1 Problem 1

Consider a two-sector version of the neoclassical growth model, where one sector produces the consumption good $c_t$ with technology

$$c_t = z_c f (k_c^t, n_c^t)$$

and the other sector produces the investment good $i_t$ with technology

$$i_t = z_i f (k_i^t, n_i^t).$$

Assume $f$ is CRS, strictly concave and increasing in both argument. Note that $f$ is the same in both sectors.

Aggregate capital is $k_t = k_c^t + k_i^t$ and accumulates according to the law of motion $k_{t+1} = (1 - \delta) k_t + i_t$. Capital used in the consumption sector is $k_c^t$ and capital used in the investment sector is $k_i^t$. Aggregate labor input is $n_t = n_c^t + n_i^t$, where $n_c^t$ is labor supplied in the consumption sector and $n_i^t$ is labor supplied in the investment sector.

The representative household has period utility $u(c_t, 1 - n_t)$ and preferences are time-separable with discount factor $\beta$. All markets are competitive and both labor and capital are free to move across sectors. Let $p_c^t$ be the price of the consumption good and $p_i^t$ be the price of the investment good.

1) Show that this economy aggregates into a one-sector growth model, i.e. the production structure can be summarized by an aggregate production technology $\tilde{z} f (k_t, n_t)$. What is the interpretation of the ratio $(p_c^t/p_i^t)$ in equilibrium?

- We want to collapse the two sectors

$$c = \tilde{z} F(K_c, N_c)$$
$$i = \tilde{z} F(K_i, N_i)$$

into a single sector with aggregate production function $y = \tilde{z} F(K, N)$, where $K = K_c + K_i$, $N = N_c + N_i$ and $\tilde{z}$ does not depend on the distribution of capital and labor across sectors.
• For simplicity, let us define the following objects

\[ k = \frac{K}{N} \]

\[ f(k) \equiv F(k, 1) \]

• Note that since \( F \) (the common production function) is CRS, it is homogeneous of degree one, hence we have the following results:

\[ F(K, N) = NF(K/N, N/N) = NF(k, 1) = Nf(k) \]

\[ F_i(K, N) = F_x(K/N, N/N) = F_i(k, 1) \text{ for } x = K, N \]

where the second result takes advantage of the fact that the partial derivatives of a function that is homogeneous of degree one will be homogeneous of degree zero.

• Assuming that households rent both labor and capital to firms (i.e. households own the capital stock), we can write the problem for a representative firm in each sector as a static one

\[ \max_{K_c, N_c} \left\{ p_c z_c F(K_c, N_c) - K_c r_c - N_c w_c \right\} \]

\[ \max_{K_i, N_i} \left\{ p_i z_i F(K_i, N_i) - K_i r_i - N_i w_i \right\} \]

where \((r_c, w_c)\) and \((r_i, w_i)\) are the (nominal) prices for capital and labor in each sector, respectively. Notice I did not include the depreciation in this problem, hence it will show up in the household one.

• Consider now the representative consumer’s problem:

\[ \max_{\{c_t, K_{c,t+1}, K_{i,t+1}, n_{c,t}, n_{i,t}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \]

subject to

\[ p_{c,t} c_t + p_{i,t} [K_{i,t+1} + K_{c,t+1} - (1 - \delta)(K_{i,t} + K_{c,t})] = w_{c,t} n_{c,t} + w_{i,t} n_{i,t} + r_{c,t} K_{c,t} + r_{i,t} K_{i,t} \]

and \(n_t = n_{c,t} + n_{i,t}\). Setting up the Lagrangean and taking the FOC’s with respect to the decision variables, where \(\lambda_t\) is the multiplier on the budget constraint, we obtain

\[(c_t) : u_1(c_t, 1 - n_t) = \lambda_t p_{c,t}\]
\[(n_{c,t}) : u_2(c_t, 1 - n_t) = \lambda_t w_{c,t}\]
\[(n_{i,t}) : u_2(c_t, 1 - n_t) = \lambda_t w_{i,t}\]
\[(K_{c,t+1}) : \lambda_t p_{i,t} = \beta \lambda_{t+1} [r_{c,t+1} + p_{i,t+1} (1 - \delta)]\]
\[(K_{i,t+1}) : \lambda_t p_{i,t} = \beta \lambda_{t+1} [r_{i,t+1} + p_{i,t+1} (1 - \delta)]\]

Clearly, the second and third FOC imply that \(w_{c,t} = w_{i,t}\), while the fourth and fifth imply \(r_{c,t} = r_{i,t}\), \(\forall t\). Why is this? We know that labor and capital are perfectly mobile across sectors. Suppose, for example, that \(w_{c,t} > w_{i,t}\). Then, the worker would decide to only work in the consumption sector. Since production is concave, this would raise (real) wages in the investment sector. The same logic goes through for the interest rate.

• We can then check that the firm’s FOC are

\[(K_c) : p_c z_c F_K(K_c, N_c) = r\]
\[(N_c) : p_c z_c F_N(K_c, N_c) = w\]
\[(K_i) : p_i z_i F_K(K_i, N_i) = r\]
\[(N_i) : p_i z_i F_N(K_i, N_i) = w\]
For each sector, divide the FOC for $K$ by the FOC for $N$ to obtain:

$$\frac{F_K(K_c, N_c)}{F_N(K_c, N_c)} = \frac{r}{w} = \frac{F_K(K_i, N_i)}{F_N(K_i, N_i)}$$

However, we know that $F_K$ and $F_N$ are homogeneous of degree zero. Therefore, we can use the initially defined notation to write

$$\frac{f_K(k_c)}{f_N(k_c)} = \frac{r}{w} = \frac{f_K(k_i)}{f_N(k_i)}$$

Notice now that, for each sector, $f_K(k)/f_N(k)$ is a strictly decreasing function of $k$. Why is this? Concavity immediately tells us that $f_K$ is decreasing in $k$. Furthermore, assuming that $F$ is continuously differentiable, we can apply the Euler Theorem and then differentiate with respect to capital, so as to obtain

$$0 = KF_{KK}(K, N) + NF_{K}(K, N)$$

Since $F_{KK} < 0$ (since it is concave), we must have that $F_N > 0$. Thus $f_N$ is increasing in $k$, making the ratio strictly decreasing.

This means that we can invert the following equality

$$\frac{f_K(k_c)}{f_N(k_c)} = \frac{r}{w} \iff k_c = g\left(\frac{w}{r}\right)$$

and, likewise for the investment sector

$$k_i = g\left(\frac{w}{r}\right)$$

Since $k_c, k_i$ depend only on aggregate variables that are not sector specific, and technology is the same across sectors, we conclude that $k_c = k_i = k$. That is, each sector uses the same capital-labor ratio. This also means that $F_{N,c} = F_{N,i}$ and $F_{K,c} = F_{K,i}$, that is, ”pure marginal productivities” (not counting with the TFP term) will be the same across sectors.

From the firm’s FOC, we can furthermore conclude that

$$z_c F_N(K_c, N_c) = \frac{w}{p_c} = \frac{p_i}{p_c} z_i F_N(K_i, N_i)$$

and

$$z_c F_K(K_c, N_c) = \frac{r}{p_c} = \frac{p_i}{p_c} z_i F_K(K_i, N_i)$$

From where we obtain that

$$\frac{z_c}{z_i} = \frac{p_i}{p_c}$$

That is, the relative price of investment and consumption will depend only on the relation between the TFP in each sector. If the consumption sector is more productive, for example, then the relative price of investment increases: this makes sense, as investment is now more expensive to produce (in the sense that allocating production factors to the investment sector is not as productive as doing it for consumption).

Now, we must think of how we should define the aggregate good. I choose to define it in terms of consumption. That is, I will define $y_t$ such that its price is the same as the price of consumption. This means that

$$p_{c,t} y_t = p_{c,t} c_t + p_{i,t} i_t$$

or

$$y = z_c F(K_c, N_c) + \frac{p_i}{p_c} z_i F(K_i, N_i)$$

$$= z_c N_c F(k_c) + \frac{z_c}{z_i} z_i N_i f(k_i)$$

$$= z_c (N_c + N_i) f(k)$$

$$= z_c F(K, N)$$
where the second line uses the fact that $p_i/p_c = z_c/z_i$ and the third uses $k_i = k_c = k$. This expression tells us that aggregate production is equal to the total production of consumption goods and consumption-equivalent investment goods, i.e., investment production adjusted by the real price of investment (relative to consumption).

- It is easy to check that the original problem corresponds to the following one-sector problem

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty u(C_t, 1 - N_t)$$

subject to

$$C_t + \frac{z_{c,t}}{z_{i,t}} [K_{t+1} - (1 - \delta)K_t] = z_{c,t}F(K_t, N_t)$$

where I use the first welfare theorem applies (markets are complete since there is no uncertainty).

- For more on this, see Greenwood, Hercowitz and Krusell (AER, 1997).
2 Problem 2

Consider a version of the neoclassical growth model where the economy is inhabited by two types of agents $i = 1, 2$ with measure $\mu_i$ where $\mu_1 + \mu_2 = 1$. Agents of type $i$ solve

$$\max_{\{c_{it}, n_{it}\}} \sum_{t=0}^{\infty} \beta^t \left( \frac{c_{it}^\alpha (1 - h_{it})^{1-\alpha}}{1 - \gamma} \right)^{1-\gamma}$$

s.t.

$$c_{it} + a_{i,t+1} = a_{i,t} [1 + r_t (1 - \tau_t)] + w_t \varepsilon_i h_{it} + T^0,$$

$$a_{i0} \text{ given for } i = 1, 2$$

where $\varepsilon_i$ are efficiency units of labor endowed to type $i$, $\tau_t$ is a capital-income tax rate and $T^0$ are lump-sum transfers. The representative firm produces with CRS technology $F(K_t, N_t)$ where $N_t$ is aggregate labor input in efficiency units. The government budget constraint is balanced in every period, i.e.,

$$T^0 = \tau_t K_t r_t$$

Finally, the aggregate resource constraint is

$$C_t + K_{t+1} = F(K_t, N_t) + (1 - \delta) K_t.$$

2.1 Indeterminate distribution of wealth in the steady state

Show that in steady-state the distribution of wealth is indeterminate.

- In this problem there are two types of individuals of measure $\mu^1, \mu^2$, where $\sum_i \mu^i = 1$.
- The agent’s budget constraint implies that households rent capital and labor to the firms. It also implies that the agent gets income from renting all his assets as capital and therefore I assume that $k_i = a_i$.
- We start with characterizing the households’ solution. At this stage I will not use functional forms. The FOCs are

$$u_c(c_i^t, 1 - h_i^t)\varepsilon w_t = u_l(c_i^t, 1 - h_i^t), \ i = 1, 2$$

(1)

$$u_c(c_i^t, 1 - h_i^t) = \beta [1 + \tau_{t+1} (1 - \tau_{t+1})] u_c(c_{i+1}^t, 1 - h_{i+1}^t), \ i = 1, 2.$$  

(2)

- We also have two budget constraints

$$c_i^t + a_{i,t+1} = [1 + r_t (1 - \tau_t)] a_i^t + h_i^t \varepsilon w_t + T^0, \ i = 1, 2.$$  

(3)

- Solve the firm problem to get prices:

$$F_n(K_t, N_t) = w_t$$

(4)

$$F_k(K_t, N_t) = r_t + \delta$$

(5)

- The government balanced budget is:

$$T^0 = r_t K_t \tau_t$$

(6)

Even though $K_t$ is the capital level of the firm, by market clearing it also is the total capital of the economy. Since all this capital is taxed, it is simpler to state this budget clearing using $K_t$. 

And finally we have three equations for aggregate values:

\[ N_t = \mu_1^1 h_t^1 + \mu_2^2 h_t^2 \]  \hspace{1cm} \text{(7)}
\[ C_t = \mu_1^1 c_t^1 + \mu_2^2 c_t^2 \]  \hspace{1cm} \text{(8)}
\[ K_t = \mu_1^1 k_t^1 + \mu_2^2 k_t^2 \]  \hspace{1cm} \text{(9)}

Since the first three equations are type specific there is a total of 12 equations. There are 12 variables in the system. These are: \( \{ c_i^t, h_i^t, a_i^t, \ldots \} \).

Note that:

1. \( T^0 \) is exogenous and therefore it is not counted as one of the variables.
2. The feasibility constraint is not included in the system of equations because it is implied by the two budget constraints of the agents and the government balanced budget.
3. To verify that all equations are independent, you can log-linearize the system and then use standard methods for linear independence.\(^1\)

The solution to this sequential problem is well defined as the number of variables equals the number of equations. The idea is that in each period (including the first one) the state of the economy \( \{ a_i^t \} \) is known, and based on that the future state \( \{ a_i^{t+1} \} \) can be found.

However, when we are asked to show that in steady-state the distribution of wealth is indeterminate, the idea is that the initial (or last period’s) conditions are unknown.

To get the steady state’s solution drop all time subscripts and note that the only important difference in the system of the 12 equations is that the Euler equation becomes the same one for both types of agents and therefore we lose one equation. Now the number of variables exceeds the number of equations by one and therefore the distribution of wealth in steady state is indeterminate, which means that there is a continuum of steady states.

How do we know that the aggregate variables are well determined?

1. The pricing equation (2) in SS pins down the rental rate and therefore the capital-labor ratio \( K/N \) as well as the wage rate \( w \).
2. The capital stock is given jointly with the total labor by the aggregate version of the FOC for labor choice (1), once we replace the functional form

\[
\alpha(1-h^i)e^i w = (1-\alpha)c^i \\
\sum_i \mu_i (1-h^i) e^i = (1-\alpha) \sum_i \mu_i c^i \\
w\alpha N = (1-\alpha) F(K, N) \\
F_N(K, N) \alpha N = (1-\alpha) F(K, N) \\
F_N(K/N, 1) \alpha N = (1-\alpha) NF(K/N, 1)
\]

which only depends on already determined \( K/N \), i.e. it defines the aggregate level of labor \( N \).

Note however that given some initial conditions, one can solve the sequential problem until convergence to a specific steady-state is reached. Please make sure that you understand the difference between ‘the distribution of wealth in steady state is indeterminate’ and ‘given initial conditions the steady state is well determinate’. This will be clear later, but we can find the aggregate capital and labor, and hence prices. Then, given an initial asset position, we can determine all we need to know about the consumers.

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\(^1\) The equations of a system are dependent if ALL the solutions of one equation are also solutions of the some other equation. Hence, the equations of a system are independent if they do not share ALL solutions.
• Figure 1 shows that the set of steady states is linear, with the slope given by the share of types and 
an intercept indicating the capital stock. Notice that negative values of assets are possible as long as 
consumption is non-negative. Also, changes in the exogenous tax rate will change the capital stock (through 
the capital-labor ratio), and will hence shift the steady-state line up or down, with unchanged slope.²

• What are the key assumptions here?

1. We could have different additive period utility functions, as long as discount factors are equal.
2. We need the time horizon to be infinite. OLG would not give rise to this kind of Euler equation.
3. We need agents to have the same return on savings, i.e. we cannot have incompleteness in the asset 
   markets, transaction costs, or non-proportional taxes.
4. We could have uncertainty, as long as we keep market completeness.

Figure 1: Different taxes, different set of steady states.

2.2 A representative agent formulation

Does this economy admit a representative agent formulation?

• For this part of the solution we need to write down the present value budget constraint. I give the derivation 
of the present value budget constraint at the end of the solution for a simpler problem. If you are unfamiliar 
with this derivation, please go over it before you continue with the solution. Then, the present value budget 
constraint for each type

\[
\sum_{s=1}^{\infty} \frac{p_s}{p_t} c_s^i = \sum_{s=1}^{\infty} \frac{p_s}{p_t} \left[ w_s e^i h_s^i + T \right] + a^i_t \left[ 1 + r_t (1 - \tau_t) \right]
\]

(recall, the wages here are real). Now add

\[
\sum_{s=1}^{\infty} \frac{p_s}{p_t} \left[ w_s e^i \cdot 1 \right]
\]

²More on this later, but I believe the figure assumes we are in a particular side of the Laffer curve.
to both sides of (10) and rearrange to get
\[\sum_{s=t}^{\infty} \frac{p_s}{p_t} c_s^i + \sum_{s=t}^{\infty} \frac{p_s}{p_t} w_s c_i^j (1 - h_i^j) = \sum_{s=t}^{\infty} \frac{p_s}{p_t} \left[w_s c_i^j \cdot 1 + T \right] + a_t^i \left[1 + r_t(1 - \tau_t)\right]. \] (11)

- The first term on the LHS is total expenditure on consumption from period \(t\) and on; the second term is total "expenditure" on leisure from period \(t\) and on. The RHS is a composite of total human wealth, transfers and initial (as of period \(t\)) financial wealth. Denote the RHS as \(\omega_t^i\), total wealth in period \(t\).

- Now, making the marginal utilities explicit we have

\[u_c(c,1-h) = \left[c^\alpha(1-h)^{1-\alpha}\right]^{-\gamma} \alpha c^{\alpha-1} (1-h)^{1-\alpha}\]
\[u_l(c,1-h) = \left[c^\alpha(1-h)^{1-\alpha}\right]^{-\gamma} c^\alpha (1-\alpha)(1-h)^{1-\alpha}\]

so (1) becomes

\[\frac{\alpha}{c_t^i} c_t^i w_t = \frac{1 - \alpha}{1 - h_t^i}\] (12)

\[\Rightarrow c_t^i = \frac{\alpha}{1 - \alpha} c_t^i w_t (1 - h_t^i) \] (13)

- From (12) we can see that preferences are intratemporal homothetic, in that the ratio of "expenditure" on leisure and consumption is fixed. Eliminate consumption in (11) by plugging in (13) to get

\[\sum_{s=t}^{\infty} \frac{p_s}{p_t} w_s c_i^j (1 - h_i^j) = (1 - \alpha)\omega_t^i\]

which means that total "expenditure" on leisure is a constant fraction on total wealth. Hence, we have the residual

\[\sum_{s=t}^{\infty} \frac{p_s}{p_t} c_i^j = \alpha\omega_t^i \] (14)

- The next step is to specify future levels of consumption in terms of current consumption. This will enable us to get an explicit solution for \(c_t^i\) and will be key to the aggregation. For this we use the explicit Euler equation. (Note that I am omitting type specific indicators for brevity of notation.)

\[\left[c_t^\alpha(1-h_t)^{1-\alpha}\right]^{-\gamma} \alpha c_t^{\alpha-1} (1-h_t)^{1-\alpha} = \beta [1 + r_t(1 - \tau_t)] \left[c_{t+1}^\alpha(1-h_{t+1})^{1-\alpha}\right]^{-\gamma} \alpha c_{t+1}^{\alpha-1} (1-h_{t+1})^{1-\alpha} \] (15)

- Using the leisure consumption ratio from (13) and \(1 + r_t(1 - \tau_t) = \frac{p_t}{p_{t+1}}\):

\[\left[c_t^\alpha (\frac{1 - \alpha}{\alpha c_t^i w_t})^{1-\alpha}\right]^{-\gamma} \alpha c_t^{\alpha-1} \left(c_t^i \frac{1 - \alpha}{\alpha c_t^i w_t}\right)^{1-\alpha} = \beta \frac{p_t}{p_{t+1}} \left[c_{t+1}^\alpha (\frac{1 - \alpha}{\alpha c_{t+1}^i w_{t+1}})^{1-\alpha}\right]^{-\gamma} \alpha c_{t+1}^{\alpha-1} \left(c_{t+1}^i \frac{1 - \alpha}{\alpha c_{t+1}^i w_{t+1}}\right)^{1-\alpha} \] (16)

\[c_t^\gamma = \frac{\beta}{p_{t+1}} c_{t+1}^\gamma \left(\frac{w_t}{w_{t+1}}\right)^{(1-\gamma)(1-\alpha)} \] (17)
\[c_{t+1} = c_t \left(\beta \frac{p_t}{p_{t+1}}\right)^{\frac{1}{\varphi}} \left(\frac{w_t}{w_{t+1}}\right)^{\varphi} \] (18)

where \(\varphi = (1 - \gamma)(1 - \alpha) / \gamma\).

\textsuperscript{3}In doing this we need to be careful with infinite sums. in general, “\(\sum_{j=0}^{\infty} a_j b_j + \sum_{j=0}^{\infty} a_j c_j = \sum_{j=0}^{\infty} a_j (b_j + c_j)\)” need not hold; but in our case, since \(a_j\) are prices and they are a strictly decreasing sequence, it does.
• Iterating forward gives a similar relation for any future period’s consumption:

$$c_{t+s} = c_t \prod_{j=t}^{s} \left( \frac{\beta p_j}{p_{j+1}} \right)^{\frac{1}{\gamma}} \left( \frac{w_j}{w_{j+1}} \right)^{\phi}$$

• Use this relation in (14) to specify current consumption of agent of type $i$ as a fraction of the agent’s wealth, a fraction which does not depend on any type specific terms:

$$c^i_t = \alpha \omega^i_t B_t$$  \hspace{1cm} (19)

Note that although $B_t$ is a complicated term (which I don’t even write here) the important point is that it does not depend on any individual-specific arguments.

• Now, use (19) in (12) to get

$$(1 - h^i_t) \epsilon^i = (1 - \alpha)\omega^i_t B_t / w_t.$$  

• At this point you should be able to see that aggregating consumption and effective labor supply will not involve and individual-specific arguments. But let us do it nonetheless.

$$C_t = \sum_i \mu^i c^i_t = \sum_i \mu^i \omega^i_t B_t$$

$$= \sum_i \mu^i \alpha \left[ \sum_{s=t}^{\infty} p_s \left[ w_s \epsilon^i \cdot 1 + T \right] + a^i_t \left[ 1 + r_t (1 - \tau_t) \right] \right] B_t$$

$$= \alpha \left[ \sum_{s=t}^{\infty} p_s \left[ \sum_i \mu^i \epsilon^i w_s + \sum_i \mu^i T \right] + \sum_i \mu^i a^i_t \left[ 1 + r_t (1 - \tau_t) \right] \right] B_t$$

$$= \alpha \left[ \sum_{s=t}^{\infty} \frac{p_s}{p_t} (w_s + T) + A_t \left[ 1 + r_t (1 - \tau_t) \right] \right] B_t.$$

• Likewise,

$$1 - N_t = \sum_i \mu^i (1 - h^i_t) c^i_t = \sum_i \mu^i (1 - \alpha) \omega^i_t B_t / w_t$$

$$= (1 - \alpha)(1 - \tau_t) B_t / w_t.$$  

• Define total aggregate wealth as follows

$$\Omega_t = \left[ \sum_{s=t}^{\infty} \frac{p_s}{p_t} (w_s + T) + A_t \left[ 1 + r_t (1 - \tau_t) \right] \right],$$

which allows us to write in compound terms

$$C_t = \alpha \Omega_t B_t$$

$$N_t = 1 - (1 - \alpha)\Omega_t B_t / w_t.$$  

So we can conclude that this economy does admit a representative entity.

• Figure 2 shows what the aggregation result means. The location of the $t+1-t+1$ line is independent of the distribution of the initial capital stock $K_0$, i.e. the point in the line $t-t$. Moreover, although two economies that only differ in the initial distribution of a given amount of total capital will have identical time paths for prices and aggregates, their asset distribution will remain different and, if total capital converges, the asset distributions will converge to different points on the SS line. In other words, there is not enough equalization of incomes and wealth in this model over time.
2.3 Transition between steady states

Suppose that the economy is, initially, in a particular steady-state with transfers $T^0$ and imagine that the government raises transfers to $T^1 > T^0$.

Does the new set of steady-states with higher transfers display higher or lower capital stock? Draw the equilibrium dynamics of this economy between steady-states in the space of wealth holding $(a_1, a_2)$ for the two agents. Is the final steady-state uniquely determined? How would you compute the dynamics?

• The set of steady states is given by all pairs $\{a^i\}_{i=1}^2$ that satisfy the condition:

$$K = \mu^1 a^1 + \mu^2 a^2$$

where $K$ is the steady state value of aggregate capital.

• In general we have two effects (substitution and wealth) that go in opposite directions so we cannot guarantee that the direction of the movement in the aggregate capital stock. On the one hand, the increase in subsidies could be financed by a decrease or an increase in proportional taxes (which depends on where we are on a Laffer curve type of argument), which would initiate the substitution effect. On the other hand, agents are richer from the subsidies (there is also a wealth effect through the change in their return on savings).

• However, assume we are under the typical scenario of being on the side of the Laffer curve where an increase in taxes increases government revenues. Recall Figure 1 (Note that this is not a formal proof). Hence, to finance the increase in subsidies we need an increase in proportional taxes. This is associated with an increase in the interest rate in SS by equation (2), which implies a decrease in the capital-labor ratio. This also leads to a decrease in the wage, which would have as a first effect a reduction in labor supply (there are secondary effects that I am skipping). Then, it must be that aggregate capital is reduced (Note that depending on the parametrization it is possible for this secondary effects to be strong).

• Nevertheless, the direction for capital cannot be determined analytically in general (at least not without parametric assumptions). But the most important thing is to know which are the forces behind, which we stated above.

• The new steady state is also indeterminate, in the same sense that the first one was. But given particular values for individual asset holdings in the previous steady state, one can uniquely determine the new one by iterating the system of equations until it converges. For more on this, see section 1 in Krusell and Rios-Rull (1999) in the AER.
3 Problem 3

Consider a pure exchange economy, where time is discrete, indexed by \( t = 0, 1, 2, \ldots \) and continues forever. The economy is populated by 2 individuals \( i = 1, 2 \) with logarithmic period utility \( \ln(c_i^t) \) and discount factor equal to \( \beta \in (0, 1) \) who trade a non-storable consumption good \( c_i \). Agents have deterministic endowment streams \( \{e_i^t\}_{t=0}^{\infty} \) of the consumption good given by

\[
e_i^t = \begin{cases} 
0 & \text{if } (t + i) \text{ is even} \\
2 & \text{if } (t + i) \text{ is odd}
\end{cases}
\]

Agents behave competitively. All markets open at time zero and contracts are traded specifying how many units of consumption good will be exchanged at each time \( t \) between the two agents.

3.1 Arrow-Debreu competitive equilibrium

Define an Arrow-Debreu competitive equilibrium and verify that the first Welfare Theorem holds.

- Note that agents alternate in terms of who gets positive endowment each period. At \( t = 0 \), agent 1 receives \( e_1^0 = 2 \), while \( e_2^0 = 0 \). At \( t = 1 \) the roles are reversed and so on.

- An Arrow-Debreu Equilibrium for this economy is a sequence of allocations \( \{c_1^t, c_2^t\}_{t=0}^{\infty} \) and a sequence of time-0 prices \( \{p_0^t\}_{t=0}^{\infty} \) such that

1. Given prices and endowments, the allocation for agent \( i \) is such that it solves the following problem

\[
\max_{\{c_i^t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_i^t) \\
\text{s.t.} \\
\sum_{t=0}^{\infty} p_0^t c_i^t \leq \sum_{t=0}^{\infty} p_0^t e_i^t
\]

2. Markets clear every period

\[c_1^t + c_2^t = e_1^t + e_2^t, \forall t \geq 0\]

- In order for the First Welfare Theorem to hold, the following conditions must be satisfied (from MWG)

1. **Complete Markets** - Trivial due to the absence of uncertainty.
2. **Price-taking** - We assume this.
3. **Locally Non-Satiated Preferences** - Satisfied, since log utility is strictly increasing, hence it also satisfies this weaker property.

- Although you were not required to check this, notice that the hypotheses for the Second Welfare Theorem hold:

1. Let the consumption possibility set: \( X = \{x \in S : x_i \geq 0\} \), same for both types. Then \( X_i \) convex:

   - take two points \( (x_1, x_2) \in X \). It follows that \( x_1, x_2 \geq 0 \). Then \( \lambda x_1 + (1 - \lambda) x_2 \geq 0, \) for \( \lambda \in (0, 1) \), thus the point \( \lambda x_1 + (1 - \lambda) x_2 \in X \) which proves convexity.
2. \( u_i \) is a sum of strictly concave functions, so it is strictly quasi-concave
3. \( u_i \) is a sum of logarithmic functions, so it is continuous
4. cheaper point: because of log preferences, \( x^*, p^* > 0 \). Consider the trivial point \( \bar{p} = 0 \). This point belongs to the consumption possibility sets and moreover, \( p^* \bar{p} < p^* x^* \), so there is an admissible cheaper point.
3.2 Characterization of the AD equilibrium allocations

Solve for the Arrow-Debreu equilibrium, i.e., characterize the equilibrium sequences of prices and allocations of consumption good among agents.

- The Lagrangian for agent $i$ is
  \[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log(c_i^t) + \lambda \left[ \sum_{t=0}^{\infty} (p_0^t e_i^t - p_0^t c_i^t) \right] \]
  (note that there is only one $\lambda$ because there is a single budget constraint).

- The FOC with respect to $c_i^t$ is
  \[ \beta^t \frac{1}{c_i^t} = \lambda^i p_0^t \]

- Set $p_0^i = 1$ and evaluate the FOC at $t = 0$ to obtain
  \[ \frac{1}{c_0^i} = \lambda^i \]

- Combining the above FOC with that for an arbitrary $t$, we obtain the following relation
  \[ \beta^t \frac{c_0^i}{c_i^t} = p_0^t \] (20)

- Since this holds for any consumer $i,j$, and since consumers are price-takers, we conclude that
  \[ \frac{c_0^i}{c_i^t} = \frac{c_0^j}{c_j^t} \]
  i.e., the consumption ratio between the two consumers is constant over time, and equal to the initial consumption ratio.

- Furthermore, market clearing tells us that $e_i^t + e_j^t - c_i^t = c_j^t$, $\forall t \geq 0$, and we also know that $e_i^1 + e_j^2 = 2$, $\forall t \geq 0$.
  Hence
  \[ \frac{c_0^i}{c_i^t} = \frac{2 - c_0^i}{2 - c_i^t} \iff c_0^i = c_i^t, \forall t \geq 0, i \in \{1,2\} \]
  so agents will consume exactly the same every period - and this level is fully determined at $t = 0$. This is the result we expected: in Arrow-Debreu economies, agents consume a constant fraction of the aggregate endowment, which is constant in our case!

- This also allows us to determine the Arrow-Debreu prices. Using (20), we get that
  \[ p_0^t = \beta^t \frac{c_0^i}{c_i^t} = \beta^t \]

- We can now use the budget constraints to pin down these initial levels of consumption. By market clearing, we just need to do this for one agent (as the other consumes the remainder). For agent 1:
  \[ \sum_{t=0}^{\infty} p_0^t c_1^t = \sum_{t=0}^{\infty} p_0^t e_1^t \]
  \[ c_0^1 \sum_{t=0}^{\infty} \beta^t = \sum_{t=0}^{\infty} [2 + 0 + 2 + \ldots] \]
  \[ c_0^1 = (1 - \beta)2[1 + 0 + \beta^2 + 0 + \beta^4 + \ldots] \]
  \[ c_0^1 = 2(1 - \beta) \sum_{t=0}^{\infty} (\beta^2)^t \]
  \[ c_0^1 = \frac{2}{1 + \beta} \]
So that $c_0^2 = 2 - \frac{2}{1+\beta} = 2\frac{\beta}{1+\beta}$. This completes the characterization of the equilibrium. Note that $c^1 > c^2$ because agent 1 is the one with positive endowment at $t = 0$.

### 3.3 Pareto optimal allocation

Define a Pareto optimal allocation for this economy. Write down the Negishi Planner’s problem where the Planner gives weight $\alpha$ to agent 1 and $(1 - \alpha)$ to agent 2. Solve the Planner’s problem for arbitrary weights $\alpha \in (0, 1)$.

- A Pareto Optimal allocation is a sequence of allocations $\{c^1_t, c^2_t\}_{t=0}^\infty$ such that
  1. At every period, the allocation is feasible
     $$c^1_t + c^2_t \leq e^1_t + e^2_t$$
  2. There is no other sequence of allocations $\{y^1_t, y^2_t\}_{t=0}^\infty$ such that $U(y^i_t) > U(c^i_t)$ and $U(y^j_t) \geq U(c^j_t)$ for $i \neq j$ and all $i$.
- The social planner maximizes the weighted sum of utilities subject to the feasibility constraint
  $$\max_{\{c^1_t, c^2_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t [\alpha \log(c^1_t) + (1 - \alpha) \log(c^2_t)]$$
  subject to
  $$c^1_t + c^2_t \leq e^1_t + e^2_t = 2, \forall t \geq 0$$
  Note that in the Planner’s problem, it doesn’t matter which agent receives the endowment. The Planner commands all the resources in the economy and reallocates them optimally.
- We can write the Lagrangean as
  $$\mathcal{L} = \sum_{t=0}^\infty \{\beta^t [\alpha \log(c^1_t) + (1 - \alpha) \log(c^2_t)] + \mu_t (2 - c^1_t - c^2_t)\}$$
  where, note, we now have a sequence $\{\mu_t\}_{t=0}^\infty$ as the planner faces a sequence of feasibility constraints.
- The FOC are
  $$(c^1_t) : \beta^t \frac{\alpha}{c^1_t} = \mu_t$$
  $$(c^2_t) : \beta^t \frac{1 - \alpha}{c^2_t} = \mu_t$$
- Naturally, we can combine them to obtain
  $$\alpha c^2_t = (1 - \alpha) c^1_t$$
- Using the resource constraint, and noting that it binds due to monotonicity of preferences, we conclude that
  $$c^1_t = 2\alpha$$
  $$c^2_t = 2(1 - \alpha)$$
  This holds for all $t$, hence fully characterizes the planner’s solution. Also note that the ratio of consumptions equals the ratio of the planner’s weights, so if $\alpha \uparrow$, the consumption of 1 increases relative to that of 2.
Finally, note that
\[ \mu_t = \beta^t \alpha c^t_i = \frac{1}{2} \beta^t \]
So that the multipliers on the resource constraint are proportional to the Arrow-Debreu prices.

How do we go from here to the competitive equilibrium allocations? The answer is that we need to choose the “right” weight \( \alpha \) in the above problem. Let us parametrize the entire family of planner’s problems above through the weight parameter \( \alpha \). Every solution to an \( \alpha \)–planner problem implies consumption allocations that in equilibrium (not in the centralized solution, where the planner owns all the resources), require an initial transfer of resources from one agent to the other in order for the path of consumption itself to be affordable.

3.4 Mapping of the Pareto optimum into a competitive equilibrium

Characterize the transfer function that allows to map any Pareto optimum, for given weights \((\alpha, 1 - \alpha)\), into a competitive equilibrium with transfers among agents and determine the specific value for the planner weights that select exactly the competitive equilibrium you have computed in point 2).

Define the transfer function \( \tau^i(\alpha) \) as the transfer of resource to agent \( i \) (from the other agent in the economy) that make the allocations that solve the \( \alpha \)–planner problem affordable in equilibrium.

For each agent, the Arrow-Debreu budget constraint can be written as
\[
p_0^0 \left[ \sum_{t=0}^{\infty} \frac{p_0^0}{p_0^t} (c^t_i - e^t_i) \right] = 0
\]

We know that \( \frac{p_{t+1}^0}{p_0^0} = \mu_{t+1}/\mu_t \). Setting \( p_0^0 = 1 \), we can then write the transfer function as
\[
\tau^i(\alpha) = \sum_{t=0}^{\infty} \frac{\mu_t}{\mu_0} [c^t_i(\alpha) - e^t_i]
\]
where \( c^t_i(\alpha) \) is the solution to the planner’s problem.

Replacing for this solution, and using the fact that \( \mu_t/\mu_0 = \beta^t \), we obtain that
\[
\tau^1(\alpha) = \sum_{t=0}^{\infty} \beta^t (2\alpha - e^1_t) = \frac{2\alpha}{1 - \beta} - \frac{2}{1 - \beta^2}
\]
\[
\tau^2(\alpha) = \sum_{t=0}^{\infty} \beta^t (2(1 - \alpha) - e^2_t) = \frac{2(1 - \alpha)}{1 - \beta} - \frac{2\beta}{1 - \beta^2}
\]

This gives us the transfer functions as a function of the Negishi weights for each agent. In order to sustain the planner’s solution as a competitive equilibrium that is consistent with the agents’ endowment processes, \( \alpha \) must be such that \( \tau^1(\alpha) = \tau^2(\alpha) = 0 \). Setting \( \tau^1(\alpha) = 0 \) yields
\[
\alpha = \frac{1}{1 + \beta}
\]
whereas \( 1 - \alpha = \frac{\beta}{1 + \beta} \). Note that \( \alpha > 1 - \alpha \), as we would expect (from our previous computation of the equilibrium).

This method of characterizing AD equilibrium allocations through the Planner’s problem is called Negishi’s method, as it was first developed by Negishi (1960), “Welfare economics and the existence of an equilibrium for a competitive economy”, Metroeconomica. Remember that it works only when the FWT and SWT hold for our economy. We can summarize the steps above as:
1. Solve the generic $\alpha$–planner problem.

2. Compute the transfers necessary to make the $\alpha$–PO allocations affordable in equilibrium, and as prices, use the multipliers on the resource constraint in each period.

3. Find the Pareto weights that makes the function $\tau^i(\alpha) = 0$, all $i$.

4. The corresponding PO allocations are the competitive equilibrium allocations. The AD prices are the Lagrange multipliers on the resource constraints of the Planner’s problem.
Problem 4

Prove that if (and only if) the indirect utility function has the form \( v_i(p, a_i) = \alpha_i(p) + \beta(p) a_i \), for an agent with utility function \( u_i \), then Engel curves (expenditures as a function of wealth) are linear in wealth.

- Assume that the indirect utility function takes the form \( v_i(p, a_i) = \alpha_i(p) + \beta(p) a_i \). By Roy’s Identity, we have
  \[
  x_i^j(p, a_i) = -\frac{\partial v_i(p, a_i)}{\partial p_j} \frac{\partial v_i(p, a_i)}{\partial a_i}
  \]
  where \( x_i^j(p, a_i) \) is agent \( i \)’s demand function for good \( j \).

- Evaluating these terms under the assumed functional form for \( v_i(p, a_i) \) yields
  \[
  \frac{\partial v_i(p, a_i)}{\partial p_j} = \alpha_{ij}(p) + \beta_j(p) a_i
  \]
  \[
  \frac{\partial v_i(p, a_i)}{\partial a_i} = \beta(p)
  \]
  where \( \alpha_{ij}(p) \) and \( \beta_j(p) \) represent derivatives of those functions with respect to their \( j \)th arguments.

- Substitution now delivers
  \[
  x_i^j(p, a_i) = -\frac{\alpha_{ij}(p)}{\beta(p)} - \frac{\beta_j(p)}{\beta(p)} a_i
  \]
  which is clearly linear in wealth. Therefore, demand and expenditures are linear in wealth under this form for the indirect utility function.

- The “only if” part of this problem is incorrect as written, which can be immediately seen by considering a single-good economy. In this setting, any utility function which is strictly increasing in the single good will lead each agent to spend all of their wealth on that good, making demand linear in wealth. However, it is clear that not every strictly increasing utility function leads to an indirect utility function of the desired form.

- Instead, the problem should be stated that demand and expenditures are linear in wealth if and only if preferences admit an indirect utility function of the form \( v_i(p, a_i) = \alpha_i(p) + \beta(p) a_i \). This problem is discussed in Mas-Colell, Whinston, and Green, and is determined (by them) to be too difficult as an exercise, so I will not attempt a solution here. However, see that text for references on where to find a proof. Similarly, the original result is proved in Gorman (Metroeconomica, 1961), “On a class of preference fields.”

Deriving the present value budget constraint

I give here the derivation of the present value budget constraint for the most simple problem. As an exercise you can derive the present value budget constraint for the agent in question 2.

The time \( t \) budget constraint is:

\[
c_t + a_{t+1} = a_t (1 + r_t) \quad \text{or} \quad a_{t+1} = a_t (1 + r_t) - c_t.
\]

Now iterate forward and use to get:

\[
a_{t+2} = a_{t+1} (1 + r_{t+1}) - c_{t+1}
\]
In a competitive equilibrium, the no arbitrage condition implies that:

\[
\frac{p_{t+1}}{p_t} = (1 + r_{t+1}), \text{ which gives:}
\]

\[
a_{t+2} = a_{t+1} \frac{p_{t+1}}{p_{t+2}} - c_{t+1}
\]

Iterate again to get:

\[
a_{t+3} = a_{t+2} \frac{p_{t+2}}{p_{t+3}} - c_{t+2}
\]

and use the previous iterations as follows:

\[
a_{t+3} = \left( a_{t+1} \frac{p_{t+1}}{p_{t+2}} - c_{t+1} \right) \frac{p_{t+1}}{p_{t+2}} - c_{t+2}
\]

\[
a_{t+3} = a_{t+1} \frac{p_{t+1}}{p_{t+2}} - c_{t+1} \frac{p_{t+1}}{p_{t+2}} - c_{t+2}
\]

\[
a_{t+3} = (a_t (1 + r_t) - c_t) \frac{p_{t+1}}{p_{t+2}} - c_{t+1} \frac{p_{t+1}}{p_{t+2}} - c_{t+2}
\]

\[
a_{t+3} = a_t (1 + r_t) \frac{p_{t+1}}{p_{t+2}} - c_t \frac{p_{t+1}}{p_{t+2}} - c_{t+1} \frac{p_{t+1}}{p_{t+2}} - c_{t+2}
\]

Multiply by \(\frac{p_{t+2}}{p_t}\) to get:

\[
\frac{p_{t+2}}{p_t} a_{t+3} = a_t (1 + r_t) - c_t - c_{t+1} \frac{p_{t+1}}{p_t} - c_{t+2} \frac{p_{t+2}}{p_t}
\]

Repeating this process infinitely yields:

\[
\lim_{j \to \infty} \frac{p_{t+j}}{p_t} a_{t+j} + \sum_{j=0}^{\infty} c_{t+j} \frac{p_{t+j}}{p_t} = a_t (1 + r_t)
\]

Since \(r_t > 0\), the limit on the LHS is zero and we get:

\[
\sum_{j=0}^{\infty} c_{t+j} \frac{p_{t+j}}{p_t} = a_t (1 + r_t)
\]

which simply states that the present value of all future consumption (with the relevant prices) equals to the current value of assets. For the problem with endowment of efficiency units, the RHS will also include the present value of the stream of labor income.