Lecture VII

Higher Order Perturbation Methods

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Quantitative Macroeconomics
Main idea of perturbation methods

• Transform the problem by rewriting it in terms of a perturbation parameter

• Approximate the solution around a particular choice of the perturbation parameter

• Perturbation parameter: SD of the shock $\sigma$

• Particular choice for local approximation: $\bar{\sigma} = 0$

• The proposed solution of this model is of the form:

$$y_t = h(x_t, \sigma)$$

$$x_{t+1} = g(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}$$

where $\eta = (0, 1)'$ and $\sigma$ is the SD of the $\varepsilon$ shock
Second-order perturbations: result

1. The coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks.

2. In other words, these coefficients must be the same in the stochastic and the deterministic versions of the model.

3. Thus, up to second order, the presence of uncertainty affects only the constant term of the decision rules.
Issues with higher order approximations

- Far from the approximation point they can be worse than linear, and non-monotonic in the states.

G. Violante, "Perturbation Methods"
Issues with higher order approximations

- **Unstable dynamics** of the approximated system
Textbook example of unstable dynamics

- Consider a second order approximation of a univariate decision rule $x_{t+1} = g(x_t)$ around $\bar{x}$ (where $\hat{x}_t = x_t - \bar{x}$):

$$
\hat{x}_{t+1} = \alpha_0 \hat{x}_t + \alpha_1 \hat{x}_t^2 \\
= \alpha_0 \hat{x}_t (1 - \alpha_2 \hat{x}_t)
$$

where $\alpha_2 = -\alpha_1 / \alpha_0$

- Logistic map: even if $\alpha_0 < 1$, it can have chaotic dynamics:

G. Violante, “Perturbation Methods” p. 6/14
Pruning (Kim, Kim, Schaumburg, and Sims, 2008)

\[ \hat{x}_{t+1}^{(n)} = g^{(n)}(\hat{x}_t, \sigma) \]

where \((n)\) denotes \(n\)-th order approxim. When we simulate the model:

\[
\begin{align*}
\hat{x}_{t+1}^{(1)} &= g^{(1)}(\hat{x}_t^{(1)}, \sigma) + \eta \sigma \varepsilon_t = \bar{g}_x \hat{x}_t^{(1)} + \eta \sigma \varepsilon_t \\
\hat{x}_{t+1}^{(2)} &= g^{(2)}(\hat{x}_t^{(2)}, \sigma) = \bar{g}_x \hat{x}_t^{(2)} + \frac{1}{2} \hat{x}_t^{(2)} \bar{g}_{xx} \hat{x}_t^{(1)} + \frac{1}{2} \bar{g}_{\sigma \sigma} \sigma^2 + \eta \sigma \varepsilon_t
\end{align*}
\]

- Under pruning:

\[
\hat{x}_{t+1}^P = g^P(\hat{x}_t^{(2)}, \sigma, \hat{x}_t^{(1)}) = \bar{g}_x \hat{x}_t^{(2)} + \frac{1}{2} \hat{x}_t^{(1)} \bar{g}_{xx} \hat{x}_t^{(1)} + \frac{1}{2} \bar{g}_{\sigma \sigma} \sigma^2 + \eta \sigma \varepsilon_t
\]

i.e., we use the dynamics from the first order approximation in the second order terms. It is easy to see that it is stationary.

- It adds new state variable / changes coefficient \(\bar{g}_{xx}\) of decision rule
Occbin

- Methodology (and suite of Matlab/Dynare codes) developed by Guerrieri and Iacoviello (JME, 2015) to apply first-order perturbation approach in a piecewise fashion in order to handle occasionally binding constraints.

- **Key insight**: occasionally binding constraints can be handled as different regimes of the same model. Under one regime, the occasionally binding constraint is slack. Under the other regime, the same constraint is binding. The piecewise linear solution method involves linking the first-order approximation of the model around the same point under each regime.

- Solution is nonlinear, i.e., decision rules parameters depend on the values of the state variables.
Occbin algorithm

- Two regimes. Reference regime (R1) that includes the SS, and alternative regime (R2). Assume the alternative regime is the one where the constraint binds and reference where it is slack.

- In the reference regime R1, the equilibrium dynamics of the endogenous state $X_t$ linearized around SS can be expressed as:

  $$ A^E_t X_{t+1} + B X_t + C X_{t-1} + F z_t = 0 $$

- In the alternative regime R2 (when constraint binds), the equilibrium dynamics of the endogenous state $X_t$ linearized around the SS can be expressed as:

  $$ A^c E_t X_{t+1} + B^c X_t + C^c X_{t-1} + D^c + F^c z_t = 0 $$

- The additional vector $D^c$ is there because the linearization is taken around a point where R1 applies.
Occbin algorithm

• Define the decision rules under the reference regime

\[ X_t = P X_{t-1} + Q z_t \]

and under the alternative regime

\[ X_t = P^c_t X_{t-1} + Q^c_t z_t + R^c_t \]

note that these function are time-dependent, i.e., depend on the values of the states.

• For a given point \((X_0, z_1)\) in the state space at \(t = 1\), how do we solve for the decision rule?
Occbin algorithm

1. **Guess the number of periods** $T \geq 0$ for which the alternative regime applies before returning to R1.

2. Use dynamics under R2 combined with the R1 decision rule at $T$:

   $$A^c E_{T-1} [P X_{T-1} + Q z_T] + B^c X_{T-1} + C^c X_{T-2} + D^c + F^c z_{T-1} = 0$$

3. Assume agents expect no shocks from $t = 1$ onward (RE violated)

   $$A^c P X_{T-1} + B^c X_{T-1} + C^c X_{T-2} + D^c = 0$$

   $$X_{T-1} = -(A^c P + B^c)^{-1} C^c X_{T-2} - (A^c P + B^c)^{-1} D^c$$

   thus

   $$P_{T-1}^c = -(A^c P + B^c)^{-1} C^c , \quad R_{T-1}^c = -(A^c P + B^c)^{-1} D^c$$
4. Using $X_{T-1} = P_{T-1}^c X_{T-2} + R_{T-1}^c$ and the dynamics under R2, obtain $X_{T-2} = P_{T-2}^c X_{T-3} + R_{T-2}^c$ and iterate back until $X_{t-1}$ where

$$X_1 = P_1^c X_0 + R_1^c + \left[ - (A^c P + B^c)^{-1} F^c \right] z_1$$

... the shock reappears in the decision rule only at $t = 1$

5. From stochastic paths for $X_t$ compute expected length of R2 to verify the current guess $T$. If the guess is verified, stop. Otherwise, update the guess and return to step 1.
Occbin algorithm

• To get the entire decision rule, you need to repeat this for every pair $(X, z)$ in the state space.

• To calculate an IRF, you need to perform this loop only once, for given initial conditions $(X_0, z_1)$.

• To compute a stochastic simulation of length $N$, you need to repeat this loop $N$ times for every pair $(X_t, z_t)$ in the state space reached by the simulation.

• ...but according to the authors, it is very fast.
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- **Main advantage**: like any linear solution method, delivers a nonlinear solution easily even for a problem with many aggregate state variables.

- **Main limit**: like any linear solution method, lack of precautionary behavior in decision rules linked to the possibility that a constraint may bind in the future.
  
  ▶ It discards all information regarding the realization of future shocks.

  ▶ E.g., in R1 the decision rule ignores that constraint may be binding under some realization of the shock.