1 Consumption and saving under uncertainty

1.1 Modelling uncertainty

As in the deterministic case, we keep assuming that agents live for two periods. The novelty here is that their earnings in the second period are uncertain. Uncertainty is defined in terms of a random variable $s \in S = \{s_1, s_1, ..., s_N\}$ which can take a finite number $N$ of values. The random variable $s$ will determine earnings: $y_2(s)$ is the earnings realization associated to state $s$ in period $t = 2$. If we let $\pi(s)$ denote the probability of state $s$ occurring, it must be that

$$\sum_{s \in S} \pi(s) = 1,$$

i.e., we sum over all the possible realizations we exhaust the probability space.

1.2 Household problem

Consider a two-period endowment economy. The representative household faces the above type of uncertainty about his period-two income $y_2$. In particular, suppose that the income realizations are ordered such that $y_2(s_N) > y_2(s_{N-1}) > ... > y_2(s_1)$. For example, state $s_N$ corresponds to a really large expansion where earnings are very high and state $s_1$ to a really severe recession where the individual becomes unemployed and his earnings fall substantially relative to period 1.

In period $t = 1$, the household can save for the next period through an asset which pays a gross interest rate $R = 1 + r$ for sure next period, independently of $s$, like a Treasury Bond (abstracting from the inflation risk).

Preferences for this household can be written as

$$u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s))$$

where the second component is expected utility in period $t = 2$.

Budget constraints for periods $t = 1, 2$ are, respectively:

$$c_1 + a_2 = y_1,$$
$$c_2(s) = y_2(s) + Ra_2,$$ with probability $\pi(s)$
so we have $N$ different budget constraints in period 2 corresponding to the $N$ states. Which one of the two budget constraints for date $t = 2$ will arise depends on the realization of the state $s$ at time $t = 2$. It is useful to simplify this problem by substituting out savings $a_2$ using the state-contingent period-two budget constraints into the period-one budget constraint:

$$c_1 + \frac{c_2(s)}{R} \leq y_1 + \frac{y_2(s)}{R}. \quad (1)$$

The constrained maximization problem of the household can still be solved with the usual Lagrangian techniques:

$$L(c_1, c_2, \lambda(s)) = u(c_1) + \sum_{s \in S} \pi(s) \left[ \beta u(c_2(s)) + \lambda(s) \left( y_1 + \frac{y_2(s)}{R} - c_1 - \frac{c_2(s)}{R} \right) \right],$$

The solution of this problem is

$$FOC(c_1) : \frac{\partial L}{\partial c_1} = 0 \rightarrow u_c(c_1) = \sum_{s \in S} \pi(s) \lambda(s) = E[\lambda(s)]$$

$$FOC(c_2(s)) : \frac{\partial L}{\partial c_2} = 0 \rightarrow \pi(s) \beta u_c(c_2(s)) = \pi(s) \lambda(s) \frac{1}{R}. $$

Adding up all the $N$ the FOC’s with respect to $c_2(s)$, we obtain

$$\beta E[u_c(c_2(s))] = \frac{1}{R} E[\lambda(s)].$$

Note that the return on bonds $R$ is outside the expectation because it is certain. Combing the FOC’s, we obtain the stochastic version of the Euler equation

$$u_c(c_1) = \beta RE[u_c(c_2(s))] \quad (2)$$

which can restated as

$$\frac{u_c(c_1)}{\beta E[u_c(c_2(s))]} = R \rightarrow \frac{MUC_1}{E[MUC_2]} = R$$

where $MUC_t$ is the marginal utility of consumption at $t$.

### 1.3 A multi-period problem with quadratic utility

For the same reasons we discussed in the deterministic case, we can generalize this derivation to a multi-period model with $t = 1, 2, 3, \ldots$ and the Euler equation determining the optimal allocation of consumption across any two periods is

$$u_c(c_t) = \beta RE_t[u_c(c_{t+1})]$$
Let’s now make two assumptions: (i) no intertemporal saving motive, i.e., $\beta R = 1$ and
(2) assume that utility is quadratic:

$$u(c_t) = b_1 c_t - (1/2) b_2 c_t^2,$$

where $b_1 > 0$, $b_2 > 0$ and $c < b_1/b_2$ to guarantee that the function is strictly increasing.

The marginal utility of consumption with quadratic utility is

$$u_c(c_t) = b_1 - b_2 c_t.$$

Using this expression for marginal utility into the Euler equation, we arrive at

$$b_1 - b_2 c_t = E_t [b_1 - b_2 c_{t+1}]$$

$$c_t = E_t [c_{t+1}]. \tag{3}$$

Equation (3) states that optimal individual consumption is a *martingale* (or a “random walk”). A *martingale* is a stochastic process (i.e., a sequence of random variables) such that the conditional expectation at date $t$ of the value of the random variable at some future date $t + j$, is equal to the realization at date $t$.

Indeed, jointly using the Euler equation and the law of iterated expectations, we obtain:

$$c_t = E_t [c_{t+1}] = E_t [E_{t+1} [c_{t+2}]] = E_t [E_{t+1} [E_{t+2} [c_{t+3}]]] = \ldots = E_t [\ldots [E_{t+j-1} [c_{t+j}] \ldots]$$

$$= E_t [c_{t+j}]$$

where the second raw holds by the law of iterated expectations. Therefore, the random walk property of consumption implies that

$$E_t [c_{t+j}] = c_t \text{ for every } j > 0. \tag{4}$$

An important implication of this theory of consumption is the following. Consider the change in consumption between $t$ and $t - 1$

$$\Delta c_t = c_t - c_{t-1} = c_t - E_{t-1} [c_{t}]$$

which means that the change in consumption between $t - 1$ and $t$ can only be affected by news accruing to the individual over that period, because all the past information (i.e., events that occurred before time $t - 1$) is already incorporated in $E_{t-1} (c_t)$.
One can show that the various history-dependent budget constraints in the stochastic case imply:

\[ \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j E_t c_{t+j} = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j E_t y_{t+j}. \]  

This expression is the equivalent of the deterministic case with the addition of the conditional expectations.

Using the property (4) of optimal consumption into (5) yields

\[ c_t \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j E_t y_{t+j} \]

\[ c_t \left( \frac{1+r}{r} \right) = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j E_t y_{t+j} \]

\[ c_t = \frac{r}{1+r} \cdot \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j E_t y_{t+j} \]

\[ H_t = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j E_t y_{t+j} \]

where

is human wealth, which equals to the expected discounted value of future earnings since earnings are uncertain. Equation (6) still states that consumption is equal to permanent income. Permanent income is now defined as the flow value, i.e. a fraction \( \frac{r}{(1+r)} \), of human wealth \( H_t \).

**Consumption dynamics**— The change in consumption between date \( t-1 \) and \( t \) equals

\[ \Delta c_t = c_t - c_{t-1} = c_t - E_{t-1} c_t = \frac{r}{1+r} (H_t - E_{t-1} H_t), \]

Therefore \( \left[ r / (1+r) \right] (H_t - E_{t-1} H_t) \) is the surprise, or the innovation, (i.e., the unexpected change) in permanent income, at time \( t \). Note that:

\[ H_t - E_{t-1} H_t = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j \left[ E_t (y_{t+j}) - E_{t-1} (E_t y_{t+j}) \right], \]

\[ = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j (E_t - E_{t-1}) y_{t+j}, \]

where in the second line we have used the law of iterated expectations \( E_{t-1} (E_t y_{t+j}) = E_{t-1} y_{t+j} \).
Combining (8) and the expression (7) for the change in consumption we arrive at

\[ \Delta c_t = \frac{r}{1 + r} \sum_{j=0}^{\infty} \left( \frac{1}{1 + r} \right)^j (E_t - E_{t-1}) y_{t+j}. \]  (9)

This equation implies another useful result: under the PIH, the change in consumption between \( t - 1 \) and \( t \) is proportional to the revision in expected earnings due to the new information (the “news”) accruing in that same time interval.

This suggests an empirical test of this hypothesis. Suppose we estimate the linear model

\[ \Delta c_t = \alpha_0 + \alpha_1 y_t + \alpha_2 y_{t-1} + \alpha_3 y_{t-2} + \ldots \]

then, according to the model, only \( \alpha_1 \) should be positive and significant, because \( y_t \) will bring some news. All the other coefficients should be zero because consumption does not respond to past information on income.

### 1.4 Risk Aversion

We introduce here the notion of risk aversion, which is key in these economies with uncertainty. Consider an individual with consumption \( c \) who faces a bet that pays a random amount \( z \) with \( E(z) = 0 \) and \( \text{var}(z) = v \). What is the premium \( \pi \) that the individual would be willing to pay to avoid this bet? The premium \( \pi \) solves

\[ u(c - \pi) = E[u(c + z)]. \]

Take a first-order Taylor expansion on the left-hand side around \( \pi^* = 0 \) and a second-order Taylor expansion on the right-hand side around \( z^* = 0 \).

\[ u(c - \pi) \approx u(c - \pi^*) - u'(c - \pi^*)(\pi - \pi^*) = u(c) - u'(c)\pi \]

\[ E[u(c + z)] \approx E \left[ u(c + z^*) + u'(c + z^*) (z - z^*) + \frac{1}{2} u''(c - z^*) (z - z^*)^2 \right] \]

\[ = E \left[ u(c) + u'(c) z + \frac{1}{2} u''(c) z^2 \right] \]

\[ = u(c) + \frac{1}{2} u''(c) E(z^2) = u(c) + \frac{1}{2} u''(c) v \]

Equating both sides, we obtain

\[ \pi = -\frac{u''(c)}{u'(c)} \cdot \frac{v}{2} \]
The term \( \frac{u''(c)}{u'(c)} \) is the coefficient of absolute risk-aversion \( \rho^{\text{abs}} \). The coefficient of relative risk aversion \( \rho^{\text{rel}} \) is
\[
\rho^{\text{rel}} = -\frac{u''(c)}{u'(c)} \]
and measures the same type of aversion to risk, but where the bet is expressed as a percentage of consumption. Extend the above derivation to the case where the premium \( \pi \) solves
\[
u(c(1-\pi)) = E[u(c(1+z))].
\]

### 1.5 Prudence and precautionary saving

We want to ask the question: what happens to saving when uncertainty about future income goes up? Consider the simple two-period consumption-saving problem
\[
\max_{\{c_0,c_1,a_1\}} u(c_0) + \beta E[u(c_1)]
\]
subject to
\[
c_0 + a_1 = y_0
\]
\[
c_1 = Ra_1 + y_1
\]
where we are using the notation \( R \equiv 1+r \). Assume that current income \( y_0 \) is given while income next period \( y_1 \) is stochastic. Define
\[
y_1 = \bar{y} + \varepsilon_1,
\]
where \( \bar{y} \) is the mean and \( \varepsilon_1 \) is the stochastic component with \( E(\varepsilon_1) = 0 \) and \( \text{var}(\varepsilon_1) = \sigma_\varepsilon \).

If we retain the assumption \( \beta R = 1 \) to simplify the algebra, the Euler equation gives
\[
u'(y_0 - a_1) = E[u'(Ra_1 + y_1)],
\]
which is one equation in one unknown, \( a_1 \). The LHS is increasing in \( a_1 \) since \( u'' < 0 \), and the RHS is decreasing for the same reason, hence \( a_1^* \) is uniquely determined.

**FIGURE HERE**

Note that current consumption \( c_0 \) is determined by the period-zero budget constraint \( c_0 = y_0 - a_1 \) hence a rise in savings \( a_1 \) leads to a fall in current consumption.
What happens to optimal consumption at $t = 0$ if the uncertainty over income next period $y_1$ rises, i.e., as future income becomes more risky? Consider a mean-preserving spread of $y_1$. The Euler equation becomes

$$u'(y_0 - a_1) = E[u'(Ra_1 + \bar{y} + \varepsilon_1)],$$

which shows that if $u'$ is convex then, by Jensen’s inequality, a mean-preserving spread of $\varepsilon_1$ will increase the value of the RHS which shifts upward, inducing a rise in $a_1^*$ and a fall in $c_0^*$.

To understand how we used Jensen’s inequality, recall that this inequality states that, given a random variable $y$:

$$\text{if } f \text{ is a convex function, then } E[f(y)] > f(E[y]),$$

and the opposite inequality holds if $f$ is concave. Start from a situation where $\text{var} (\varepsilon_t) = 0$ and $y_1 = \bar{y}$. The right hand side of the Euler equation in this case is

$$E[u'(Ra_1 + \bar{y})] = u'(Ra_1 + \bar{y}),$$

since all is deterministic. Now add some uncertainty, so $\text{var} (\varepsilon_t) > 0$. The new RHS of the Euler equation becomes

$$E[u'(Ra_1 + y_1)] > u'(E[Ra_1 + y_1]) = u'(E[Ra_1 + \bar{y} + \varepsilon_1]) = u'(Ra_1 + \bar{y})$$

where the first inequality follows from Jensen’s inequality and the convexity of $u'$. The above equation shows that, when we add uncertainty, the RHS of the Euler equation increases and savings $a_1^*$ go up (so $c_0^*$ falls).

The convexity of the marginal utility (or $u'' > 0$) is called “prudence” and is a property of preferences, like risk aversion: risk-aversion refers to the curvature of the utility function, whereas prudence refers to the curvature of the marginal utility function. If the marginal utility is convex ($u'' > 0$), then the individual is “prudent” and a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption. Prudence induces saving in order to take precaution against possible future negative realizations of the income shocks. Savings induced by prudence are called “precautionary savings.”
1.6 Overview of saving motives

This is a good time to make a short remark about “saving motives”, i.e. reasons why households save:

1. The saving motive associated to $\beta R > 1$ in the deterministic model without uncertainty, which pushes the individual to postpone consumption because of high patience and/or high returns to savings is called *intertemporal motive*.

2. The saving motive pushing households to smooth consumption through income shocks is called *smoothing motive*.

3. The saving motive which pushes the individual accumulate assets as a precaution against future income uncertainty is called *precautionary motive*.

4. We add that in a life-cycle model where the individual faces a retirement period, during the working stage of the life-cycle the individual would have a *life-cycle motive* for saving associated to the desire of smoothing consumption throughout her life, across the working-life where she earns income and retirement where she does not.

5. *Bequest motive* that induces altruistic households to leave part of their assets as bequest to their offsprings upon death.

**FIGURE HERE**