1 The Permanent Income Hypothesis

1.1 A two-period model

Consider a two-period model where households choose consumption \((c_1, c_2)\) to solve

\[
\max_{\{c_1, c_2\}} \log c_1 + \beta \log c_2 \quad \text{s.t.} \quad c_1 + \frac{c_2}{1 + r} = \left(1 + \frac{1}{1 + r}\right) y^p + y^T
\]

where \(\beta\) is the discount factor, \(r\) the interest rate. The term \(y^p\) is the permanent component of income, i.e., income that accrues to the individual in every period, whereas \(y^T\) is the transitory component, i.e. the component that is earned only in period \(t = 1\) but not in period \(t = 2\). Think of \(y^p\) as a base salary and \(y^T\) as commissions or as a lottery win. Clearly, \(y^T\) could be negative, e.g. a capital loss on a financial investment, or a temporary shift from full-time work to part-time work.

Let’s solve this problem.

\[
L(c_1, c_2, \lambda) = \log c_1 + \beta \log c_2 + \lambda \left[\left(1 + \frac{1}{1 + r}\right) y^p + y^T - c_1 - \frac{c_2}{1 + r}\right]
\]

\[
\begin{align*}
FOC(c_1) & : \frac{\partial L}{\partial c_1} = 0 \rightarrow \frac{1}{c_1} = \lambda \\
FOC(c_2) & : \frac{\partial L}{\partial c_2} = 0 \rightarrow \frac{\beta}{c_2} = \frac{\lambda}{1 + r}
\end{align*}
\]

which yields the Euler equation

\[
\frac{c_2}{c_1} = \beta (1 + r).
\]

And using the Euler equation into the lifetime budget constraint, we arrive at the optimal consumption \(c^*_1\) (and similarly we arrive at \(c^*_2\))

\[
\begin{align*}
\frac{c^*_1 \beta (1 + r)}{1 + r} & = \left(1 + \frac{1}{1 + r}\right) y^p + y^T \\
c^*_1 (1 + \beta) & = \left(1 + \frac{1}{1 + r}\right) y^p + y^T \\
c^*_1 & = \frac{1}{1 + \beta} y^p + \frac{1}{1 + \beta} y^T.
\end{align*}
\]

Now, let’s make the simplifying assumption that \(\beta (1 + r) = 1\). From the Euler equation, this means that the offsetting effects of patience and interest rate on saving exactly cancel
out and the household optimally chooses to perfectly smooth consumption over the two periods and \( c_1 = c_2 = c \). Note that \( y^T \) could be very large, which means that the income profile could be very steep, and yet consumption is perfectly smoothed. The individual will borrow enough in the first period against future income to equalize consumption across dates \( t = 1 \) and \( t = 2 \).

Imposing the parametric restriction \( \beta (1 + r) = 1 \) or \( \beta = 1 / (1 + r) \), we arrive at

\[
c^* = y^P + \frac{1 + r}{2 + r} y^T.
\]

This equation explains how consumption reacts differently to permanent and transitory changes in income. The marginal propensity to consume \((MPC)\) out of permanent income is one and the MPC out of transitory income is \( (1 + r) / (2 + r) \).

Suppose \( y^P \) and \( y^T \) both change by an amount \( \Delta Y \), then consumption will change, respectively, by

\[
\Delta c^* = \Delta Y
\]

if the change in income is permanent and

\[
\Delta c^* = \frac{1 + r}{2 + r} \Delta Y \approx \frac{\Delta Y}{2}
\]

if the change in income is transitory where the approximation sign holds for \( r \approx 0 \).

In other words, consumption responds a lot more to permanent changes in income compared to transitory changes in income. This result is the heart of the so called “permanent income hypothesis” formulated by the Nobel Prize winner Milton Friedman: the MPC out of permanent income changes is one and the \( MPC \) out of transitory income changes is always smaller than one.

### 1.2 The infinite horizon case

The infinite horizon assumption is made in macroeconomics for two main reasons. First, it is a more convenient assumption than the one of a finite horizon (of length \( N \)), since the problem is easier to solve. Second, the infinite horizon problem corresponds to an individual that has perfect altruism towards her offsprings. To see this, consider instead the case of an agent who lives for \( N \) periods from \( t = 0, ..., N - 1 \). The discounted stream of utility coming from consumption during her own \( N \) period life is:

\[
U = \sum_{t=0}^{N-1} \beta^t u(c_t).
\]
Now, suppose, she has an offspring born at \( N \) who also lives for \( N \) periods and discounts her own utility at the same rate \( \beta \). Denote the utility of the offspring as

\[
U^{off} = \sum_{t=T}^{2N-1} \beta^{t-N} u(c_t).
\]

Suppose the parent discounts the utility of the offspring \( U^{off} \) at rate \( (\delta \beta)^N \), i.e. \( \delta \in [0, 1] \) is the relative weight she puts on the happiness of her offspring relative to her own. If \( \delta = 0 \), she does not care for the offspring (no altruism), while if \( \delta = 1 \), she cares for the offspring as much as she cares for herself, i.e., she is perfectly altruistic. In general

\[
U^{altr} = U + (\delta \beta)^N U^{off}
\]

If \( \delta = 1 \), we have

\[
U^{altr} = \sum_{t=0}^{N-1} \beta^t u(c_t) + \beta^N \sum_{t=N}^{2N-1} \beta^{t-N} u(c_t) = \sum_{t=0}^{N-1} \beta^t u(c_t) + \sum_{t=N}^{2N-1} \beta^t u(c_t) = \sum_{t=0}^{2N-1} \beta^t u(c_t)
\]

which means that, with perfect altruism, it is as if the individual lives for \( 2N \) periods. If we keep iterating, assuming that every generation is perfectly altruistic and cares about offsprings with weight \( \delta = 1 \), we arrive at

\[
U^{altr} = \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

i.e., it is as if the individual has an infinite planning horizon. This combined utility of the individual and all her descendants is called dynastic utility.

Consider now an agent with an infinite planning horizon who faces an infinite sequence of earnings \( \{y_t\}_{t=0}^{\infty} \) which is perfectly known already at time \( t = 0 \). And it is equal to

\[
y_t = \begin{cases} 
y^P + y^T & \text{for } t = 0 \\
y^P & \text{for } t > 0
\end{cases}
\]

Thus, again in period \( t = 0 \), there is an income component \( y^T \) that the agent only receives at \( t = 0 \), i.e., it is a transitory income change.

This agent solves the following consumption/saving problem:

\[
\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \\
s.t. \\
\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} c_t = y^T + \left(\frac{1+r}{r}\right) y^P
\]
How did we obtain that budget constraint? Consider the two period model budget constraint
\[
c_0 + \frac{c_1}{1 + r} = y^T + \left(1 + \frac{1}{1 + r}\right)y^P
\]
and add one extra period
\[
c_0 + \frac{c_1}{1 + r} + \frac{c_2}{(1 + r)^2} = y^T + \left[1 + \frac{1}{1 + r} + \frac{1}{(1 + r)^2}\right]y^P
\]
and keep adding periods. It is easy that, as \(t\) goes to infinity, the left hand side becomes
\[
\sum_{t=0}^{\infty} \frac{1}{(1 + r)^t}c_t.
\]
(1)
Consider the right hand side. The infinite geometric series
\[
1 + \frac{1}{1 + r} + \frac{1}{(1 + r)^2} + \ldots + \frac{1}{(1 + r)^t} + \ldots = \frac{1}{1 - \frac{1}{1 + r}} = \frac{1 + r}{r}.
\]
and therefore the lifetime budget constraint can be written as
\[
\sum_{t=0}^{\infty} \frac{1}{(1 + r)^t}c_t = y^T + \frac{1 + r}{r}y^P.
\]
To derive the Euler equation in the infinite horizon model, write the Lagrangean:
\[
L \left(\{c_t\}, \lambda\right) = \sum_{t=0}^{\infty} \beta^t \log(c_t) + \lambda \left[y^T + \left(\frac{1 + r}{r}\right)y^P - \sum_{t=0}^{\infty} \frac{1}{(1 + r)^t}c_t\right]
\]
The FOC’s with respect to \(c_0, c_1, c_2\) are:
\[
\frac{\partial L}{\partial c_0} = 0 \rightarrow \frac{1}{c_0} = \lambda
\]
\[
\frac{\partial L}{\partial c_1} = 0 \rightarrow \beta \frac{1}{c_1} = \frac{\lambda}{1 + r}
\]
\[
\frac{\partial L}{\partial c_2} = 0 \rightarrow \beta \frac{1}{c_2} = \frac{\lambda}{(1 + r)^2}
\]
Thus, combining the first two equations we arrive at:
\[
\frac{c_1}{c_0} = \beta (1 + r)
\]
and combining the second and third FOCs:
\[
\frac{c_2}{c_1} = \beta (1 + r)
\]
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In general, the Euler equation determining the optimal consumption path across two consecutive periods is given, at every \( t \), by:

\[
\frac{c_{t+1}}{c_t} = \beta (1 + r).
\]

Imposing \( \beta (1 + r) = 1 \) we arrive at the same result we had in the two period model: consumption \( c_t \) is constant and equal to \( c^* \) in both periods. Therefore, we can pull \( c^* \) off the infinite sum on the left hand side of the budget constraint and obtain:

\[
c^* \frac{1 + r}{r} = y^T + y^P \frac{1 + r}{r}.
\]

This solution shows the permanent income hypothesis at work in an even more extreme way. Permanent changes in income are reflected “one for one” into consumption, whereas transitory changes are reflected proportionally to the factor \( r/(1 + r) \) which for \( r \approx 0 \) is also close to zero. Hence, in the infinite horizon model, consumption, basically, does not respond to transitory changes in income.

### 1.3 Generalization to \( N \) period economy

In a finite horizon model with \( N \) periods, it is easy to see that the lifetime budget constraint reads

\[
\sum_{t=0}^{N} \frac{1}{(1 + r)^t} c_t = y^T + y^P \sum_{t=0}^{N} \frac{1}{(1 + r)^t}.
\]

The Euler equation is always the same. Hence, if \( \beta (1 + r) = 1 \), we always have \( c_t = c_{t+1} = c^* \). And using this result in the above equation, we have:

\[
c^* \sum_{t=0}^{N} \frac{1}{(1 + r)^t} = y^T + y^P \sum_{t=0}^{N} \frac{1}{(1 + r)^t}.
\]

\[
c^* = \frac{1}{N} y^T + y^P
\]

\[
= \frac{1}{1 - \left( \frac{1}{1 + r} \right)^N} y^T + y^P
\]

\[
= \frac{1}{1 - \left( \frac{1}{1 + r} \right)^N} \left( \frac{r}{1 + r} \right) y^T + y^P
\]
Consider the case $N = 2$.

$$1 - \left( \frac{1}{1 + r} \right)^2 = 1 - \frac{1}{1 + r^2 + 2r} = \frac{(2 + r) r}{(1 + r)^2}$$

and using this result in the lifetime budget constraint:

$$c^* = \frac{1}{(2 + r)r} \left( \frac{r}{1 + r} \right) y^T + y^P = \frac{(1 + r)^2}{(2 + r)r} \left( \frac{r}{1 + r} \right) y^T + y^P$$

$$= \frac{1 + r}{2 + r} y^T + y^P$$

as we showed earlier.

Consider the case $N = 3$.

$$1 - \left( \frac{1}{1 + r} \right)^3 = 1 - \frac{1}{1 + 3r^2 + 3r + r^3} = \frac{(3 + 3r + r^2) r}{(1 + r)^3}$$

and using this result in the lifetime budget constraint:

$$c^* = \frac{1}{(3 + 3r + r^2)r} \left( \frac{r}{1 + r} \right) y^T + y^P = \frac{(1 + r)^2}{3 + 3r + r^2} y^T + y^P.$$ 

Let’s take stock. Assume that $r \simeq 0$. From our derivations, when $N = 2$ the MPC out of transitory income is $1/2$. When $N = 3$, it is easy to see that it becomes $1/3$. In the infinite horizon model, it is $1/\infty = 0$. So, it is easy to generalize this as a rule of thumb. In an $N$ period model, a change in income that takes place in only one of the $N$ periods affects consumption proportionately to $1/N$. If the change occurs for $M < N$ periods, the transmission coefficient to consumption is approximately $M/N$ and if $M = N$ (and hence the change in income is permanent), this transmission coefficient is one. Independently of the horizon, the MPC out of $y^P$ is always one.

The following table summarizes these results:

<table>
<thead>
<tr>
<th>$N$</th>
<th>MPC out of $y^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{2 + r}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{(1 + r)^2}{3 + 3r + r^2}$</td>
</tr>
<tr>
<td>$N$</td>
<td>$\frac{1}{1 - \left( \frac{1}{1 + r} \right)^N} \left( \frac{r}{1 + r} \right)$</td>
</tr>
<tr>
<td>$N \to \infty$</td>
<td>$\frac{r}{1 + r}$</td>
</tr>
</tbody>
</table>