to equal the value of the output they produce. The worst-quality capital that will be employed is determined by

$$\pi(k_{\min}) = 0 \quad \Rightarrow \quad F(k_{\min}, s_{\min}) = w(s_{\min}), \tag{3}$$

where $s_{\min} = \phi(k_{\min})$.

The worst machine must yield zero profits because perfect substitutes for it are freely available on the scrap heap. Arrow [3, Eq. (13)] has the same condition.

Labor Market Clearing

Let $m(k)$ denote society's endowment of machines of quality $k \in [k_{\min}, k_{\max}]$ and let $n(s)$ denote its endowment of labor with skill level $s \in [s_{\min}, s_{\max}]$. If ϕ is a monotone increasing function, the assignment is positive so that the best machines match with the best people. For the labor market for each skill s to clear, we need, for any k, that the number of machines of quality exceeding k equal the labor allocated to such machines:

$$\int_{k}^{k_{\max}} m(v)\, dv = \int_{\phi(k)}^{s_{\max}} n(v)\, dv \quad \text{for all } k. \tag{4}$$

For now, m and n are two atomless densities that we take as given. This defines ϕ uniquely in terms of m and n, and allows us to recover the equilibrium wage function. Since $w(s) = w(s_{\min}) + \int_{s_{\min}}^{s} w'(v)\, dv$, (3) and (1) imply that the wage function is

$$w(s) = F(k_{\min}, s_{\min}) + \int_{s_{\min}}^{s} F_2(\phi^{-1}(v), v)\, dv \tag{5}$$

for $s \in [s_{\min}, s_{\max}]$.

The equilibrium can be calculated recursively: (4) yields ϕ and then (5) yields the wage function.

A Conjectured Equilibrium Assignment

I have so far assumed that ϕ is an increasing function. In fact, it will, in equilibrium, turn out that k/s will be constant across matches. So let $k = xs$, where x is a constant. Then,

$$\phi^{-1}(s) = xs. \tag{6}$$

I shall show in the next section that on the steady-state growth path $m(\cdot)$ and $n(\cdot)$ will indeed have the required properties for (6) to hold. Since F is linear homogeneous, $w(s) = s_{\min} F(x,1) + \int_{s_{\min}}^{s} F_2(x,1)\, dv$. Write $F(k,s) \equiv sf(k/s)$ so that $f(x) = F(x,1)$. Then

$$w(s) = s_{\min} f(x) + [f(x) - xf'(x)](s - s_{\min}) \tag{7}$$

for $s \in [s_{\min}, s_{\max}]$ and

$$\pi(k) = f'(x)(k - k_{\min}) \tag{8}$$

for $k \in [k_{\min}, k_{\max}]$. Since $w(\cdot)$ is a linear function and since $F_{22} < 0$, the first-order condition (1) is also sufficient for a strict maximum at $s = \phi(k)$.

A Momentary Equilibrium

A momentary equilibrium consists of an assignment function $\phi(k)$, a rental function $\pi(k)$, and a wage function $w(s)$ for which (1), (4), and (7) hold for all k and s. Moreover, all combinations of (k, s) other than $(k, \phi(k))$ yield negative profits. In the next section I will show that (6) holds and that, hence, a momentary equilibrium exists.

3. GROWTH

Although the qualities of workers and machines are both endogenous, it will turn out that the long-run growth rate is determined in the human capital sector. Inequality, on the other hand, is determined largely by the production technology for machines. I shall analyze a long-run equilibrium in which output and the stocks of each input type grow at the rate g, as do their wages and rentals.

3.1. Normalization of Variables

The functions $m(\cdot)$ and $n(\cdot)$ will refer to the distributions of endowments at date zero. Let k_t and s_t denote the qualities of capital and labor at date t. If the stocks of all types of capital and labor grow at the rate g, the distributions of the detrended qualities will be the same as their period-zero distributions. In other words, for any $t > 0$, $m(\cdot)$ will be the distribution of the variable $e^{-gt}k_t$ and $n(\cdot)$ the distribution of the variable $e^{-gt}s_t$.

The function $w(\cdot)$, defined in (7) for $s \in [s_{\min}, s_{\max}]$, will denote the wage function that prevails at date zero. If the wage of each type of normalized labor grows at the rate g, a worker of type s_t will at date t receive a wage of $e^{gt}w(e^{-gt}s_t)$. When this is the case, the distribution of detrended wages will also be invariant.

3.2. Accumulation of Skill

Accumulation Technology

A worker's supply of skill depends on his human capital, h, and on the fraction, u, of time that he works. Specifically,

$$s = uh.$$

The rest of his time he spends learning, and this augments his human capital as in Lucas [28],

$$\frac{dh}{dt} = \eta(1-u)h, \tag{9}$$

where η is a parameter. The output of a firm that employs this worker is $F(k, uh)$.

Assume that the best worker at date zero has $h = 1$. Suppose all workers choose the same value of u. Then $s_{\max} = u$, and hence $k_{\max} = ux$.

Wealth Maximization

A worker that at date t supplies skill $s_t = u_t h_t$ will receive the wage

$$\begin{aligned} W(t, s_t) &\equiv e^{gt}w\left(e^{-gt}s_t\right) \\ &= e^{gt}\left\{s_{\min}f(x) + [f(x) - xf'(x)]\left(e^{-gt}s_t - s_{\min}\right)\right\} \\ &= e^{gt}s_{\min}xf'(x) + [f(x) - xf'(x)]s_t. \end{aligned} \tag{10}$$

The first equality in (10) follows from (7). The worker behaves as if this is the wage contract for all feasible (s_t) sequences.[4] The worker wishes to

[4] The problem is therefore defined formally only on the set of (u_t) sequences for which $e^{-gt}u_t h_t \in [s_{\min}, s_{\max}]$. In other words, the worker is constrained to supplying skill only in markets that are open in equilibrium. A similar issue comes up in the treatment of machine producers, and I discuss it in the Appendix.

maximize the present discounted value of his wealth, $\int_0^\infty e^{-rt} W(t, u_t h_t)\, dt$, but since the first term on the right-hand side of (10) does not depend on s_t, the problem is equivalent to the problem

$$\max_{(u_t, h_t)_0^\infty} \left\{ \int_0^\infty e^{-rt} [f(x) - xf'(x)] u_t h_t \, dt \right\} \quad \text{s.t. } \frac{dh_t}{dt} = \eta(1 - u_t) h_t,$$

with $h(0) \leq 1$, and with r and x given.

The Hamiltonian is $e^{-rt}[f(x) - xf'(x)]uh + \bar{\lambda}\eta(1 - u)h$, where $\bar{\lambda}$ is the multiplier on the constraint. Let $\lambda = e^{rt}\bar{\lambda}$ be the current value multiplier. Then evaluated at a point at which $d\lambda/dt = 0$, the conditions of optimality are

$$f(x) - xf'(x) - \lambda\eta = 0 \tag{11}$$

and

$$\lambda[-r + \eta(1 - u)] + [f(x) - xf'(x)]u = 0. \tag{12}$$

Together these two conditions imply that

$$r = \eta \tag{13}$$

and then (16) below implies a growth rate of human capital of

$$g \equiv \eta(1 - u) = \frac{\eta - \rho}{\gamma}. \tag{14}$$

This also is the equilibrium growth rate in Lucas' model when population is fixed and when external effects are absent.

3.3. Saving

People value consumption streams (c_t) by the utility function

$$\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1 - \gamma}\, dt. \tag{15}$$

A person can borrow and lend at the rate r, and so his decisions decompose into two steps. He first chooses a lifetime human capital investment plan so as to maximize his wealth and he then maximizes utility given that discounted consumption must equal wealth.[5] If c is to grow at the rate g, we must have

$$g = \frac{r - \rho}{\gamma}. \tag{16}$$

3.4. The Distributions $m(\cdot)$ and $n(\cdot)$

Since $m(\cdot)$ and $n(\cdot)$ are distributions of detrended quality, and since they are invariant, it suffices to derive them for date $t = 0$. To ease notation, let $k_{\max} \equiv z$, so that the date-zero frontier machine quality is

$$z = ux.$$

Suppose that the quality of an installed machine depreciates at the rate δ, and let T denote the age at which machines are retired. Then

$$k_{\min} \equiv ze^{-(g+\delta)T}.$$

T-period-old machines are steadily replaced by frontier machines that improve at the rate g. Let τ denote the age of a machine at $t = 0$. Then that machine's quality is

$$k_\tau = uxe^{-(g+\delta)\tau}. \tag{17}$$

Since machines are replaced at a constant rate, the age distribution of machines is uniform. A uniformly distributed τ then implies that the density of k is log-uniform:

$$m(k) = \left[\frac{1}{(g+\delta)T}\right]\frac{1}{k} \quad \text{for } k \in \left[uxe^{-(g+\delta)T}, ux\right]. \tag{18}$$

Since s is proportional to k, it too is log-uniform:

$$n(s) = \left[\frac{1}{(g+\delta)T}\right]\frac{1}{s} \quad \text{for } s \in \left[ue^{-(g+\delta)T}, u\right]. \tag{19}$$

[5] Because people's skills differ, so will their wealth. But Caselli and Ventura [10] show that with perfect capital markets, with the homothetic preferences in (15), and with an accumulation of earning power implied by (9), aggregate consumption depends only on aggregate wealth, and not on its distribution among consumers. That is, aggregate consumption coincides with what a fictional representative agent would consume if he were endowed with average wealth.

Although values for T, u, and x are yet to be determined, these forms for $n(\cdot)$ and $m(\cdot)$ meet two necessary conditions of equilibrium:

(a) The assignment in (6) and an equilibrium as described in Section 2, and

(b) A steady replacement of T-period-old machines by frontier machines that improve at the rate g.

3.5. Machine Production

The Cost Function for k

External effects will allow returns to remain constant as the economy grows, but to diminish at the firm level. Each period, a machine producer can make one machine of *any* quality k, at a cost (in goods) equal to

$$y = z\xi\left(\frac{k}{z}\right), \qquad (z,k) \in R_+^2,$$

where $\xi(0) = 0$ and where ξ is increasing, convex, and twice differentiable. Therefore $\partial y/\partial z = \xi - (k/z)\xi' < 0$,[6] and hence aggregate knowledge bestows a cost-reducing externality on each machine producer.

Inversion of the cost function yields the *production function* for k:

$$k = z\xi^{-1}\left(\frac{y}{z}\right).$$

Returns diminish at the firm level because ξ^{-1} is convex. Let

$$\sigma \equiv \frac{\xi(1)}{\xi'(1)}.$$

This is the elasticity of k with respect to y at the point $k = z$, and it is a measure of the returns to scale (with respect to private inputs y) at that point. The constant elasticity Cobb–Douglas form,

$$k = z^{1-\sigma}y^{\sigma},$$

is an example I shall work out later, and it implies that the fraction of productivity growth in the machine sector that comes through "spillovers" is $1 - \sigma$. In this case, $\xi(v) = v^{1/\sigma}$.

[6] For any $s > 0$, the mean value theorem implies that $\xi(s) = \xi(0) + s\xi'(s^0)$ for some $s^0 \in [0, s]$. Strict concavity of ξ implies $s^0 < s$. Hence, $s\xi'(s^0) < s\xi'(s)$ and the claim follows.

Rentals

I shall normalize capital quality by dividing it by e^{gt}, as I did with s. Normalized quality will have a stationary distribution $m(\cdot)$. In the steady state, age-specific rentals grow at the rate g. Let a machine be of quality k at date zero. By date t, its quality will have depreciated to $ke^{-\delta t}$. Hence the period-t rental of the machine will be $e^{gt}\pi(ke^{-(g+\delta)t})$. At date zero the present value of these rentals is

$$P(k) = \int_0^{L(k)} e^{-(r-g)t}\pi(ke^{-(g+\delta)t})\,dt, \tag{20}$$

where $L(k)$ is the remaining lifetime of the machine. It is scrapped when its quality falls to $uxe^{-(g+\delta)T} = k_{\min}$, so that $L(k)$ must solve $k_{\min} = ke^{-(g+\delta)L(k)}$. Then $(k/ux)e^{-(g+\delta)L(k)} = e^{-(g+\delta)T}$, i.e.,

$$L(k) = T + \frac{1}{g+\delta}\ln\left(\frac{k}{ux}\right). \tag{21}$$

A newly built machine of quality $k < ux$ is just like a machine that was once on the frontier but that has since aged $T - L(k)$ periods down to quality k. Let $P(k)$ be the price of the machine that is of quality k when built. In Section 2, I assumed that final goods producers could freely enter by using a machine from the scrap heap. They can also freely enter by purchasing a newly produced machine. The purchase price of the machine, $P(k)$, will therefore be bid up to a level equal to the discounted present value of its rentals, so that $P(k)$ is indeed given by (20).

A Machine Producer's Decision Problem

A machine producer therefore chooses k to maximize his instantaneous profits for that period:

$$\max_k \left\{P(k) - z\xi\left(\frac{k}{z}\right)\right\}. \tag{22}$$

For $k = ux$ $(= z)$ to be the optimal decision, the first-order condition is

$$P'(ux) = \xi'(1). \tag{23}$$

Assuming free entry into machine production, profits must be zero:

$$P(ux) - ux\xi(1) = 0. \tag{24}$$

The problem and its solution at $k = ux$ are illustrated in Fig. 1.

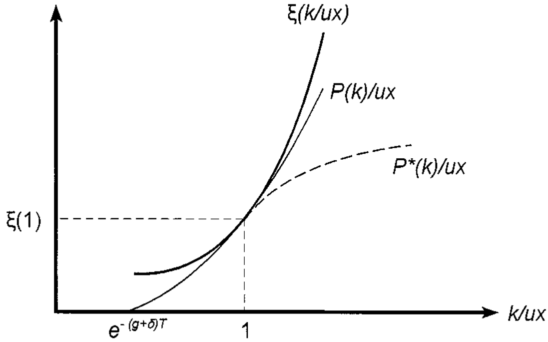

FIGURE 1

Because of its technical nature, I defer the discussion of the sufficiency of conditions (23) and (24) for a maximum until Section 3.9.

3.6. Characterizing $P(k)$

I shall now show that $P(\cdot)$ is increasing and convex, as illustrated in Fig. 1. With (8), (20) gives us

$$P(k) = kf'(x)\left\{\frac{1-e^{-(r+\delta)L(k)}}{r+\delta} - e^{-(g+\delta)L(k)}\frac{1-e^{-(r-g)L(k)}}{r-g}\right\}. \quad (25)$$

Second, since $\pi(ke^{-(g+\delta)L(k)}) = 0$, and since $L'(k) = 1/((g+\delta)k)$, differentiation of (20) shows that the value of a machine is strictly increasing in its quality:

$$P'(k) = \int_0^{L(k)} e^{-(r-g)t}\pi'(ke^{-(g+\delta)t})\,dt = f'(x)\frac{1-e^{-(r+\delta)L(k)}}{r+\delta}. \quad (26)$$

Moreover the relation is convex:

$$P''(k) = f'(x)\frac{e^{-(r+\delta)L(k)}}{(g+\delta)k} > 0. \quad (27)$$

Since $\pi(\cdot)$ is linear in k, convexity arises solely because a machine's remaining life grows with its quality; that is, $L'(k) > 0$.

From (21) and (25) it is clear that $(P(k))/ux$ depends on k via the ratio k/ux only. The functions ξ, P, and P^* (which will be introduced in Section 3.9) are depicted in Fig. 1. The figure shows that the capital goods producer optimally chooses quality $k = ux$, which also is the unique optimum, at which he also breaks even.

3.7. The Implicit Function that Defines T

All the equilibrium conditions reduce to a single equation in one unknown, T. Combining (25) and (26) with (23) and (24) yields

$$\sigma = \frac{P(ux)}{uxP'(ux)} = 1 - \frac{\left[\left(e^{-(g+\delta)T}\right)\left(1 - e^{-(r-g)T}\right)\right]/(r-g)}{\left(1 - e^{-(r+\delta)T}\right)/(r+\delta)}$$
$$= 1 - \left(\frac{r+\delta}{r-g}\right)\frac{e^{(r-g)T} - 1}{e^{(r+\delta)T} - 1}. \tag{28}$$

Since $r = \eta$ and (setting $\gamma = 1$) $g = \eta - \rho$, we can express the right-hand side of (28) in terms of the exogenous parameters, and end up with an equation in one unknown, T:

$$\sigma = 1 - \left(\frac{\eta + \delta}{\rho}\right)\left(\frac{e^{\rho T} - 1}{e^{(\eta+\delta)T} - 1}\right) \equiv \psi(T). \tag{29}$$

3.8. Equilibrium

Equilibrium consists of five nonnegative scalars, g, r, x, u, and T, and a function $P(\cdot)$ that satisfy (13), (14), (16), (23), (24), and (25). With these, $m(\cdot)$ and $n(\cdot)$ are defined in (18) and (19), and $w(\cdot)$ and $\pi(\cdot)$ are defined in (1) and (2).[7]

The equilibrium can be determined recursively: (29) has one unknown, T. Since ψ is continuous and strictly increasing in T from $\psi(0) = 0$ to $\lim_{T\to\infty} \psi(T) = 1$, a unique equilibrium exists as long as $\sigma < 1$. With T and u [from (14)], we can solve for x in (25). Of course, ξ must be specified so that things indeed look as shown in Fig. 1. This means that the second-order conditions must hold too. These conditions are laid out in the next section. Because they are involved, I will offer no general existence result. Instead I shall construct an equilibrium by example in the subsequent section.

[7] Although it is not part of the definition of equilibrium, one can derive the income identity as follows: The measure of machines in use is 1, and each is scrapped when it reaches age $T > 0$. Only frontier machines, of quality z, are built. New machines come in at a constant rate and their age distribution is uniform. To keep the number of machines at 1, the arrival rate must be $\overline{m} \equiv 1/T$. The cost of building each machine is $z\xi(1)$, and so aggregate investment is $\overline{m}z\xi(1)$. Aggregate output then is $c + \overline{m}z\xi(1)$, where c is aggregate consumption.

3.9. The Sufficiency of Conditions (23) and (24)

Conditions (23) and (24) are necessary for a maximum at $k = ux$. For sufficiency, one must show that the expression in (22) is strictly negative for all $k \neq ux$, namely, that

$$\frac{P(k)}{ux} - \xi\left(\frac{k}{ux}\right) < 0 \tag{30}$$

for $k \neq ux$. The revenue from choosing $k < ux$ is $P(k)$. But $P(k)$ is a fictional concept for $k > ux$. The rental markets for $k > ux$ are inactive, but it seems reasonable to assume that a $k > ux$ type of machine would match with the best labor and that during its "avant garde" epoch, the machine would rent for the value of its output net of the wage of the best labor. I define the present value of a $k > ux$ machine to be $P^*(k)$, and in the Appendix I prove the following claim.

CLAIM 1. (*a*) $P^*(k) < P(k)$ *for* $k > ux$.
(*b*) $P^{*\prime}(ux) = P'(ux)$.

The claim implies that conditions (23) and (24) are necessary for a maximum at $k = ux$, and that (30) is sufficient.

3.10. Example for Which Equilibrium Exists

Let us return to the Cobb–Douglas form for the production function for machines: $k = z^{1-\sigma}y^{\sigma}$. Let $v \equiv k/z$ and let $\xi(v) = v^{1/\sigma}$, where $\sigma < 1$. Let $b \equiv (r + \delta)/(g + \delta)$ and $B \equiv e^{-(r+\delta)T}$. Then $v \geq B^{1/b}$ and $e^{-(r+\delta)L(k)} = B\exp\{\ln(v^{-(r+\delta)/(g+\delta)})\} = Bv^{-b}$. Equation (26) reads

$$P'(k) = f'(x)\frac{1 - Bv^{-b}}{r + \delta} \equiv \beta(v),$$

which increases monotonically from zero, when $v = B^{1/b}$, to $f'(x)/(r + \delta)$ as v becomes large. Then $P(k) = P(uxv)$ starts from zero and is asymptotically linear. This means that $\xi(v)$ starts out above P and eventually ends up above it because $\sigma < 1$, and therefore the first-order condition is

$$\beta(v) = \xi'(v). \tag{31}$$

Optimality at $v = 1$ requires that $\beta(1) = \xi'(1)$, i.e., that

$$f'(x)\frac{1 - B}{r + \delta} = \frac{1}{\sigma} \tag{32}$$

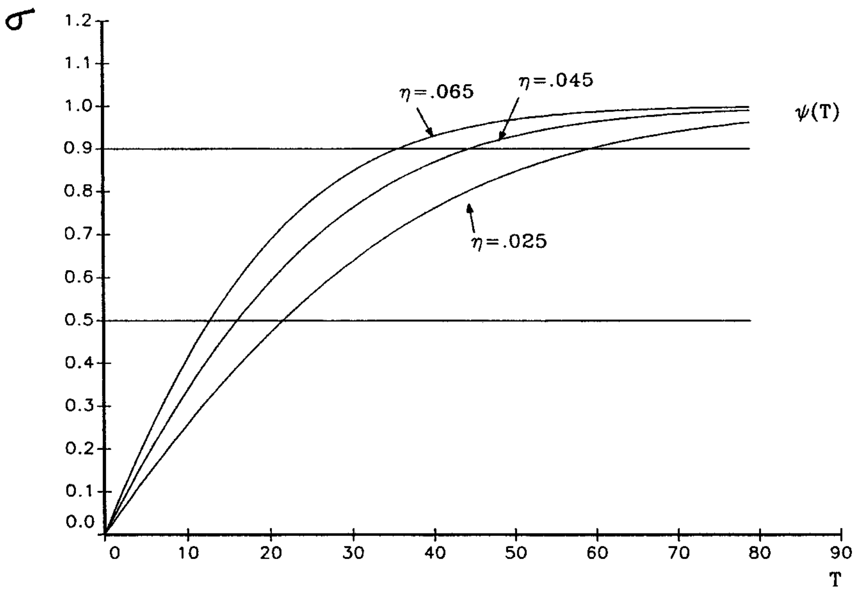

FIG. 2. Determination of T in Eq. (29).

and that the second-order condition hold:

$$\frac{bB}{1-B} < \frac{1}{\sigma} - 1. \tag{33}$$

CLAIM 2. *If* (32) *and* (33) *hold, then the maximum at* $v = 1$ *is global.*

The proof is given in the Appendix. Condition (33) is hard to check analytically, but numerical simulations show that it holds for all $\sigma \in [0.01, 0.99]$ when $\eta = 0.04$, $\delta = 0.05$, and $\rho = 0.02$.[8]

3.11. The Importance of σ

Inequality depends critically on σ. The studies that Griliches [23] surveys tend to imply that σ is between 0.5 and 1. Figure 2 plots the two sides of (29); the intersection of the two lines takes place at the equilib-

[8] Alternatively, we can replace (33) with a more restrictive condition, but one that is easier to analyze: Proposition 2 implies that $B < (1-\theta)^b$ and $1 - B^{1/b} > \theta$. Since $b > 1$ and $B < 1$ we have $B^{1/b} > B$, which implies that $1 - B > 1 - B^{1/b} > \theta$. These bounds on B and $1 - B$ imply that the second-order condition is met if $b(1-\theta)^b < 1 - \theta$ or $1/(1-\theta) > b^{1/(b-1)}$. If we take the empirically valid values $\eta = 0.04$, $\delta = 0.05$, and $\rho = 0.02$, this condition holds for all $\theta \in [0.58, 1)$.

rium value of T. Figure 2 plots three versions of $\psi(T)$. Each has $\rho = 0.02$ and $\delta = 0.05$, but the lowest curve has $\eta = 0.025$, the middle curve has $\eta = 0.045$, and the top curve has $\eta = 0.065$. The rates of growth implied by these parameter values are $\frac{1}{2}, 2\frac{1}{2}$, and $4\frac{1}{2}\%$.

Recall that η is the rate of return on human capital formation and that from (13) this also equals the rate of interest. Of the three, the middle curve therefore fits recent world experience the best in terms of the rate of growth of $2\frac{1}{2}\%$ and the interest rate, which at $4\frac{1}{2}\%$ is not far from the midpoint of the long-run rate of return on bonds, on the one hand, and equity, on the other.

The solution for T depends largely on σ. First, let $\sigma = 0.5$. Then $T = 22$, 16, and 13 years, respectively. The relative productivity differential between the best and worst worker, $e^{(g+\delta)T}$, hovers around 3.3 for all three cases. This is not that big. Next, raise σ to 0.9. Then $T = 60$, 45, and 36, and $e^{(g+\delta)T} = 27$, 29, and 31, respectively! Finally, as $\sigma \to 1$, inequality becomes infinite, and if $\sigma = 1$, equilibrium does not exist.[9]

3.12. Is the Balanced Growth Path Unique?

Although the path I have described is unique in its class, I have to this point maintained three assumptions which I shall now defend in more detail.

(a) *x is the same for each machine–worker pair*: If the distribution of h is to be invariant, all workers must invest at the same rate. If x differed over assignments, then in (10) x would depend on s and the function $W(t, s)$ would be nonlinear in s. But given the linear accumulation technology for h, a worker's optimal u would then depend on his level of h, and hence some workers would invest faster than others and the distribution of h would not be stable.

(b) *u does not depend on time*: First, if u varied over time and if this variation was correlated over workers, the rate of growth of the population average h would vary over time and so would the economy's growth rate. Second, if the variation was *un*correlated over workers, the distribution of h could be invariant only if there was some turnover in the distribution. This implies that we would see overtaking behavior in h. But in a continuous time accumulation model this is impossible, for at some date the two workers' h levels would be the same, and from that point on they would remain the same.

[9] Or, rather, there is no steady-state equilibrium. There may exist an equilibrium with an ever-expanding skill distribution.

(c) *Only frontier machines are built*. Can there be a steady state in which, say, two machine qualities, z_1 and z_2 (with $z_1 < z_2$) are built? Then $m(k)$ would have two distinct portions. Its right tail would remain log-uniform as in (18) if in that formula we replace z by z_2, but there would now be a jump at z_1. If $x = z_2/u$, $n(\cdot)$ would have to have a jump in it and be of the same form as $m(\cdot)$. Given that $z = z_2$, however, the value of x is the same and hence the price functions $w(\cdot)$, $\pi(\cdot)$ and $P(\cdot)$ would then be the same because they depend on $m(\cdot)$ and $n(\cdot)$ only through x. Then u too would be the same.

With more than two candidate zs, the same remarks apply. Let N different qualities be built at $z_1, \dots, z_N$, and suppose that z_N is the largest. As in the previous paragraph, we construct a hypothesized equilibrium in which u and $P(\cdot)$ are the same as they would be if only quality z_N were built. But in fact, it is easy to see that if ξ is convex, they are unlikely to arise. Certainly they cannot arise for the equilibrium constructed in Section 3.10: In the version of Fig. 1 that applies to the Cobb−Douglas case, there can be at most one tangency between the two solid curves. Perhaps one could cook up an example with two or even more tangencies, but it certainly would be hard, and perhaps impossible.

These arguments make a pretty strong case that the balanced growth path is indeed unique.

4. PROPERTIES OF THE MODEL

First I shall discuss properties of the model on a given growth path, and then I shall examine how steady-state behavior changes in response to exogenous, once-and-for-all shifts in parameters.

4.1. A Lower Bound on Productivity Differentials

Balanced growth must involve inequality in both k and h. Otherwise the solution for T in (28) would have to be zero, which cannot happen as long as $\xi'(1) < \infty$. The extent of inequality depends inversely on the degree to which private returns to creating machine quality diminish.

Productivity Differentials

The productivity of the best worker relative to that of the worst worker is

$$\frac{(1/u)F(ux, u)}{(1/u)F(uxe^{-(g+\delta)T}, ue^{-(g+\delta)T})} = e^{(g+\delta)T}. \tag{34}$$

Productivity differentials are proportional to skill differentials. A lower bound on productivity differentials easily emerges:

PROPOSITION 1.

$$e^{(g+\delta)T} > \frac{1}{1-\sigma}. \qquad (35)$$

Proof. Since $P(k_{\min}) = 0$, $ux\xi(1) = P(ux) = \int_{k_{\min}}^{ux} P'(v)\,dv < P'(ux)(ux - k_{\min})$, because $P'' > 0$. Now $k_{\min} = uxe^{-(g+\delta)T}$ and $P'(ux) = \xi'(1)$. This yields the inequality $\xi(1) < \xi'(1)(1 - e^{-(g+\delta)T})$. Rearrangement yields the claim. Q.E.D.

So while growth depends on the human capital technology alone, productivity differentials depend at least in part on the machine production technology. If σ is close to unity, it is relatively easy to move beyond the frontier, and this stretches out the variance of machine quality and with it the variance of skills. On the other hand, if σ is small, it is hard to move ahead of the pack, the variance of machine quality is small, and so is steady-state inequality. In this sense, spillovers in machine production reduce steady-state inequality. In sum, the flatter are the marginal costs of improving machines, the more inequality we will see in the end.

My interpretation of the proposition emphasizes the steepness of the machine producer's marginal costs as he perceives them. But one might be tempted to interpret the result as a restatement of the age-old view that technological spillovers "hold the world together" by preventing the technological leaders from pulling away from the rest of the pack. I am not convinced by this view because machine producers are identical here, and they remain so even as $\sigma \to 1$. In any event, I see no way to investigate this issue further without relaxing the constraint that the production function for k has constant returns to scale in y and z, and this would require a rewriting of the entire model.

If the residents of each country all had (for a reason not modelled here) the same skills, then (34) would be the ratio of output per head in the richest country relative to that of the poorest country.

4.2. Skill Premia, Vintage Premia, and the Diffusion of Technology

Skill Premia

Wages vary *less* than skills. From (7), the wage of the best worker relative to that of the worst is

$$\frac{w(u)}{w(ue^{-(g+\delta)T})} = 1 + (e^{(g+\delta)T} - 1)\left[1 - x\frac{f'(x)}{f(x)}\right]. \qquad (36)$$

If $F(k, s) = k^{\alpha} s^{1-\alpha}$, $f(x) = x^{\alpha}$, and $x(f'(x))/(f(x)) = \alpha$, the relative wage ratio is $1 + (1 - \alpha)(e^{(g+\delta)T} - 1)$. For this example, then, income inequality is monotonically (and linearly) related to productivity differentials.

Vintage Premia

Machine rentals and prices vary proportionally *more* than machine quality. From (2), the ratio of the rental of a frontier machine relative to that of a τ-year-old machine is

$$\frac{\pi(ux)}{\pi(uxe^{-(g+\delta)\tau})} = \frac{e^{(g+\delta)T} - 1}{e^{(g+\delta)(T-\tau)} - 1}. \tag{37}$$

As a machine ages and τ increases from 0 to T, the relative rental rises from 1 to ∞, at an increasing rate. Therefore machine obsolescence rates as usually measured increase with machine age. This result and the result that wages vary less than productivity stem from the assumption of free entry into final goods production, and the free availability of scrapped machines which forces the rental of the worst machine in use to zero.

Diffusion of Technology

It takes T periods for a technology to diffuse fully. Technological users—workers—are at different points of the technological ladder, each climbing it at the same speed. Only frontier labor uses frontier methods. The rest adopt and use only technologies inside the frontier, which seems to be so in poor countries. And while unskilled labor earns less than skilled labor, it gets a larger share of the output that it produces—the least skilled labor gets all its output. The investment−output ratio rises with development—unskilled labor works with cheap capital. This helps explain why the investment rate is low in poor countries.

4.3. Comparative Steady-State Analysis

4.3.1. *The Parameters* η, δ, ρ, *and* σ

Differentials in productivity, wages, and profits depend partly on the parameters and partly on the endogenous variables T and x. I now examine how the steady-state values of these two variables depend on η, δ, ρ, and σ. It turns out that T and x move together in response to a change in each parameter. The algebra gets involved, and so comparative steady-state analysis was done only for the case in which utility is logarithmic ($\gamma = 1$), and I report it in an appendix available on request.

Implicit differentiation of (29) yields the following responses of T to the parameters:

$$\frac{\partial T}{\partial \eta} < 0, \qquad \frac{\partial T}{\partial \delta} < 0, \qquad \frac{\partial T}{\partial \rho} > 0, \quad \text{and} \quad \frac{\partial T}{\partial \sigma} > 0.$$

None is surprising. Faster growth and faster depreciation hasten replacement, whereas higher discounting retards it. The fourth result restates the intuition behind Proposition 1.

For the response of x, evaluate (26) at x to get the first-order condition

$$\xi'(1) = f'(x)\frac{1 - e^{-(r+\delta)T}}{r + \delta}. \tag{38}$$

Substitute into it the solutions for r and g, and calculate the response of x. The strict concavity of f and the comparative statics results on T allow us to sign the response of x as follows:

$$\frac{\partial x}{\partial \eta} < 0, \qquad \frac{\partial x}{\partial \delta} < 0, \qquad \frac{\partial x}{\partial \rho} > 0, \quad \text{and} \quad \frac{\partial x}{\partial \sigma} > 0.$$

Therefore T and x move in the same direction in response to each parameter. Changes in δ and in σ do not affect the growth rate. Their effect on the level of world output is ambiguous. A higher x raises the output of an individual of given ability, but the accompanying rise in T means that average ability is lower.

4.3.2. *The Effect of Capital–Skill Complementarity*

From (5) and (3), wage inequality in (36) can be written as

$$\begin{aligned} \frac{w(s_{\max})}{w(s_{\min})} &= 1 + \frac{1}{F(k_{\min}, s_{\min})} \int_{s_{\min}}^{s_{\max}} F_2(\phi^{-1}(v), v)\, dv \\ &= 1 + \left(\frac{s_{\max} - s_{\min}}{s_{\min}}\right)\left[1 - \frac{xf'(x)}{f(x)}\right]. \end{aligned}$$

Holding $s_{\max}$, $s_{\min}$, and $k_{\min}$ constant, wage inequality will rise when machines and skills are stronger complements in the sense of a larger value of F_{12}. But the story does not end there because the rise in wage differentials encourages more investment in h and leads to a higher g, and this has repercussions for x. In the end, I cannot show that equilibrium income inequality will rise.

4.3.3. *The Effect of Progress in the Technology of Machine Production*

Neutral shifts of the machine production technology affect x but not T. Let μ be a Hicks-neutral shift parameter in the cost function $\xi(\cdot)$. That is, let

$$\xi(\cdot) \equiv \mu\zeta(\cdot),$$

so that technological progress in the technology for producing machines is represented by a fall in μ. Since $\sigma = (\zeta(1))/(\zeta'(1))$, μ does not enter (29). Hence μ does not affect T and it does not affect productivity differentials.

Technological progress does, however, lead to a higher x, and if the elasticity of substitution in F is less than unity, it also leads to higher income inequality. From (38),

$$x = f'^{-1}\left[\left(\frac{(r+\delta)\zeta'(1)}{1 - e^{-(r+\delta)T}}\right)\mu\right]$$

and x is therefore decreasing in μ.

A change in μ does affect income inequality, however. The higher capital−labor ratio will, however, raise wage inequality if the elasticity of substitution in $F(\cdot)$ is less than 1. To see why, consider the Constant Elasticity of Substitution (CES) example

$$F(k,s) = [\alpha k^{\varepsilon} + (1-\alpha)s^{\varepsilon}]^{1/\varepsilon} = s[\alpha x^{\varepsilon} + 1 - \alpha]^{1/\varepsilon} = sf(x),$$

where $x = k/s$. Then

$$\frac{xf'(x)}{f(x)} = \frac{\alpha x^{\varepsilon}}{\alpha x^{\varepsilon} + 1 - \alpha},$$

and it is decreasing in x if $\varepsilon < 0$ (which is equivalent to an elasticity of substitution of less than unity). From (36), we then find that wage inequality increases when μ falls. On the other hand, if $\varepsilon > 0$ and the elasticity of substitution exceeds unity, a fall in μ would reduce income inequality.

The model therefore suggests the following interpretation of the industrial revolution: At the turn of the nineteenth century the West was only slightly richer than China. If μ had experienced a once-and-for-all reduction around then, there would be a transition to higher value of x, a higher growth in per capita output for a while, and, notably, a rise in income inequality. So, as long as the elasticity of substitution between k and s is less than unity (for which I cannot provide any direct evidence), the model is consistent then with the rise in inequality which followed industrial revolutions as documented by Greenwood and Yorukoglu [21].

5. DISCUSSION

The previous analysis rests on four premises. First, new technology is embodied in machines. Second, *quantities* of capital and labor are matched in fixed proportions. Third, capital *quality* and labor *skill* are complements. And, fourth, assignment is frictionless. I discuss each assumption next, in light of the evidence.

5.1. Capital−Skill Complementarity

Griliches [22] finds that capital intensive sectors pay a higher premium for skill. But payments to capital may reflect its quantity, not its quality, and the assumption that $\partial^2 F / \partial k \, \partial s > 0$ is about the latter. Bartel and Lichtenberg [6], however, find that sectors where capital is older and hence of poorer quality employ fewer skilled workers. And Siegel [38], and Doms, Dunne, and Troske [15], actually identify new technologies, and find that firms that adopt them use skilled workers. Finally, Huggett and Ospina [24] find that plants that employ more skilled workers invest more in new equipment (as opposed to used equipment).

5.2. Inventions Embodied in Capital Goods

Machines of all kinds are obviously improving every year; the only question is how fast. In their study of a large number of plants, Bakh and Gort [5] find that embodied technological change of capital is associated with between 2.5 and 3.5% change in output for each 1-year change in vintage. And Greenwood, Hercowitz, and Krusell [20] find that the decline in the relative price of capital implies that embodied technological change may be about 3.2% per year.

5.3. Fixed Proportions between Machines and Workers

If proportions were not fixed, and if workers could operate an unlimited number of machines of more than one vintage at a time, we would not see widespread scrappage of still-usable equipment! Now, if proportions are indeed fixed, the magnitude of those proportions still remains to be determined. Because I fix the proportions at 1 : 1, my model allows no interaction between capital of different vintages, an assumption that Gort and Boddy [19, p. 395] criticize:

> The trouble with defining production units in a way that limits the scope of each to one vintage of capital is that, in fact, a large proportion of capital goods of differing vintages perform interdependent functions. Consequently, they are inputs into a common production process . . . the best level of detail in the choice of production units depends on two conditions: namely, the homogeneity of the physical process and the degree of interdependence among individual capital goods.

It would be tedious but inconsequential to assume that the worker can combine with *two* pieces of capital instead of one (a building and a computer, say). This would allow the interaction that Gort and Boddy emphasize, and yet it would retain fixed proportions—2 : 1 instead of 1 : 1. A harder extension would be to assume that the *outputs* of several worker–machine pairs interact nonadditively in a multiworker multimachine production function.

5.4. Frictions

In the Introduction, I argued that in Arrow's and Parente's models there is excessive switching of ranks among workers (or countries), a feature we do not observe in the data. This is true, but in my model the increased stability of inequality in productivities and earnings comes at the price of excessive switching of technologies, another feature we do not observe in the data. Since I assume that reassignment is frictionless and that skill is general, as it ages, a machine changes hands repeatedly. In fact, however, frictions such as transportation and installation costs and poor information about quality will impede reassignment. The most significant friction of all is probably technology-specific human capital, an acquired skill that a reassignment to a better technology at least partially destroys. This specialization attaches people to what they do, and Zeckhauser [42], Chari and Hopenhayn [11], Parente [30], and Jovanovic and Nyarko [26, Case A] all stress this.

But as Jovanovic and Nyarko [26, Case B] emphasize, far from creating attachment, learning can also do the opposite: it will promote technological switching when it is transferable from one technology to a more productive one! Indeed, Jovanovic and Nyarko [27] argue that this force helps explain upward "career" mobility—a pilot, for example, learns to fly a small plane before being entrusted with a Boeing 747.

Frictions, in any event, are not so large as to choke of all reassignment, which in practice occurs in three ways:

(a) *Labor turnover*: When upgrading its technology, the firm typically hires skilled and releases unskilled labor, as Siegel [38] finds. This may be why gross flows of labor among sectors, firms, and plants exceed net flows. And it is probably the best labor from poor countries that migrates toward the best capital in rich countries.

(b) *Used equipment sales*: To make room for a new machine, a firm will sell its used machines. Used equipment and structures comprise just under 10% of all investment in the U.S. industry, but up to 50% for some smaller firms in Japan (Shinohara [37, Table 11]) and well over 50% for some smaller Columbian plants (Huggett and Ospina [24]). Used equipment also flows across borders; Goolsbee [18] reports that upon their retirement from use in the U.S., 43% of all Boeing 707 jet planes were sold abroad, Ramey and Shapiro [32] report that when a defense contractor held a closing auction of its capital goods, 6% by value was sold abroad and Rosenberg [35, pp. 271–272] describes the extensive buying of used equipment by several developing countries. Perelman [31] provides further evidence on used-equipment investment in the U.S. and elsewhere.

(c) *Reallocation within firms*: A firm may be able to save on reallocation costs by organizing many grades of capital and labor under one roof and reassigning internally. In fact, the typical firm's capital stock comprises many vintages of machines, and its workforce is heterogeneous—Davis and Haltiwanger [13] find that the within-plant variance of wages is as much as one-third of its between-plant variance.

All this notwithstanding, the model as it stands implies far too much reallocation—no match lasts longer than an instant. But this can, in principle, be fixed; the instantaneous abandonment of a technology right after its adoption occurs because the model is cast in continuous time and because technology improves at every instant.[10] In a discrete time model one could choose the length of the period to coincide with the worker's average tenure on a technology. Or, in a continuous time model one could assume that improvements arrive at randomly spaced intervals whose average length could be calibrated to equal the average duration of worker–machine matches.

Finally, even as it stands, the model should be useful for analyzing *levels* of variables such as productivity, earnings, and wealth. The "menu cost" literature tells us is that for a single agent, at least, small lumpy adjustment costs induce big ranges of inaction (which here would mean holding on to a machine for a while), but small effects on the agent's payoff level [14]. Moreover, the technology adoption "game" without externalities on the user side is one in which adoptions of frontier machines are strategic substitutes—if I choose not to adopt, I make it cheaper for you to adopt, and so I would not expect that in this model small effects would snowball.[11]

5.5. Relation to Vintage Capital Models

Most vintage capital theory takes the efficiency of investment as an exogenous function of the vintage of investment. A comprehensive treatment is provided in Benhabib and Rustichini [8].[12] Full dynamics are

[10] I thank Daron Acemoglu for pointing this out.

[11] Two recent analyses of frictions in the assignment of people to technologies are Stolyarov [39] and Violante [41].

[12] The papers in this issue by Dwyer and by Klenow are partial equilibrium versions of this type of model.

notoriously difficult in such models, and the stability of the balanced (exogenous) growth path depends on what one assumes about the profile of vintage and depreciation components. Using numerical methods in a model related to mine, Greenwood and Yorukoglu [21] find that the stationary growth path is indeed stable.

In my model, like those of Parente [29] and Cooley, Greenwood, and Yorukoglu [12], the efficiency of each vintage of capital is endogenous, and it varies when the economy is not on its balanced growth path. Cooley et al. find that the balanced growth path is locally stable, at least for the parameter values they used.

5.6. Remarks on Full Dynamics: Income Inequality versus Inequality in Wealth and Consumption

My model explains inequality in incomes. To get implications for inequality in consumption and wealth, one would need to carry out the full dynamics, presumably from an initial condition under which all agents start out equal. But it is easy to see what a fully dynamic analysis would lead to. Indeed, to turn the inequality of income paths into consumption inequality, one needs to introduce leisure into preferences or some form of market incompleteness. While I cannot do this here, I would like to elaborate briefly on these ideas.

For specialization-based models like Lucas' homogeneity is not necessarily the appropriate initial condition. Switzerland made watches and Norway made ships, and it would, in view of their respective geographical locations, be surprising if the opposite had turned out to be the case. But the logic does not extend to my model in which there is just a single consumption good. So suppose everyone starts with the same skill and identical machines, and with access to the machine production and human capital technologies assumed above. There will be no symmetric growth equilibrium. The only possible outcome is an *asymmetric* equilibrium in which some agents get new machines and accumulate skills faster than others. Now, if the capital markets are perfect and since leisure is excluded from the utility function, this creates a problem for the analysis. Initially identical people may well opt for different income trajectories, but they would at the outset have to be indifferent between them, which can only mean that their initial wealth must be same. But then if discount rates do not differ, their consumption and wealth will always be equal. Those that opt for the low training strategy will collect higher incomes in the short run, but lower incomes in the long run.

Since inequality in wealth and consumption is in fact roughly the same as inequality in incomes, this is an undesirable implication. There are two ways to change it. The first is an imperfect capital market. One solution is to shut off all borrowing, in which case an agent's consumption would equal his income. Those that wished to lead would then have to train harder, receive lower wages, and sacrifice some consumption early on, for which they would be compensated by higher incomes and hence higher consumption later.

The second solution is to assume that leisure is valued in the utility function. If a strategy of rapid development involved the optimal sacrifice of leisure early on, the eventual leaders would need to be compensated for it by higher equilibrium wealth and a permanently higher level of consumption. A model with this flavor is Rios-Rull [33].

Either of these two features would complicate the analysis. For instance if the worker's consumption were to equal his wage, the rewards to accumulating h would cease to be linear. Adding leisure would be harder still. In any event, it should be clear that any equilibrium trajectory would necessarily involve inequality in incomes, consumption, and wealth.

6. CONCLUSION

The puzzle of underdevelopment is, in a sense, tantamount to the puzzle of the reasons for sustained inequality. Attributing this to persistent differences in human capital or in government policies that explain inequality begs the following questions:

(a) Why did such differences arise in the first place?

(b) Why are they as large as they are?

I offer simple answers to both questions. First, (a) inequality can originate in the finite capacity of the capital goods sector. Machines are unequal because it costs too much to replace everyone's current machine with a new one all the time. This induces workers to invest unequally in skill because capital quality and skill are complements. Thus inequality arises even if everyone starts out in the same place.

As for (b), Proposition 1 suggests that inequality depends above all else on how easily better machines are produced: The flatter the marginal costs of raising the quality of machines, the greater the long run equilibrium inequality. But in the model these marginal costs are steeper precisely when spillovers of knowledge among machine producers are bigger, and so the quicker productive knowledge travels among capital goods producers, the smaller will inequality be in the long run. The model implies that the ongoing information technology revolution will ultimately reduce the income differentials to levels below what they are today.

APPENDIX

Proof of Claim 1. The "avante garde" phase of a $k > ux$ machine lasts for $L(k) - T$ periods. Assuming that such a machine works with the best labor, it would fetch a rental of $\pi^*(k,t) \equiv F(ke^{-\delta t}, ue^{gt}) - w(u)e^{gt}$ when t periods old. The present value of its rentals is

$$P^*(k) = \int_0^{L(k)-T} e^{-rt}\pi^*(k,t)\,dt + \exp(-(r-g)[L(k)-T])P(ux). \tag{A1}$$

I now show that (a) $P^*(k) < P(k)$ and (b) $P^{*\prime}(ux) = P'(ux)$.

(a) It suffices to show that $\pi^*(k,t) < e^{gt}\pi(ke^{-(g+\delta)t})$:

$$\begin{aligned} e^{-gt}\pi^*(k,t) &\equiv F(ke^{-(g+\delta)t}, u) - w(u) = uf\left(\frac{k}{u}e^{-(g+\delta)t}\right) - w(u) \\ &= uf(x) - w(u) + u\int_x^{(k/u)\exp(-(g+\delta)t)} f'(v)\,dv \\ &< uf(x) - w(u) + uf'(x)\int_x^{(k/u)\exp(-(g+\delta)t)} dv \\ &= \pi(ux) + uf'(x)\left[\frac{k}{u}e^{-(g+\delta)t} - x\right] \\ &= f'(x)\big(ux - k_{\min} + ke^{-(g+\delta)t} - ux\big) \\ &= \pi(ke^{-(g+\delta)t}). \end{aligned}$$

The strict inequality holds because f is concave. An avant garde firm has a higher capital–labor ratio than other firms, and so diminishing returns to capital set in.

(b) Note that in (20), $\pi(ke^{-(g+\delta)t}) = \pi(ux\exp(-(g+\delta)[t+T-L(k)]))$. Changing the variable of integration to $\tau \equiv t + T - L(k)$ yields $P(k) = \int_{T-L(k)}^{T}\exp(-(r-g)[\tau - T + L(k)])\pi(uxe^{-(g+\delta)\tau})\,d\tau$, and, therefore,

$$P'(k) = \frac{1}{(g+\delta)k}[\pi(k) - (r-g)P(k)]. \tag{A2}$$

From (A1), since $L(ux) = T$,

$$P^{*\prime}(ux) = \frac{1}{(g+\delta)ux}\left[\pi^*(ux,0) - (r-g)P(ux)\right]. \qquad \text{(A3)}$$

Evaluating (A2) at $k = ux$ and observing that $\pi^*(ux,0) = \pi(ux)$ proves (b). Q.E.D.

Discussion of Claim 1. Note that P^* and π^* are price functions defining terms of trade for machines that are not produced in equilibrium and that the exercise holds u constant at its stationary level. This seems appropriate if dealings between the avant garde machine user and his workers are impersonal, as they would be if each period a different worker was hired on the "best worker spot market." No worker would then want to invest differently than he would working in any other firm. We might say that P^* and π^* are based on a "spot labor market" conjecture.

A thorny alternative is a "contract market" conjecture: An avant-garde machine producer writes a long-term contract with a top worker that gives him tenure with the machine during its *entire* avant garde phase, and allows him to either

(a) train harder and take advantage of the capital−skill complementarity or

(b) raise u_t, stop investing, and supply more effective skill for a while.

If the contract offers the worker the same lifetime wealth as he gets from wages paid by ordinary firms, this course of action defines a machine value $P^0(k)$. Since the policy $u_t = u$ is feasible for the tenured worker, certainly $P^0(k) \geq P^*(k)$ for $k \geq ux$. Nevertheless, one might be able to show[13] that $P^0(k) \leq P(k)$, and that claims (a) and (b) of this Appendix are therefore true even if we replace P^* with P^0. That is, even if long-term contracts are allowed, it does not pay a machine producer to move ahead of the pack. Therefore the equilibrium might well survive even the contract market conjecture.

[13] The procedure I used was to maximize [over $k \geq ux$ and over (u_t), with the initial condition $h(0) = 1$] the discounted output of the avant garde match plus the discounted price of the machine plus the worker's discounted wealth as of the time the avant garde era ends. Apparently the maximum of this problem takes place at $k = ux$, so that the duration of the avant garde era is zero.

Proof of Claim 2. Write $\beta(v)$ as $\gamma_0 - \gamma_1 v^{-b}$, so that (31) reads

$$\gamma_0 = \frac{1}{\sigma} v^{1/\sigma - 1} + \gamma_1 v^{-b}. \tag{A4}$$

Since $\xi(v)$ starts out above P and eventually also ends up above it, if the claim is false, there are at least three solutions to (31), two of which occur at values other than $v = 1$. Call these solutions v_1 and v_2. Since (A4) also holds at $v = 1$, we have $1/\sigma + \gamma_1 = (1/\sigma) v_i^{1/\sigma - 1} + \gamma_1 v_i^{-b}$, $i = 1, 2$, or $(1/\sigma)(1 - v_i^{1/\sigma - 1}) + \gamma_1(1 - v_i^{-b}) = 0$, $i = 1, 2$, or

$$\frac{1 - v_i^{\sigma - 1}}{1 - v_i^{-b}} = -\sigma\gamma_1, \qquad i = 1, 2. \tag{A5}$$

But the function $(1 - v^{\sigma - 1})/(1 - v^{-b})$ is strictly decreasing in v, while the right-hand side is a constant. So (A5) cannot hold at both v_1 and v_2, and therefore the claim is true. Q.E.D.

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