1 Modelling borrowing constraints in Bewley models

Consider the problem of a household who faces idiosyncratic productivity shocks, supplies labor inelastically and can save/borrow only through a risk-free asset. This household problem, in recursive form, is

\[
v(a, \varepsilon) = \max_{c, a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E} v(a', \varepsilon') \pi(\varepsilon', \varepsilon) \right\}
\]

s.t.

\[
c + a' = (1 + r) a + \omega \varepsilon
\]

Besides the budget constraint, we argued that households may face a debt limit. We have studied two types of borrowing constraints, so far. The first class of borrowing constraints is “ad-hoc”. Let \( a' \) be the asset position of the household next period, \( \varepsilon \) be its idiosyncratic productivity level, \( \omega \) be the equilibrium wage rate, and \( r \) the equilibrium interest rate.

Then, two types of ad-hoc constraints are, for example,

\[ a' \geq -b, \] (1)

or also

\[ a'(\varepsilon) \geq -\varphi \omega \varepsilon. \] (2)

Constraint (1) sets the borrowing limit to an exogenous debt level \( b \). Constraint (2) sets it to a multiple \( \varphi \) of current labor income. Note that this latter borrowing limit is state-dependent but straightforward to compute.

The second class of borrowing constraints was the one suggested by Aiyagari (1994) – we called it the “natural” borrowing constraint. The idea is that if preferences satisfy the Inada condition

\[
\lim_{c \to 0} u(c) = -\infty,
\]

then households will never choose to be in an asset position that may induce, with positive probability, a future state where they end up with zero consumption. Thus, this borrowing constraint is directly implied by the condition \( c > 0 \) holding in every state of the world. If we let \( \varepsilon_{\text{min}} \) be the lowest realization of individual productivity, then the natural borrowing limit becomes

\[ a' \geq \frac{\omega \varepsilon_{\text{min}}}{r}. \]
Note that what these borrowing constraints do is imposing limits on “net worth” (i.e., assets minus liabilities), because in this economy there is only one asset. For example, \( a' > -b \) means that the agent can cumulate unsecured (or non-collateralized debt) up to \( b \).

An implicit assumption that we have made so far is that, once households have taken on some debt, they commit to repay. Somewhere in the background of our economy there is an enforcement agency that makes sure that all the trades at every date \( t \) are respected and carried out as promised [think “The Sopranos”]. In actual economies, enforcement institutions exist, but they only have limited power. Unlike the Sopranos, they will not shoot you in your legs if you don’t repay... For example, many households with negative asset positions choose to default on their debt. In the United States, for example, every year almost 1 percent of consumers declares bankruptcy and defaults on its debts.

### 1.1 No-default borrowing constraints

Suppose households have the option default on their debt. However, if they do an enforcement agency excludes the defaulting household from future participation in the asset market. In other words, the household must live in financial autarky forever after. Let the value of autarky be

\[
v^\text{aut} (\varepsilon) = u (w \varepsilon) + \beta \sum_{\varepsilon' \in \mathcal{E}} v^\text{aut} (\varepsilon') \pi (\varepsilon', \varepsilon),
\]

i.e. households can participate to the labor market, but they must consume their earnings every period.

In order to avoid default, lenders limit the amount of funds lended such that the household has always the incentive to pay back. This goal is achieved by choosing the limit so that the discounted lifetime utility from participating in the asset market is at least as high as that of autarky defined above. This version of the Bewley model with no-default constraints is studied by Zhang (1997).

Because the borrowing limit depends on the value of participating, it is endogenous. Since both the value of participating and the value of autarky depend on the individual exogenous state \( \varepsilon \), the borrowing constraint is state-dependent. We can write the no-default debt constraint as

\[
a' (\varepsilon) \geq -b^* (\varepsilon)
\]
where
\[ b^* (\varepsilon) = \max \left\{ b (\varepsilon) : v (-b (\varepsilon), \varepsilon') \geq v^{aut} (\varepsilon'), \forall \varepsilon' : \pi (\varepsilon', \varepsilon) > 0 \right\} . \] (4)

So, \( b^* (\varepsilon) \) is the largest amount of debt that a household with shock \( \varepsilon \) can take on so that for every possible productivity realization next period, she will always find optimal to repay than to default and revert to financial autarky. Note that if the Markov chain is such that \( \pi (\varepsilon', \varepsilon) > 0 \) for all pairs \( (\varepsilon, \varepsilon') \), then \( b^* (\varepsilon) \) is a constant.

How do we know that such a limit exists? We want to rule out that the above set is empty or that the solution for \( b^* \) is negative.

\[
\text{FIGURE}
\]

In the above figure, we plot \( v^{aut} (\varepsilon) \) and \( v (-b, \varepsilon) \) as a function of \( b \) in the domain \([0, \infty)\) for \( b \). The value of autarky is independent of \( b \). The value of participating is decreasing in \( b \), since it is increasing in asset holdings. It is obvious that \( v (0, \varepsilon) > v^{aut} (\varepsilon) \) because the participating household has the same income as the household in autarky, but he can always save in order to smooth consumption and achieve a higher expected lifetime utility. Now, let \( -b^{nat} \) be the natural debt limit, and assume that

\[ v (-b^{nat}, \varepsilon) < v^{aut} (\varepsilon), \forall \varepsilon \in E \]

Recall that the natural limit is the maximum amount households will choose to borrow. The above condition says that, at \( -b^{nat} \) households are better off defaulting. Then, by the mean value theorem, there is a value \( b^* \in (0, b^{nat}) \) such that \( v (-b^*, \varepsilon) = v^{aut} (\varepsilon) \). This value is the no-default constraint for an income shock \( \varepsilon \).

**Solution method:** How do we solve the *household problem* (for a given \( r \)) with the no-default debt limit?

1. Compute the value of autarky \( v^{aut} (\varepsilon) \) which is a trivial fixed point problem solved once and for all by iterating over (3). Guess an interest rate \( R \).
2. Need to guess a constraint function \( b^* (\varepsilon) \in (0, b^{nat}) \). With a guess for our no-default borrowing constraints, we can now solve the household problem and obtain decision rules.
3. Simulate a long history for the household starting in states \( (-b^* (\varepsilon), \varepsilon) \) for all \( \varepsilon \), and compute her discounted lifetime utility of participating, i.e. \( v (-b^* (\varepsilon), \varepsilon) \).
4. Verify if \( b^* (\varepsilon) \) satisfies the definition of \( b^* (\varepsilon) \) in equation (4). If not, update the new guess for \( b^* (\varepsilon) \). For example, if at your guess the value of participating exceeds the value of autarky, you can loosen the constraint, etc...

5. Verify asset market clearing and update your guess of \( R \).

2 An economy with default occurring in equilibrium

The Zhang economy has two big shortcomings. First, default does not occur in equilibrium. Second, the asset market is not truly competitive. In the Zhang economy there are arbitrage opportunities in offering loans. A financial intermediary could enter the market offering to lend more than the no-default limit at a higher interest rate, i.e., an interest rate that internalizes the probability that the household may default and not repay next period. The household may also find such contract profitable, so there would be gains from trade.

We now formalize an endowment economy with default occurring in equilibrium, following Livshits, McGee, and Tertilt (2006).

**Demographics and preferences:** There is a continuum of agents of measure one, infinitely lived. Agents’ period utility is \( u(c) \) with \( u' > 0, u'' < 0 \). Future utility is discounted at rate \( \beta \). Preferences are separable over states and over time.

**Uncertainty:** Let \( y \in Y \) be the labor income of the an agent, where \( y \) follows a Markov chain with typical element \( \pi (y', y) \).

**Default:** Agents can choose to default on their debt. If they do, their debts are fully discharged, but in the period following default: 1) they are excluded from borrowing through financial markets, 2) a fraction \( \gamma \) of their income is seized by the intermediary and used as partial repayment. These are some feature of the U.S. consumers’ bankruptcy law [called “Chapter 7”].

**Individual states:** The individual state is the couple \((a, y)\), where \( a \) is the agent’s asset position. Moreover, we need an additional state variable indicating whether the agent declared bankruptcy the previous period. Let \( D(a, y) \) be the indicator function denoting the decision to default, i.e., \( D(a, y) = 1 \) means default.
Financial markets: Agents can save/borrow through financial intermediaries which act competitively as price takers. The interest rate on savings is risk-free and equal to $1 = \bar{\theta}$, i.e., the saver deposits $\bar{q}$ units of consumption this period into the bank and next period receives one unit of consumption for sure. Borrowers can subscribe loan contracts through the same banking intermediaries. The price of a loan for a given individual will depend on the individual income shock and on the amount of debt issued, because they’re both predictors of its default probability at $t + 1$. Let $\theta (a', y)$ denote this default probability. An individual who borrows $a'$, and has current income shock $\gamma$ (assumed to be observable by the financial intermediaries) pays a price $q (a', y) < 1$.

Household problem: Let $v$ be the value function of a household in good standing and $\omega$ be the value function of a household in bankruptcy. We have that $v$ is given by

$$
v (a, y) = \max_{c, a'} \left\{ u (c) + \beta \max_{y' \in Y} \left[ v (a', y'), \omega (0, y') \pi (y', y) \right] \right\} \quad \text{(HP)}
$$

subject to

$$
a + y - c = \begin{cases} 
\bar{q}a' & \text{if } a' \geq 0 \\
q (a', y) a' & \text{if } a' < 0 
\end{cases}
$$

where the second max operator captures the bankruptcy choice $D (a', y')$ which is taken after the income shock is observed.

The value of bankruptcy $\omega$ is given by

$$
\omega (0, y) = \max_{\tilde{c}, \tilde{a}'} \left\{ u (\tilde{c}) + \beta \sum_{y' \in Y} v (\tilde{a}', y') \pi (y', y) \right\} \quad \text{(DHP)}
$$

subject to

$$
\tilde{c} + \bar{q}\tilde{a}' = (1 - \gamma) y
$$

$$
\tilde{a}' \geq 0
$$

where $\tilde{c} (y)$ and $\tilde{a}' (y)$ are the consumption and saving rules during default. Note that, during bankruptcy a household can save (but cannot borrow). Moreover, given that the following period she will have non-negative assets, she would not have incentives to default, so next period value function is $v$ for sure.

---

1This means that she will receive $q (a', y) < 1$ units of consumption this period and repay one unit of consumption next period in absence of default.
Stationary equilibrium: A stationary recursive competitive equilibrium for this economy is: (i) value functions \( \{v_\sigma, \omega\} \); (ii) decision rules \( \{c, a', D; \bar{c}, \bar{a}'\} \); (iii) prices \( \{q\} \); (iv) default probabilities \( \theta \); (iv) invariant distribution \( \lambda(a, y, D) \) such that:

- Given prices, the decision rules \( \{c, a', D\} \) and \( \{\bar{c}, \bar{a}'\} \) solve the household problems (HP) and (DHP), and \( \{v_\sigma, \omega\} \) are the associated value functions.

- The intermediaries make zero profits on each loan type \( (a', y) \), i.e.

\[
\frac{1}{\bar{q}} = \frac{1}{q(a', y)} \left\{ [1 - \theta(a', y)] \cdot 1 + \sum_{y' \in \mathcal{Y}} D(a', y') \left( \frac{\gamma y'}{a'} \right) \pi(y', y) \right\} \text{ for all } (a', y).
\]

The LHS denotes the return on each unit saved, i.e. a cost for the intermediary who used these funds to finance borrowers. The RHS denotes the expected return for the intermediary on the loan contract: if the agent does not default, then the intermediary gets one unit of consumption back. If she defaults, the intermediary can seize a fraction \( \gamma \) of income as repayment.\(^2\)

- Default probabilities used by the intermediary sector are consistent with the agents’ decisions, i.e.

\[
\theta(a', y) = \sum_{y' \in \mathcal{Y}} D(a', y') \pi(y', y).
\]

- The asset market clears

\[
\int_{A \times \mathcal{Y} \times \mathcal{D}} ad\lambda = 0.
\]

- The invariant distribution \( \lambda \) solves the right fixed-point problem.

Computation of equilibrium: The computation is different from the standard Aiyagari model for two reasons. First, it is convenient to guess and iterate over value functions, since the default decision involves a comparison between \( v \) and \( \omega \). Second, the equilibrium price we need to loop over is not just a number, but a function \( q(a', y) \) for \( a' < 0 \) and a number \( \bar{q} \) (the function \( q \) for \( a' \geq 0 \)). With \( \bar{q} \) and \( q(a', y) \) in hand,

\(^2\)This separating equilibrium where zero-profits holds for each type \( (a', y) \) is the consequence of perfect competition. If there was only one intermediary pooling all the risk and charging only one price, then there would be scope for arbitrage. Indeed, in the pooling equilibrium the individuals with low default probability cross-subsidize the types with high default probability. An outside institutions could offer to charge less to the types with low default probability and poach them away from the pooling contract.
we can solve the household problem and obtain decision rules for consumption, saving and default. Then, given $\bar{q}$, the zero-profit condition (5) combined with the aggregate consistency condition (6) is used to verify that, with the derived default decision rules, the function $q(a',y)$ is the right one. If not, one keeps iterating on $q$ (given $\bar{q}$) until convergence is reached. Then, market clearing condition (7) is used to verify that our guess of $\bar{q}$ is the right one.

References
