This document supplements Appendix A in *Optimal Welfare-to-Work Programs*, by Nicola Pavoni and Giovanni L. Violante.

**Sequential Formulation**

**History:** The world starts with an exogenously given initial condition \((z_0, h_0, y_0)\), where \(z_0 \in \{z^e, z^u\}\) is the initial employment status of the worker, \(h_0 \in H\) is its level of human capital, and \(y_0 \in \{s, f\}\) is the outcome of the worker’s activity. At the beginning of each period \(t\), the outcome \(x_t\) of a uniform [0, 1] random variable \(X_t\) is publicly observed. The random variables \(\{X_t\}\) are serially uncorrelated and independent of any choice made by the agent or the planner. Let \(\sigma_t = \{z_0, h_0, y_0, x_0, ..., z_t, h_t, y_t, x_t\}\) be a history up to time \(t\).

**Contract:** Let \(W(z_0, h_0, y_0) = \{c, a, m\} = \{c_t(\sigma^t), a_t(\sigma^t), m_t(\sigma^t)\}_{t=0}^{\infty}\) to denote the contract, where

- \(c_t(\sigma^t)\) is the transfer function, with \(c_t(\sigma^t) \geq 0\) for any \(\sigma^t\). Denote by \(c(x^T)\) the continuation plan of transfers after history \(x^T\), i.e., \(c_t(x^T) = \{c_{t+n}(\sigma^{t+n})\}_{n=0}^{\infty} / \sigma^t\)

- \(a_t(\sigma^t)\) is the action (effort choice), where

\[
a_t(\sigma^t) \in \begin{cases} 
0, e & \text{if } z_t = z^u, \\
e & \text{if } z_t = z^e,
\end{cases}
\]

i.e., employment is defined as a state where the worker is productive and production requires the high effort level \(e\). Denote by \(a_t(\sigma^T)\) the continuation plan of effort choices after node \(\sigma^T\) and by \(A_t(\sigma^T)\) the set of all admissible continuation plans, after history \(\sigma^T\).

- \(m_t(\sigma^t) \in \{0, 1\}\) is a dummy variable for the use of the search-effort monitoring technology, with \(m_t(\sigma^T)\) denoting the continuation plan contingent on history \(\sigma^T\).

Define the expected continuation utility promised in equilibrium by the contract \(W\) after history \(\sigma^t\) as

\[
U_t(W; \sigma^t) = \mathbb{E} \left[ \sum_{n=0}^{\infty} \beta^n u(c_{t+n}(\sigma^{t+n})) - v_{z_{t+n}}(a_{t+n}(\sigma^{t+n})) \mid a_t(\sigma^t), m_t(\sigma^t), \sigma^t \right].
\]
we assume that $U_t (W; \sigma^t)$ is well defined for all $(W; \sigma^t)$.

**Incentive compatibility:** In our framework, $(z_t, h_t, y_t, x_t)$ is fully observable. Because of the existence of the monitoring technology, at every node with $m_t (\sigma^t) = 1$, the effort chosen by the agent should be included in the set of contractible variables.

Define by $a^m_t (\sigma^t) \subset a_t (\sigma^t)$ the sub-plan of actions which are not contractible under the monitoring plan $m$. We then have that $a^m_t (\sigma^t) = a_t (\sigma^t)$ if and only if $m_t (\sigma^t) = 0$. In order to generate the sub-plan $a^m$ we simply delete the element $a_t (\sigma^t)$ from $a$ whenever $m_t (\sigma^t) = 1$. We are now ready to define the set of incentive compatibility constraints. For all $\sigma^t$ we require

$$U_t (c, a, m; \sigma^t) \geq U_t (c, \hat{a}, m; \sigma^t), \quad (IC(x^t))$$

where, $\hat{a}_t (\sigma^t)$ can differ from $a_t (\sigma^t)$ only on the non-contractible components $a^m_t (\sigma^t)$. Notice that in order to lighten notation, we have omitted the argument $(\sigma^t)$ from the continuation plans.

**Planner problem:** In the sequential representation of the contractual relationship, the planner solves

$$V^* (U_0, z_0, h_0, y_0) = \sup_{W} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \left( r (h_t, z_t, m_t (\sigma^t)) - c_t (\sigma^t) \right) \mid a, m, (z_0, h_0, y_0) \right],$$

$$s.t.: \quad \int_0^1 U_0 (W; \sigma_0) \, dx_0 \geq U_0 \text{ and } IC(\sigma^t) \text{ for all } \sigma^t \mid \sigma_0,$$

where the return function during employment is $r (h_t, z_t, m_t (\sigma^t)) = w (h_t)$, and during unemployment is $r (h_t, z_u, m_t (\sigma^t)) = -\kappa (m_t (\sigma^t))$, with the costs given by $\kappa (0) = 0$, and $\kappa (1) = \kappa > 0$.

**Options of the contract during unemployment:** There are four admissible combinations of effort and monitoring the planner can implement at every node: positive or zero unmonitored effort, and positive or zero monitored effort. Note however, that choosing zero search effort and at the same time monitoring workers’ effort is not optimal since in this case the moral hazard problem disappears: because $\pi (0, h_t) = 0$, $y_{t+1} = s$ is never an equilibrium outcome, the planner can implement $a_t = 0$ by threatening an infinite punishment, for example, no benefits, off the equilibrium (i.e., whenever $y_{t+1} = s$).

The planner can therefore restrict attention to the three remaining options: 1) positive unmonitored effort, 2) positive monitored effort, and 3) zero unmonitored effort. These are labelled, respectively, 1) Unemployment Insurance (UI), 2) Job-search Monitoring (JM), and 3) Social Assistance (SA), and are described in more detail in the main text.

**Recursive formulation:** The state space can be described as a correspondence $\Gamma (h, z)$ from all the pairs of human capital and employment status $(h, z) \in H \times \{z^e, z^u\}$ to the set of attainable workers’ lifetime utility given by

$$\Gamma (h, z) = \left\{ U : \exists \ W \text{ satisfying } IC(\sigma^t) \forall \sigma^t \mid \sigma_0; \int_0^1 U_0 (W; \sigma_0) \, dx_0 = U, \ (h_0, z_0) = (h, z) \right\},$$
where we have omitted $y_0$ from the initial conditions since it is payoff irrelevant for both the agent and the planner.

A straightforward extension of the standard recursive-contracts methodology (e.g., Spear and Srivastava, 1987) delivers the recursive formulation of the principal-agent problem in terms of the triple $(U, h, z)$ we propose in the text. Below we will show that the functions solving the Bellman equation are bounded and continuous. Moreover, it is easy to show that by the Maximum Theorem the policy correspondence admits a (Borel) measurable selection. The usual verification theorem (Theorem 9.2 in Stokey, Lucas and Prescott, 1989, thereafter SLP) hence implies that the recursive formulation of the problem fully characterizes the optimal program.

**Proof of Proposition 0**

Most of the proof will follow standard techniques, namely SLP. Let $z = u(c)$, and $g = u^{-1}$. Given the function $W$ and a generic $V$, the value function associated to policy $i$ takes the form

$$V^i_{W, V}(U, h) = T^i(V, W)(U, h) = \max_{(z, U') \in \Gamma^i(U, h)} \kappa^i - g(z) + \beta \left[p^i(h) W(U', h^*) + (1 - p^i(h)) V^f(U^f, h^f)\right]$$

s.t. $IC^i(U, h), PK^i(U, h)$

with $\kappa^i = 0$ if $i \neq JM$ and $\kappa^{JM} = \kappa$. Moreover, $p^i(h) = \pi(h)$ if $i = UI$, $JM$, and $p^i(h) = 0$ if $i = SA$. Analogously, we have different incentive constraints for different policies $i$ (see section 3.2 in the main text). Standard techniques can be used to show that under our assumptions the operators $T^i$ are contractions, mapping continuous (jointly in $U, h$), concave, monotone and bounded functions into functions with the same characteristics.\(^3\) With respect to boundedness, notice that since $\omega(\cdot)$ is bounded and $\lim_{c \to \infty} u'(c) = 0$ one can show that there is a $U^* < \infty$ such that for $U \geq U^*$ the only optimal policy is $SA$ with full insurance.\(^4\) This implies that $V$ is bounded below (say by $V_{\min}$). Since $c \geq 0$ $V$ is also bounded above by $V^{\max} = w_{\max}/(1 - \beta)$. Joint continuity is easy to see since both $g$ and $p^i$ are continuous functions (in $z$ and $h$ respectively).\(^5\) Moreover, notice that in the above expression for $T^i$ for each given $i$ both the constraints $IC^i(U, h), PK^i(U, h)$ and $\Gamma^i(U, h)$\(^6\) are linear in $z$ and $U^y$, $y = s, f$; and $g$ is a convex function. This implies that whenever both $W$ and $V$ are concave, then $V^i$ is concave as well. Monotonicity in $U$ is also easy to see. Whenever $c^i(U, h) > 0$ and $U$ decreases (marginally) to $U'$ the planner can replicate exactly the same payment

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\(^3\)In particular, each $T^i$ satisfies the Blackwell sufficiency conditions (SLP, Theorem 3.3).

\(^4\)A sufficiently high value for $U$ is that where

$$\frac{w_{\max}}{1 - \beta} - \frac{g((1 - \beta)(U + e))}{1 - \beta} \leq - \frac{g((1 - \beta)U)}{1 - \beta}$$

with $g = u^{-1}$. Such value exists since by the mean value theorem we have

$$g((1 - \beta)(U + e)) - g((1 - \beta)U) = g'((\xi)(1 - \beta)e$$

for some $\xi \in [(1 - \beta)U, (1 - \beta)(U + e)]$, and $g'((\xi)e = \frac{1}{u'((c_\xi))}$ with $c_\xi \to \infty$ as $\xi \to \infty$.

\(^5\)The feasibility correspondence can also be shown to be continuous. The details of this part of the proof follow SLP, Theorems 3.4 and 3.5, and can be made available upon request.

\(^6\)This constraint only defines the domain of $U^y$, which is an interval whenever $u$ is unbounded below.
scheme for continuation utilities as that under $U$ and reduce $c$ so to satisfy the new $PK^i(U', h)$. This possibility is always available to the planner since it is easy to see that $IC^i(U, h)$ does not depend on $z = u(c)$ or on $U$. Finally, that $T^i$ preserves monotonicity in $h$ can be shown using the fact that $W(U, h^i) \geq V(U, h^f)$ and that the incentive constraint is relaxed when $h$ increases and both $W$ and $V$ are concave.\(^7\) Since this set of functions when endowed with the sup metric constitutes a complete metric space, we can directly apply the theorems in SLP, Chs. 4.2 and 9.2. The non-standard part of the proof consists in showing that the maximization and integration operator stated in (3) in the main text preserves such properties as well. It is easy to see that monotonicity and boundedness are preserved by such operator. We will hence focus on concavity and joint continuity.

(i) The concavity of $W$ is obvious. Concavity of $V$ can be guaranteed under very general conditions. The idea is that with a continuum of shocks we can ‘convexify’ the upper envelope function $V$. Clearly, we are done if we show that the ipo-graph of $V(\cdot, h)$ is a convex set for all $h$.

**Lemma A1:** The ipo-graph of $V(\cdot, h)$ is a convex set for all $h$.

**Proof:** Let $D$ be the domain of $U, B$ a large number, and define the ipo-graph of $V$ given $h$ as

$$F^V(h) = \text{Gr}V(\cdot, h) = \{ x \in \mathbb{R}^2 : y_1 \in D, -B \leq y_2 \leq V(x_1, h) \}.$$  

Now, for each $h \in H$ and $y = s, f$, the correspondence $F^V(h)$ generates the following set

$$A(h) = \left\{ y \in \mathbb{R}^2 : \exists \text{ a pair of integrable functions } (u, v) \text{ such that } (u, v)(x) \in F^V(h) \text{ for all } x \in [0, 1] \text{; and } \int_0^1 u(x) \, dx = y_1; \int_0^1 v(x) \, dx = y_2 \right\}. \tag{1}$$

Notice the two following facts. First, the function $V(U, h)$ defined in problem (3) in the main text is the upper boundary of such set. That is, $A(h)$ is the graph of $V(\cdot, h)$. Second, $A(h)$ is the convex hull of the correspondence $F^V(h)$. For example a particular $y \in A(h)$ can be such that it is derived by selecting $y \in F^V(h)$ whenever $x \in [0, \lambda]$ and $y \in F^V(h)$ whenever $x \in [\lambda, 1]$. In

\(^7\)For example, assume that $UI$ is implemented at $(U, h)$, and consider $h' \geq h$. We want to satisfy the promise keeping constraint with the same $U^f$, i.e.:

$$U = u(c) - e + \beta \left[ \pi(h)U^* + (1 - \pi(h))U^f \right] = u(c) - e + \beta \left[ \pi(h')U^* + (1 - \pi(h'))U^f \right].$$

In order to compute $\hat{U}^*$ assume that in case of successful search the planner gives $U^*$ to the agent with probability $\gamma$ so that $\gamma \pi(h') = \pi(h)$, and $U^f$ otherwise (notice that $U^* \geq U^f$ since it might be the case that at some point during unemployment the agent supplies positive effort). Let $(h')^f = (1 - \delta) h'$. The net return for the planner in the next period is

$$\pi(h') \left[ \gamma W(U^*, (h')^f) + (1 - \gamma) W(U^f, (h')^f) \right] + (1 - \pi(h'))V(U^f, (h')^f) = \pi(h) W(U^*, (h)^f) + (1 - \pi(h)) V(U^f, (h')^f) + \left[ \pi(h') - \pi(h) \right] \left[ W(U^*, (h')^f) - V(U^f, (h')^f) \right],$$

which is greater than the next period planner’s return under $h^f = (1 - \delta) h$ because obviously $W(U, (h')^f) \geq W(U, h^f)$ for any $U$, $V$ is increasing in $h$ by the inductive assumption, and clearly $W(U, h) > V(U, h)$ for any pair $(U, h)$. 


this case, \( y = \lambda \bar{y} + (1 - \lambda) y \). This shows that \( coF^V(h) \subset A(h) \). Moreover, it can be shown that \( A(h) \subset coF^V(h) \).\(^8\)

(ii) We now turn to the issue of joint continuity. That \( W \) is jointly continuous is obvious. Moreover, notice that whenever each is jointly continuous then \( V = \max_i V^i \) is jointly continuous. Consider again the correspondence \( F^V(h) \). Since \( V \) is jointly continuous then \( F^V(h) \) changes continuously with \( h \) as well, and so does its convex hull \( A(h) \) (Aumann, 1965, Corollary 5.2). We hence have shown that the ipo-graph of \( V(\cdot, h) \) describes a continuous correspondence as a function of \( h \). But this implies that \( V \) is jointly continuous. In summary, we have shown the following:

**Lemma A2:** Whenever the upper envelope function \( V \) is jointly continuous in \((U, h)\), \( V \) is jointly continuous. Q.E.D.

(iii) Given concavity, differentiability of \( V^i \) together with the envelope theorem can now be shown by using standard arguments (see SLP, Theorems 4.11-12). Regarding the differentiability of \( V \) notice, that \( V \) is clearly differentiable at all interior points where it is linear and at all interior points where it coincides with one specific \( V^i \). It remains to show that it is differentiable also at all boundary points of its linear portions. Denote \( U_0 \) one of such points. In this case we can apply directly the Benveniste and Scheinkman Lemma (e.g., see SLP, Theorem 4.10, page 84). To see that the conditions for its application are met notice that \( V(U, h) \geq V^i(U, h) \) for all \( U \) and that \( V(U_0, h) = V^i(U_0, h) \) with \( V \) concave and \( V^i \) concave and differentiable by the previous argument. The Benveniste and Scheinkman Lemma furthermore implies \( V_U(U_0, h) = V^i_U(U_0, h) \). Q.E.D.

**Proof of Proposition 1A**

For the sake of contradiction, suppose (without loss of generality) that there is an optimal plan \( W \) implementing \( SA \) in period \( t \) almost surely for all \( x_t \), and policy \( i \neq SA \) in period \( t + 1 \) for a positive measure of shocks \( x_{t+1} \). We claim that the stated sequence (\( SA \) followed by policy \( i \neq SA \)) cannot be part of an optimal program by showing that the planner can provide the agent with the same expected utility as under \( W \) at a lower cost, by designing an alternative plan \( W' \).

Notice first the following. Because of the optimality of full insurance if we let \( c_t \) the payment made in period \( t \) (when policy \( SA \) is implemented), then, \( c_t \) must be the same for all \( x_t \). Moreover, this same payment must be delivered (only for one period) in the following period of unemployment for all \( x_{t+1} \) (when possibly some other policy is implemented) as well.

The alternative plan \( W' \) uses a randomization between two branches. **Branch 1:** This branch is implemented with probability \( \beta \) (i.e. equal to the discount factor). In this branch, the new program \( W' \) implements exactly the same (randomization of) policies following \( SA \) in the original program \( W \), and it delivers the same lifetime utilities delivered there. **Branch 2:** With probability \( (1 - \beta) \),

\(^8\)There are several ways of showing this property (e.g., Abreu et al., 1990, Theorem 3, use the Krein-Milman theorem together with Proposition 6.2 in Aumann, 1965). By reading \( A(h) \) as the Aumann integral of \( coF(h) \), we can directly apply Theorem 3’ in Aumann (1965), and immediately get that \( A(h) = coF(h) \) (This is a very general property of the Aumann integral. Formal proofs of (extended) versions of this statement can be found in Richter, 1963; and Debreu, 1967, pages 369-370).
the planner implements SA forever and transfers to the agent consumption $c_t$ with certainty in each period.

Operationally, branch 2 is simple to implement and it delivers an ex-ante utility for the agent equal to $(1 - \beta) \frac{u(c_t)}{1 - \beta} = u(c_t)$. The easiest way of describing how the new plan is implemented in branch 2 is the following. Given $h_t$ and for each $x_t$ we can consider a new random variable $X_i$, whose realization will be denoted by $x_{t+1}$. Now, denote by $U_i^j(x_t, h_{t+1}, x_{t+1})$ the promised utility delivered to the agent by the original contract $W$ after history $(x_t, h_{t+1}, x_{t+1})$ following node $(U_t, h_t)$, where $h_{t+1} = (1 - \delta) h_t$, and $i$ is the policy implemented in the original program at this node. The new contract $W'$ delivers exactly this utility in branch 1, after the sequence of shocks $x_t$, and $x_{t+1}$ (remember that we are still in period $t$, so the level of human capital is $h_t$). Since utilities coincide, we will now slightly abuse notation. By construction, the new plan $W'$ hence provides the agent with the same ex-ante utility as under the original program $W$. Namely,

$$u(c_t) + \beta \int_0^1 \sum_{i \in \{SA, UI, JM\}} \int_{X^i(x_t)} U_i^j(x_t, h_{t+1}, x_{t+1}) \, dx_t \, dx_{t+1},$$

where $X^i(x_t)$ is the set of $x_{t+1}$ shocks for which the program implements policy $i$ in period $t + 1$.

As we explained above, the first addend represents the expected utility the agent obtains in branch 2, while the second component is obtained ex-ante by the agent in branch 1 (notice that it is multiplied by $\beta$ because branch 1 only occurs with probability $\beta$).

We will now show that $W'$ is also cost-reducing compared to $W$. It is easy to see that, if from period $t + 1$ onward the planner optimizes over policies and payments, the new plan delivers the following net returns to the planner

$$-c_t + \beta \int_0^1 \sum_{i \in \{SA, UI, JM\}} \int_{X^i(x_t)} V_i^j(U_i^j(x_t, h_{t+1}, x_{t+1}), h_t) \, dx_t \, dx_{t+1}. \quad (2)$$

while the net planner’s returns implied by the original program cannot be higher than

$$-c_t + \beta \int_0^1 \sum_{i \in \{SA, UI, JM\}} \int_{X^i(x_t)} V_i^j(U_i^j(x_t, h_{t+1}, x_{t+1}, h_{t+1}) \, dx_t \, dx_{t+1}. \quad (3)$$

The only difference between the two expressions appears in the second argument of the value functions $V_i^j$. It is now easy to see that the monotonicity of each $V_i^j$ in $h$ and the fact that $h_{t+1} = (1 - \delta) h_t \leq h_t$ means that the value implied by (2) dominates that in (3).\(^9\) The dominance is strict whenever for some

\(^9\)In general, let $X^SA$ be the set of $x_t$ shocks for which the original plan implements $SA$ in the first period, and denote by $\mu$ the measure of the set

$$\cup_{i=UI, JM} \cup_{x \in X^{JM}} X^i(x_t).$$

Then if $\mu > 0$, the whole proof goes through with three branches. One, with probability $1 - \mu$, where we follow the old plan. Another where we follow branch 1, with probability $\mu \beta$; and another branch - with probability $\mu (1 - \beta)$ - where we implement branch 2. Of course, if after $SA$, the policy $i \neq SA$ is implemented for the first time only after $n$ periods, than the randomization across the last two branches must be adjusted so that we have $\mu \beta^n$ and $\mu (1 - \beta^n)$ respectively instead, and so on.

\(^{10}\)Notice that the monotonicity in $h$ in Proposition 0 does not require positive consumption.
\(i \neq SA\), the set \(\cup_{x_t \in [0,1]} X^i (x_t)\) has positive measure. The idea is simple: the new program dominates the old essentially because it uses policies \(i \neq SA\) (i.e. \(UI\) or \(JM\)) for higher levels of human capital, which relaxes the incentive compatibility (IC) constraint and increases expected returns to search.

The final part of the proof hence requires to show that because of the concavity of the problem, such extra randomizations over different policies and utilities do not improve the planner’s net value.

Let us first assume, that in constructing the plan \(W'\), the planner does not necessarily stick to the policies suggested above, while keeping the same allocation of lifetime utilities. This modified new plan improves the planner returns even further while leaving the worker indifferent. Formally, since \(V \geq V^i\) for all \(i\), the following expression dominates (2):

\[
-c_t + \beta \int_0^1 \int_0^1 V \left( U^i (x_t, h_{t+1}, x_{t+1}), h_t \right) dx_t dx_{t+1}.
\]

(4)

Finally, we allow the planner to re-allocate utilities as well. Recall that \(U_t\) is the (common to both \(W\) and \(W'\) plans) ex-ante lifetime utility delivered to the agent. The value represented in (4) is hence dominated by

\[
-c_t + \sup_{\hat{U}(x_t, x_{t+1}) \in D} \int_0^1 \int_0^1 V \left( \hat{U} (x_t, x_{t+1}), h_t \right) dx_t dx_{t+1}
\]

s.t.

\[
\int_0^1 \int_0^1 \hat{U} (x_t, x_{t+1}) dx_t dx_{t+1} = U_t - u(c_t).
\]

Since \(h_t\) is constant and \(X\) is a continuous random variable with density, it is easy to see that it cannot matter for the value if the payoff irrelevant randomization is bi-dimensional or one-dimensional. In other terms, we have shown that the planner value at node \((U_t, h_t)\) in the original plan is dominated by the quantity

\[
(1 - \beta) V \left( \hat{U}_t^{B2}, h_t \right) + \beta V \left( \hat{U}_t^{B1}, h_t \right),
\]

where \(\hat{U}_t^{B2} = u(c_t) \frac{1}{1 - \beta}\), while \(\hat{U}_t^{B1} = U_t - u(c_t)\) as been defined above. In turn, since obviously

\[
(1 - \beta) \hat{U}_t^{B2} + \beta \hat{U}_t^{B1} = U_t,
\]

the concavity of \(V\) implies that this quantity is dominated by \(V (U_t, h_t)\), i.e. since \(V\) is concave, the initial randomization across the two branches is not necessary either. This concludes the proof since we have generated a contradiction to the fact that the original program was optimal, only by using the fact that \(W\) contemplated to switch between \(SA\) and some other policy \(i \neq SA\). Q.E.D.

**Proof of Proposition 2**

See Appendix A of the main text for \(SA\) and \(JM\). Here, we focus on the absorbing nature of \(UI\) in absence of human capital dynamics. Let us start by stating the first-order and envelope conditions
under $UI$:

\[-V_U(U) = -V_U^{UI}(U) = \frac{1}{u'(c)}, \]

\[-V_U(U^f) = \frac{1}{u'(c)} - \mu \frac{\pi}{1 - \pi}, \]

\[-W_U(U^s) = \frac{1}{u'(c)} + \mu, \]

where $\mu \geq 0$ is the multiplier on the incentive compatibility constraint. It is useful to begin by stating the conditions under which the incentive constraint is binding and promised utility declines.

**Lemma A3:** At any $U_0$ where $UI$ is optimal, we must have $U^f < U_0$. Moreover, if $V$ is strictly concave to the left of $U_0$, then it must be that $\mu > 0$.

**Proof.** It is immediate from (5) and the concavity of $V$ that if the incentive compatibility constraint holds, then $U^f < U_0$. Now assume $\mu = 0$. Note first that incentive compatibility implies that $U_0 \geq u(c) + \beta U^f$. Secondly, when $\mu = 0$ the special form of $W$ together with the last condition in (5) imply that $u(c) = (1 - \beta)U^s$. It is now clear that if $U^f \geq U_0$ then from the incentive compatibility constraint we would have $U^s > U_0$ as well, and the inequality $U_0 \geq u(c) + \beta U^f$ could never be satisfied.

Finally, consider the case where $V$ is strictly concave to the left of $U_0$. Since when $\mu = 0$ the first two conditions in (5) imply that $V_U(U_0) = V_U(U^f)$ and by the strict concavity to the left we get $U_0 \geq U^f$, the first part of the proof leads to a contradiction. Q.E.D.

Since each function $V^i$ is continuous, if for different levels of utility different policies are preferred, the value functions must cross each other. We now present two lemmas. Lemma A4 shows that the slope of $V^{UI}$ is always more negative than that of $V^{SA}$. Lemma A5 ranks the slopes of $V^{UI}$ and $V^{JM}$ at the crossing point (if any). In both cases, the implication is that $V^{UI}$ crosses from above (hence only once) the other two functions. In turn, these properties imply that $UI$ can never be followed by either $SA$ or $JM$. The reason is simple. First, the stated ranking of slopes implies that at the left of a $U_0$ for which $UI$ is optimally implemented (with probability one) we can only have $UI$. But then $V$ must be strictly concave at the left of such $U_0$ since for all such utility values we have $V(U) = V(U) = V^{UI}(U)$. From Lemma A3 we have that in this case $\mu > 0$ and $U^f < U$. The same ranking then implies that at $U^f$ the only optimal program is $UI$, and so on.$^{11}$

We now show the mentioned ranking on the slopes.

**Lemma A4:** For every $U$, we have that $V^{SA}_U(U) \geq V^{UI}_U(U)$.

**Proof.** The first order and envelope conditions for the program $SA$ are:

$$V^{SA}_U(U) = -\frac{1}{u'(c^{SA})} = V_U(U^f_{SA}).$$

$^{11}$Formally, whenever $\mu > 0$ the optimality conditions (5) imply $V_U(U) = V^{UI}_U(U) < V_U(U^f)$, and any lottery (indexed by $x$) implementing $U^f$ solves $V_U(U^f) = V_U^0(U(x)) = V_U(U(x))$ for (almost) all $U(x)$ promised utilities in such a lottery. The concavity of $V$ hence implies that $U(x) < U_0$ and the above result implies that neither $SA$ nor $JM$ can be implemented at any of such $U(x)$ (almost surely).
where \( c^{SA} \) and \( U^f_{SA} \) are the optimal consumption and continuation utility under \( SA \), and solve \( U = u(c^{SA}) + \beta U^f_{SA} \). The optimality conditions (5) for \( UI \) imply
\[
V^UI_U(U) = -\frac{1}{w'(c^{UI})} \leq V_U(U^f_{UI}),
\]
where \( c^{UI} \) and \( U^f_{UI} \) are respectively the optimal consumption and continuation utility under \( UI \). Notice that if \( c^{SA} \leq c^{UI} \) the envelope condition and the concavity of \( u \) implies the desired result.

We now show that it cannot be that \( c^{SA} > c^{UI} \). Rearranging the incentive and promise keeping constraint we get
\[
U \geq u(c^{UI}) + \beta U^f_{UI} \tag{6}
\]
If \( c^{SA} > c^{UI} \) in order to satisfy (6) and \( U = u(c^{SA}) + \beta U^f_{SA} \) with the same \( U \) it must be that \( U^f_{UI} > U^f_{SA} \). The concavity of \( V^f \) and the envelope conditions however imply that whenever \( U^f_{UI} > U^f_{SA} \), we have
\[
-\frac{1}{w'(c^{UI})} \leq V_U(U^f_{UI}) \leq V_U(U^f_{SA}) = -\frac{1}{w'(c^{SA})}.
\]
This implies \( u'(c^{UI}) \leq u'(c^{SA}) \). Since \( u \) is strictly concave we have \( c^{UI} \geq c^{SA} \). A contradiction. Q.E.D.

**Lemma A5:** Let \( U_0 \) be such that \( V^{JM}(U_0) = V^{UI}(U_0) \). Then, we have \( V^{JM}_U(U_0) \geq V^{UI}_U(U_0) \). Hence, \( V^{JM} \) and \( V^{UI} \) can cross each other at most once.

**Proof.** The intuition of the proof is the following. We know that \( JM \) is an absorbing policy. This implies that when the worker enters into \( JM \), although he will be asked to always supply positive effort, he will never be subject to random consumption due to the presence of the incentive constraint. On the other hand, from Lemma A3 and A4, we know that after implementing \( UI \), an optimal program never uses the policy \( SA \). This implies that once the worker is assigned to policy \( UI \), he will always be required to supply positive effort. In addition to that, he will possibly face random consumption. This extra randomization on consumption will induce extra costs which make \( V^{UI} \) more negatively sloped than \( V^{JM} \). Formally, we have:

**Lemma A6.** Consider the problems of implementing \( UI \) and \( JM \), respectively, at a generic level of utility \( U_0 \). In the following two cases we have that \( V^{JM}_U(U_0) \geq V^{UI}_U(U_0) \). (i) If at \( U_0 \) after \( JM \) the program implements \( SA \) almost surely (no matter what is implemented after \( UI \)). (ii) If at \( U_0 \), after \( JM \), \( UI \) is never used, while after \( UI \), \( SA \) is never used.

We postpone the proof of Lemma A6. Let us first continue with the proof of Lemma A5, taking for granted the results of Lemma A6.

Clearly, if at \( U_0 \) \( V^{JM} \) and \( V^{UI} \) have the same slope we are done. So assume they have different slopes. In this case, none of the two policies \( UI \) or \( JM \) can be optimally implemented with probability one at this point.\(^{12}\) We are hence forced to study what happens ‘off the optimum’.

\(^{12}\)It is easy to see it graphically: since at \( U_0 \) the slope of the two functions is different, in any sufficiently small neighborhood of \( U_0 \) a ‘straight line’ induced by a random assignment clearly dominates the deterministic assignment to either policy. The graphical representation is formally correct since \( V^i \) are continuously differentiable.
First of all notice, that \( V^{JM}(U_0) = V^{UI}(U_0) \) implies that the (‘off the optimum’) problem of implementing policy \( UI \) at \( U_0 \) must display \( \mu > 0 \). Otherwise, we would have \( V^{JM}(U_0) < V^{UI}(U_0) \) since the same contract will be implemented under both \( UI \) and \( JM \), and by implementing \( UI \) the planner saves \( \kappa^{JM} \). If we compare the first order conditions across the two policies, we hence get

\[
V^{JM}_U(U_0) = V_U\left(U^{f}_{JM}\right), \quad \text{and} \quad V^{UI}_U(U_0) < V_U\left(U^{f}_{UI}\right), \tag{7}
\]

where \( U^{f}_{JM} \) and \( U^{f}_{UI} \) are the optimal continuation utilities delivered under the policies \( JM \) and \( UI \) respectively. If \( U^{f}_{JM} \leq U^{f}_{UI} \) we are done since from (7) and the concavity of \( V \) we get

\[
V^{JM}_U(U_0) = V_U\left(U^{f}_{JM}\right) \geq V_U\left(U^{f}_{UI}\right) > V^{UI}_U(U_0),
\]

as desired.

We will hence consider the case \( U^{f}_{JM} > U^{f}_{UI} \), and study the combination of policies implementing these different levels of continuation utilities.

Let us start with \( U^{f}_{JM} \). If \( U^{f}_{JM} \) is delivered by implementing \( SA \) almost surely, part \( (i) \) of Lemma A6 delivers the desired result. The complementary case is when \( U^{f}_{JM} \) is delivered by using policy \( UI \) with positive probability. We first claim that \( U^{f}_{JM} \) cannot be delivered by implementing \( JM \) with positive probability. If this was the case, then the first order conditions would imply the following condition for all \( U(x) \) included in the randomization

\[
V^{JM}_U(U_0) = V_U\left(U^{f}_{JM}\right) = V^{JM}_U(U(x)).
\]

Since \( V^{JM} \) is strictly concave, we would have \( U_0 = U(x) \); and this would contradict the fact that at \( U_0 \) the policy \( JM \) is not optimally implemented.

Since both \( V^{UI} \) and \( V^{SA} \) are strictly concave the randomization implementing \( U^{f}_{JM} \) must take the following simple form. Let \( \lambda > 0 \) be the measure of \( x \) shocks for which \( UI \) is implemented. We have \( U^{f}_{JM} = \lambda U^{f,UI}_{JM} + (1 - \lambda) U^{f,SA}_{JM} \), where \( U^{f,UI}_{JM} \) and \( U^{f,SA}_{JM} \) are the continuation utilities delivered under policies \( UI \) and \( SA \) respectively after \( JM \). Optimality requires

\[
V^{JM}_U(U_0) = V_U\left(U^{f}_{JM}\right) = V^{UI}_U\left(U^{f,UI}_{JM}\right) = V^{SA}_U\left(U^{f,SA}_{JM}\right), \tag{8}
\]

which implies \( U^{f,SA}_{JM} \geq U^{f,UI}_{JM} \geq U^{f,UI}_{JM} \) by Lemma A4 and the definition of \( U^{f}_{JM} \).

Now notice that if \( U^{f,UI}_{JM} \leq U_0 \), we are done since from (8) and the fact that each \( V^{UI} \) is strictly concave, we get \( V^{JM}_U(U_0) = V^{UI}_U\left(U^{f,UI}_{JM}\right) \geq V^{UI}_U(U_0) \) as desired.

We now aim at ruling out the alternative possibility - \( U^{f,UI}_{JM} > U_0 \) - through a contradiction. Consider first how \( U^{f}_{UI} \) is implemented, and note that, from (8), (7) and the strict concavity of \( V^{UI} \), \( U^{f,UI}_{JM} > U_0 \) implies \( V^{JM}_U(U_0) = V_U\left(U^{f}_{JM}\right) = V^{UI}_U\left(U^{f,UI}_{JM}\right) < V^{UI}_U(U_0) < V\left(U^{f}_{UI}\right) \). It is hence easy to see that policy \( SA \) is not used in implementing \( U^{f}_{UI} \).\(^{13}\)

\(^{13}\) Let \( U^{f,SA}_{UI} \) be the promised utility under policy \( SA \) implementing \( U^{f}_{UI} \). If \( SA \) is used with positive probability in
We hence must randomize over UI and JM. If we denote by $U^{f,UI}_0$ and $U^{f,JM}_0$ the respective continuation utilities, we get $U^{f,UI}_0 = qU^{f,UI}_{U1} + (1-q)U^{f,JM}_{U1}$, with
\[
V^{UI}_U(U_0) < V_U(U^{f,UI}_0) = V^{UI}_U(U^{f,UI}_0) = V^{JM}_U(U^{f,JM}_0).
\]

(9)

Finally, consider the possibilities for implementing $U_0$. First of all notice, that since, by definition, at $U^{f,JM}_0$ policy UI is implemented with probability one, Lemma A4 excludes the possibility that SA could be optimally implemented at any $U \leq U^{f,UI}_0$. Recall that we are considering the case where $U^{f,JM}_0 > U_0$, hence the program will not use SA to implement $U_0$. Since none of the two policies is optimal we must have
\[
V^{UI}_U(U_0) > V_U(U_0) > V^{JM}_U(U_0).
\]

(10)

Now $U_0 = pU^{f,UI}_0 + (1-p)U^{f,JM}_0$ with $p \in (0,1)$ is the expression of $U_0$ in terms of its continuation utilities $U^{f,JM}_0$ and $U^{f,UI}_0$ delivered with policies JM and UI respectively. Optimality implies
\[
V_U(U_0) = V^{UI}_U(U^{f,UI}_0) = V^{JM}_U(U^{f,JM}_0).
\]

(11)

Consider now the value $U^{f,JM}_0$ more in detail. This value exists since by assumption the program uses a randomization involving JM with positive probability otherwise at $U_0$ UI would be optimal. On one hand, by concavity of $V^{JM}$, condition (10) implies that $U_0 > U^{f,JM}_0$. On the other hand, since at $U^{f,JM}_0$ the implemented policy is JM, and we know that JM is absorbing, the function (and its slope) at this point corresponds to that of implementing JM forever with probability one. Now consider the slope of $V^{UI}$ at $U^{f,JM}_0$. Since by assumption $U^{f,JM}_0 < U_0$, the same argument that rules out the use of SA for $U_0$ also rules out the use of SA for $U^{f,JM}_0$. In this case, we can hence use part (ii) of Lemma A6 to show that the slope of $V^{UI}$ at this point is less than that of $V^{JM}$. Formally, we have
\[
V^{UI}_U(U^{f,JM}_0) \leq V^{JM}_U(U^{f,JM}_0).
\]

This, together with (11), and the strict concavity of $V^{UI}$ implies that $U^{f,UI}_0 \leq U^{f,JM}_0$. However, since $U^{f,JM}_0 < U_0$, by the definition of $U_0$ we get $U^{f,UI}_0 > U_0$, a contradiction.

To conclude the proof we show Lemma A6.

Proof of Lemma A6. (i) Consider first the case where at $U_0$, we start with JM and after this first period of job monitoring the program implements utility $U^{f,JM}_0$ by using policy SA almost surely. The optimality conditions in this case will be $V^{JM}_U(U_0) = V_U(U^{f,JM}_0) = W_U(U^{s,JM}_0) = V^{SA}_U(U^{f,JM}_0) = -\frac{1}{u^{L}\sigma^{M}}$. Moreover, recall that SA is an absorbing policy. Since the agent will be fully insured both implementing $U^{f,JM}_0$, we have
\[
V^{SA}_U(U^{f,SA}_{U1}) = V_U(U^{f,SA}_{U1}) = V_U(U^{f,UI}_0) > V^{UI}_U(U^{f,UI}_0) = V_U(U^{f,JM}_0) = V_U(U^{f,JM}_0).
\]

Since $V$ is concave we have $U^{f,SA}_{U1} < U^{f,UI}_0$. We hence have that $V^{SA}$ is dominated by $V^{UI}$ for a large $U = U^{f,UI}_0$, and at the same time $V^{SA}$ dominates $V^{UI}$ for a smaller $U = U^{f,SA}_{U1}$. This contradicts Lemma A4.

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across the different policies and across the different states during $JM$, in this case, the promised utility can be expressed as\(^{14}\)

$$U_0 = \frac{z_{0JM}}{1 - \beta} - e. \tag{12}$$

where $z_{0JM} = u(c_{0JM})$.

We now have to describe how $U_0$ is generated when the first policy to be implemented is $UI$. Since both $SA$ and $JM$ are absorbing policies we can describe, without loss of generality, the whole set of possible sequences of policies delivering $U_0$ as follows.

Let $\mu_i^t$ with $i = JM, SA, UI$ be the probability of switching to (or remaining in) policy $i$ at time $t \geq 0$. Denote by $z_i^t = u(c_i)$ the utility payment during unemployment, and by $z_i^t$ the utility payment during employment if the implemented policy is $UI$. Moreover, for $t \geq 1$, denote by $U_i^t$ the lifetime utility during unemployment and by $U_i^t = \frac{z_i^t}{1 - \beta}$ the lifetime utility during employment whenever the policy implemented in the previous period was $UI$. Finally, recall that during $JM$ the worker is fully insured across time and states. The utility delivered in an absorbing $JM$ node is then $\frac{z_i^t}{1 - \beta} - \frac{e}{1 - \beta (1 - \pi)}$.

When the worker enters into $SA$, her utility is obviously $\frac{z_i^t}{1 - \beta}$. Since in period zero this program implements $UI$, we have

\[
U_0 = z_{0UI}^t - e + \beta \pi U_1^s + \beta (1 - \pi) U_1^f
\]

\[
= z_{0UI}^t - e + \beta \pi U_1^s + \beta (1 - \pi) \left\{ \mu_1^{JM} \left[ \frac{z_1^f}{1 - \beta} - \frac{e}{1 - \beta (1 - \pi)} \right] + \mu_1^{SA} \right\}
\]

\[
= z_{0UI}^t - e + \beta \pi U_1^s + \beta (1 - \pi) \mu_1^{UI} \left[ z_1^f - e + \beta \pi U_2^s \right] + \beta^2 (1 - \pi)^2 \mu_1^{UI} U_2^f
\]

\[
+ \mu_1^{JM} \left[ \frac{z_1^f}{1 - \beta} - \frac{e}{1 - \beta (1 - \pi)} \right] + \mu_1^{SA} \frac{z_1^f}{1 - \beta}
\]

\[
= \ldots
\]

\[
= z_{0UI}^t - e + \beta \pi \frac{z_1^s}{1 - \beta} +
\]

\[
+ \sum_{t=1}^{\infty} ((1 - \pi)\beta)^t \left( \mu_1^{UI} \mu_2^{UI} \cdots \mu_t^{UI} \right) \left[ z_t^f - e + \beta \pi \frac{z_{t+1}^s}{1 - \beta} \right]
\]

\[
+ \sum_{t=1}^{\infty} ((1 - \pi)\beta)^t \left( \mu_1^{UI} \cdots \mu_{t-1}^{UI} \cdot \mu_t^{JM} \right) \left[ z_t^f - e + \beta \pi \frac{e}{1 - \beta (1 - \pi)} \right]
\]

\[
+ \sum_{t=1}^{\infty} ((1 - \pi)\beta)^t \left( \mu_1^{UI} \cdots \mu_{t-1}^{UI} \cdot \mu_t^{SA} \right) \left[ \frac{z_t^f}{1 - \beta} \right].
\]

Notice that convergence of the series is not an issue as all payments are bounded above by $z_1^s$, and the other multiplicative term at time $t$ is non-negative and bounded above by $((1 - \pi)\beta)^t$, which converges to zero.\(^{15}\) Now notice, that the expression reported above in the last line describes the

\(^{14}\)Notice that the form of $W$ implies that $U_{JM}^s = \frac{z_{JM}^s}{1 - \pi}$.

\(^{15}\)Hence the series must have a limit. Since $U_0 > -\infty$ this limit must be finite.
utility value $(1 - \beta) U_0$ as a convex combination of payments $z_0^{UI}, z_t^f$ and $z_t^s$, with weights $(1 - \beta)$ for $z_0^{UI}; (1 - \beta) \beta^t (1 - \pi)^t \mu_{1^{UI}}^{t} \ldots \mu_{i-1}^{UI} \mu_{i}^{t}$, $\beta^t (1 - \pi)^t \beta \pi \mu_{1^{UI}}^{t} \ldots \mu_{i-1}^{UI} \mu_{i}^{t}$, and $\beta^t (1 - \pi)^t \beta \pi \mu_{1^{UI}}^{t} \ldots \mu_{i-1}^{UI} \mu_{i}^{t}$; $t = 1, 2, ...$ for $z_t^f$ (we will denote them as $k_t^{i,f}$), and weights $\beta^t (1 - \pi)^t \mu_{1^{UI}}^{t} \ldots \mu_{i-1}^{UI} \mu_{i}^{t}$; $t = 1, 2, ...$ for $z_t^s$ (denoted, for notational simplicity, as $k_t^{i,s}$). These weights obviously sum to one. The value $(1 - \beta) U_0$ is generated by all this, minus an expected discounted value of search effort, multiplied by $(1 - \beta)$, which can be denoted by $(1 - \beta) \Gamma e$, with $\Gamma \geq 1$. Let us now compare this expression with (12). First, observe that $z_0^{JM}$ in (12) can always be decomposed as a convex combination of $z_t^{JM}$'s with the same weights as in the expression above. That is:

$$(1 - \beta) U_0 = (1 - \beta) z_0^{JM} + \sum_{t,i} k_t^{i,f} z_t^{JM} + \sum_{t,i} k_t^{i,s} z_t^{JM} - (1 - \beta) e. \quad (13)$$

Second, under any circumstance (no matter whether the incentive compatibility constraint is binding or not) optimality implies the following first order conditions for $UI$

$$g'(z_t^f) = \pi g'(z_{t+1}^f) + (1 - \pi) g'(z_{t+1}^f), \text{for any } t = 0, 1, ...$$

Now, appropriately decomposing the constant payments (both across policies each period and across states whenever either $JM$ or $SA$ are implemented), one can set the following recursive formula that links envelope conditions at each period of $UI$:

$$g'(z_t^f) = (1 - \beta) \left\{ g'(z_t^f) + \frac{\beta \pi}{1 - \beta} g'(z_{t+1}^f) + \beta (1 - \pi) \left( \mu_{1}^{UI} g'(z_{t+1}^f) + \frac{\mu_{1}^{JM} + \mu_{1}^{SA}}{1 - \beta} g'(z_{t+1}^f) \right) \right\}. \quad \text{This formula applied infinitely many times delivers}^{16}$$

$$g'(z_0^{UI}) = (1 - \beta) g'(z_0^{UI}) + \sum_{t,i} k_t^{i,f} g'(z_t^f) + \sum_{t,i} k_t^{i,s} g'(z_t^s)$$

as a convex combination of $g'$'s where the weights are exactly those associated to the $z_t^f$ and $z_t^s$ above. Since $g'$ is convex, it must be that $z_0^{UI} \geq (1 - \beta) z_0^{UI} + \sum_{t,i} k_t^{i,f} z_t^f + \sum_{t,i} k_t^{i,s} z_t^s$. But then notice that we have

$$(1 - \beta) U_0 = (1 - \beta) z_0^{UI} + \sum_{t,i} k_t^{i,f} z_t^f + \sum_{t,i} k_t^{i,s} z_t^s - (1 - \beta) \Gamma e \quad (14)$$

with $\Gamma \geq 1$. By comparing (14) with (13) it is now easy to see that in order for $U_0$ to be the same number it must be that $g'(z_0^{UI}) \geq g'(z_0^{JM})$. The envelope condition then delivers the desired result.

$(ii)$ The proof of the second part is similar. The only difference is that for the $U_0$ implemented starting with $JM$, instead of (12) we now have

$$U_0 = \frac{z_0^{JM}}{1 - \beta} - G e$$

---

16: Notice again that we do not have any technical issue about convergence since by convexity the derivatives are bounded above by $g'(z_0^f)$ (and below by zero), and the weights are all bounded by coefficients converging to zero.
where $1 \leq G \leq \frac{1}{1-\beta(1-\pi)}$, with the last inequality strict if the program uses $SA$ with positive probability along the plan. The same argument used above also delivers the same expression as (14) with now $\Gamma = \frac{1}{1-\beta(1-\pi)} \geq G$ because $SA$ is never used in this case. So, exactly the same argument as that followed above delivers $g'(z_{0}^{UI}) \geq g'(z_{0}^{JM})$ as desired. Q.E.D.

PROOF OF COROLLARY TO PROPOSITION 2.

The ranking between $V^{UI}$ and $V^{JM}$ at the crossing point has been shown in Lemma A5. The proof that $\hat{V}_{U}^{SA}(U) \geq V_{U}^{JM}(U)$ follows exactly the same lines of Lemma A6. Under $SA$ the agent is fully insured and $a = 0$ forever, hence $U = \frac{z_{0}^{SA}}{1-\beta}$, while under $JM$ the agent is required to supply positive effort at least in the first period. It really does not matter what the exact sequence is, we will always have $U_{0} = \frac{z_{0}^{JM}}{1-\beta} - Ge$ with $G > 0$. The result is then obtained by the envelope theorem. Q.E.D.

DYNAMICS OF $U$ DURING POLICY UI

The proof of Proposition 5 required a characterization on the dynamics of $U$ during $JM$. The dynamics during $SA$ are obvious. For completeness, we now briefly describe the dynamics of $U$ during $UI$.

Lemma A9: If at $(U, h)$ the implemented policy is $UI$, then $U_f \leq U$. Moreover, if $V$ is either strictly submodular or strictly concave or both then the incentive constraint must be binding.

Proof (Sketch): The idea of the proof is quite simple, and it can be made available upon request. From the incentive compatibility constraint, the law of motion for $U_0$ during $UI$ solves $U_0 \geq u(c_0) + \beta U_f$. Moreover, in all subsequent periods the law of motion for $U$ can take two forms. It can follow $U_t = u(c_t) + \beta U_{t+1}^f$ -this happens whenever we implement $SA$, or we implement $UI$ and the incentive compatibility constraint is binding. Alternatively, during $JM$ or whenever in $UI$ the incentive constraint is slack we have $U_t = u(c_t) - e + \beta \left[ \pi(h) \frac{u(c_t)}{1-\beta} + (1-\pi(h_t)) U_{t+1}^f \right]$. Since by the envelope conditions $c_t \leq c_0$ for all histories, by induction it is easy to show that we must have $U_f < U$.\footnote{The formal proof is now straightforward but notationally quite heavy because we must work with histories of $x$ shocks (along the lines of Lemma A6). The key step is to show that $U_T$ is bounded above for all histories so that $\lim_{T \to \infty} \beta^T U_T \leq 0$. But we know this is the case since for sufficiently large levels of utility the only optimal program is $SA$, with full insurance (see discussion in Proposition 0).}

If the incentive constraint is slack, i.e. $\mu = 0$, the first order conditions imply $V_{U}(U, h) = V_{U}^{UI}(U, h) = V_{U}(U_f, h_f)$. Whenever $V$ is submodular they imply $U_f > U$ which contradicts the above property, hence we must have $\mu > 0$. Q.E.D.

PROOF OF PROPOSITION 6.

The line of proof is an extension to that adopted for Proposition 1 (SA absorbing). For the sake of contradiction, assume without loss of generality that at node $(U_t, h_t)$ there is an optimal plan $W$
implementing \( JM \) in period \( t \) almost surely for all \( x_t \), and \( UI \) in period \( t + 1 \) for a full measure of shocks. We allow for any random plan from period \( t + 1 \) onward.

The idea of the proof is again that we can construct an alternative plan delivering the same ex-ante expected utility to the agent, and larger net returns for the planner. This plan can be generated as follows. First, construct a plan \( W' \) delivering the same ex-ante expected utility to the agent, and same net returns for the planner. Next, show that in the new plan the IC constraints are relaxed with respect to \( W \) in some states, thus costs can be further reduced, thus \( W \) cannot be optimal.

The program \( W' \) consists again of two branches. \textit{Branch 1:} This branch is implemented with probability \( \beta (1 - \pi) \). To construct this branch, we start from period \( t + 1 \) and look at what policy is implemented in period \( t + 1 \) in the original program \( W \). Each time that in the original plan we see \( UI \) or \( JM \) we retain \( UI \) or \( JM \), but whenever in the old plan we see the sequence \( SA \rightarrow SA \rightarrow SA \ldots \) (notice that from Proposition 1 this is the only possibility) then we substitute this sequence with \( JM \rightarrow SA \rightarrow SA \ldots \). Next, we shift the whole plan backward by one period, so this branch will start with \( UI \) at time \( t \). \textit{Branch 2:} This branch has probability \( 1 - \beta (1 - \pi) \). To construct this branch, we replicate exactly branch 1 with the following amendment: each time in branch 1 we see \( UI \), we replace it with \( JM \). In particular, since branch 1 starts with \( UI \), in branch 2 the new contract \( W' \) starts with \( JM \) at time \( t \).\footnote{In general, let \( X_{JM} \) be the set of \( x_t \) shocks for which the original plan implements \( JM \) in the first period, \( X_{UI} (x_t) \) the set of \( x_{t+1} \) shocks for which the original plan implements \( UI \) at date \( t + 1 \), and denote by \( \mu \) the measure of the set \( \cup_{x \in X_{JM} \cap X_{UI} (x_t)} \). Then if \( \mu > 0 \), the whole proof goes through with three Branches. One, with probability \( 1 - \mu \) where we follow the old plan. Another one where we follow what we described above form branch 1, with probability \( \mu \beta (1 - \pi) \); and yet another branch–with probability \( \mu (1 - \beta (1 - \pi)) \)–where we implement the same plan as that described in branch 2. Of course, if after \( JM \), the policy \( UI \) is implemented for the first time only after \( n \) periods, then the randomization across the last two branches must be adjusted so that we have \( \mu \beta^n (1 - \pi)^n \) and \( \mu (1 - \beta^n (1 - \pi)^n) \) respectively instead, and so on.}

We now show that both the planner net returns and the agent ex-ante expected utility are the same in the two programs.

\textbf{Monitoring costs:} We now prove equivalence of the two programs with respect to the monitoring costs for the planner in two steps.

\textit{Step 1):} In branch 1, the only additional monitoring costs with respect to the continuation of the old plan \( W \) is borne whenever the latter contemplated \( SA \) for the first time. Let \( q \) be the probability of such event. Note now that this same additional cost occurs also in the second branch, which has weight \( 1 - \beta (1 - \pi) \). So, up to now, we conclude that the new plan yields additional costs with respect to the \( t + 1 \)-continuation of the old plan equal to \( q \kappa \).

\textit{Step 2):} Recall that in the first period of program \( W \) the planner pays an initial monitoring cost \( \kappa \) with certainty. Consider now branch 2. If the plan always implemented \( JM \), we had a cost \( \kappa \) with probability \( (1 - \pi) \) every period, so we would get precisely a discounted present value of the monitoring costs equal to \( \kappa \), which is the initial cost of the old plan. But the cost in branch 2 is smaller since there is a tail, occurring with probability \( q \), where the cost is not paid because the new
plan picks SA from then onward. The reduction in cost associated to this tail, compared to the the unlimited implementation of JM, is $-q \frac{\kappa}{1-\beta(1-\pi)}$, which, weighted with the probability of branch 2 occurring, yields exactly $-q\kappa$, the excess cost of the new plan computed in Step 1, which proves that the expected discounted monitoring cost in the two programs must be the same.

**Expected wage:** The expected value of wage returns for the planner must also be the same in the two programs. The argument is as follows. Notice two facts. First, both branches of the new plan are constructed by looking at what happens in period $t+1$, with human capital $h_{t+1}$. Clearly, the effort level is $a = 0$ only under social assistance. In cross sectional terms, implementing JM for one more period whenever the old plan implemented SA hence guarantees that - again - $a = 0$ is chosen in the new plan only for values of human capital for which the old plan recommended $a = 0$, with exactly the same probabilities in the two programs. See more on that below in the effort cost computation.

**Payments and Agent Utility:** Let $c_t$ be the payment made under $W$ in period $t$ (notice that because of full insurance during JM, this must be the same for all $x_t$ and across states). In branch 2, the new plan transfers to the agent $c_t$ in any period, history of shocks and states. Her expected utility is hence $u(c_t) / (1 - \beta (1 - \pi))$. In branch 1 we have the following. First, every time we implement UI we pay the agent exactly what was paid to him in the old plan during this policy, both in case of success and of failure of search. The same is true for SA. That is, whenever we implement SA in the new plan we pay the agent exactly the same consumption she received in the old plan. The payments under JM are as follows: whenever JM is implemented in order to replace JM in the old plan, the payments are again exactly the same as those in the old plan, in all states. Finally, when JM is implemented (for one period) in order to replace the first period of SA then the transfers are those used in SA in the old plan.

It is easy to see that the payments are the same in the two programs. Branch 1 of the new plan can be thought of as the continuation of the old plan from time $t + 1$ onward, since it has weight $\beta (1 - \pi)$. In branch 2 payments are like in the old plan, so the discounted present value of payments in branch 2 equals $c_t$, the consumption of the first period $t$ in the old plan.

We now verify in detail that the ex-ante expected utility of the agent in the two programs is the same as well.

**Old Plan:** Let us start by describing the utility implied by the original plan $W$. By assumption, in the first two periods we have JM and then UI with probability one. This simplifies the notation a bit, as we can denote $z(h_t, x_t) = u(c_t)$ for all $x_t$ because of full insurance. We hence have

$$U^{JM}(h_t, x_t) = u(c_t) - e + \beta \left[ \frac{u(c_t)}{1-\beta} + (1-\pi) U^f(h_{t+1}, x_t) \right]$$

with

$$U^f(h_{t+1}, x_t) = \int_0^1 U^{UI}(h_{t+1}, x_t, x_{t+1}) \, dx_{t+1}$$

(15)

$$U^{UI}(h_{t+1}, x_t, x_{t+1}) = u(c_t) - e + \beta \left[ \frac{z^s(h_{t+2}, x^1)}{1-\beta} + (1-\pi) U^f(h_{t+2}, x^1) \right].$$

Notice that also during the first period of UI consumption is $c_t$. This follows trivially from the envelope
condition.\(^{19}\)

The agent’s utility in branch 2 is easily described cross-sectionally. Given \(h_t\), for all \(x_t \in [0, 1]\) we implement \(JM\). Now, let \(h_{t+n} = (1 - \delta)^n h_t\) and consider any history of shocks \(x^n\) following \(h_t\). Then, we implement \(SA\) if and only if \(SA\) was implemented in the original plan at node \((h_{t+n}, x^n)\), while we implement \(JM\) otherwise. The payments are all constant and equal to \(c_t\), no matter what the state or the history of shocks are. We are done since there are no incentive constraint that must be satisfied here. The ex-ante utility the agent gets in this node, can be denoted as

\[
[1 - \beta (1 - \pi)] \frac{u(c_t)}{1 - \beta} - Me = u(c_t) + \beta \pi u(c_t) - Me,
\]

where \(M\) will be specified below. For now, notice that from the first equation in (15), the consumption utility delivered to the agent in this branch coincides with what the agent gets in the old plan in the first period \((u(c_t))\) plus what he gets (in expected terms) during employment if a job is immediately found: \(\beta \pi u(c_t)\).

**Effort cost:** We now show that the effort cost is the same in the old and new plans. The argument resembles to that for the monitoring cost.

First of all note that if the old plan never implemented \(SA\) then the total effort cost of the agent would be \(\frac{e}{1 - \beta}\). And this would also be true for the new program.

Assume that actually the old program implements \(SA\) at some nodes, and let \(q\) be the ‘discounted’ probability of a generic node that in the original program \(W\) implements \(SA\) (i.e., \(a = 0\)) for the first time. Since \(SA\) is absorbing, the total discounted effort cost for the agent would be \(\frac{e}{1 - \beta} - q \frac{e}{1 - \beta}\).

If we show that the new plan has the same effort cost in this special case, we are done since the node and its probability are generic, so by following the same argument one can show that the two programs impose the same effort cost on the agent, no matter how many these nodes are and what their respective probabilities are.

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\(^{19}\)The notation for the old plan is as follows. Let \(x^n\) be the history of \(X\) shocks from period \(t\) till period \(t+n\) emanating from node \((U_t, h_t)\). The law of motion describing the agent’s utility at node \((h_{t+n}, x^n)\), is:

\[
U^{JM}(h_{t+n}, x^n) = z(h_{t+n}, x^n) - e + \beta \left[ \pi \frac{z(h_{t+n+1}, x^n)}{1 - \beta} + (1 - \pi) U^{JM}(h_{t+n+1}, x^n) \right]
\]

if the implemented policy is \(JM\), with \(h_{t+n+1} = (1 - \delta) h_{t+n}\). While the law of motion is

\[
U^{UI}(h_{t+n}, x^n) = z(h_{t+n}, x^n) - e + \beta \left[ \pi \frac{z(h_{t+n+1}, x^n)}{1 - \beta} + (1 - \pi) U^{UI}(h_{t+n+1}, x^n) \right]
\]

if the implemented policy is \(UI\). In both cases, if we denote by \(X^i(h_{t+n+1}, x^n)\) the set of \(x_{t+n+1}\) shocks for which at the beginning of this node program \(i\) is implemented, we have

\[
U^{f}(h_{t+n+1}, x^n) = \int_{0}^{1} \sum_{i \in \{SA, UI, JM\}} \int_{X^i(h_{t+n+1}, x^n)} U^{f}(h_{t+n+1}, x^n, x_{t+n+1}) dx_{t+n+1}.
\]

Finally, because of its absorbing nature, under \(SA\) we have:

\[
U^{SA}(h_{t+n}, x^n) = \frac{z(h_{t+n}, x^n)}{1 - \beta}.
\]
Recall that the new plan is constructed by looking at policies implemented at period \( t+1 \) in the original plan. In branch 1 of the new program, since we started from period \( t \), the same events described above, i.e. the occurrence of \( SA \) for the first time, has a discounted probability of \( \frac{q}{\beta(1-\pi)} \). However, since the sequence \( SA \rightarrow SA \rightarrow SA \ldots \) is always replaced by the sequence \( JM \rightarrow SA \rightarrow SA \ldots \) the discounted probability of this event is in fact \( q \) in the new program for this branch. The expected discounted value of the effort cost saved by the agent with respect to \( \frac{e}{1-\beta} \) in this branch is \( q \frac{e}{1-\beta} \).

Hence, in branch 1 the total discounted effort cost for the agent is \( \frac{e}{1-\beta} - q \frac{e}{1-\beta} \).

Let’s consider now branch 2. To construct the new plan we look at the original contract in period \( t+1 \), so the discounted probability of the event is \( \frac{q}{\beta(1-\pi)} \), but since we postpone \( SA \) for one period, where the implemented policy is \( JM \) instead, we have that this same event has a discounted probability of \( q \), implying a total effort cost saving of \( q \frac{e}{1-\beta} \) with respect to \( \frac{e}{1-\beta} \), i.e., a total cost of only \( \frac{e}{1-\beta} - q \frac{e}{1-\beta} \).

We are hence done since the probabilities of two nodes sum to one.

We are now ready to study branch 1 in more detail. Here, we must show that in this branch of the new plan the incentive compatibility constraints can be satisfied by improving the planner’s net profits. This is done analogously to the proof of Proposition 1. Given \( h_t \) and for each \( x_t \) we can consider a new random variable \( X \), whose realization will be denoted by \( x_{t+1} \). Denoting by \( U^I (x_t, h_{t+1}, x_{t+1}) \) the promised utility delivered by the original program after history \((x_t, h_{t+1}, x_{t+1})\) following node \((U_t, h_t)\), the utility of the agent in the new plan can be computed recursively as follows.

Denote by \( \hat{U}^{B1}_t \) the expected utility the agent gets in branch 1. Then we have

\[
\hat{U}^{B1}_t = \int_{0}^{1} \int_{0}^{1} \hat{U}^{UI} (h_t, x_t, x_{t+1}) \, dx_t \, dx_{t+1}
\]

where

\[
\hat{U}^{UI} (h_t, x_t, x_{t+1}) = u(c_t) - e + \beta \left[ \frac{\hat{z}^s (h_{t+1}, x^1)}{\frac{1}{1-\beta}} + (1 - \pi) \hat{U}^f (h_{t+1}, x^1) \right].
\]

Notice that, as described above, we have \( \hat{z}^s (h_{t+1}, x^1) = z^s (h_{t+2}, x^1) \), that is, the payment in employment is the same. Also \( u(c_t) \) is the same in the two programs. In general, it is useful to compare \( \hat{U}^i (h_{t+n-1}, x^n) \) and \( \hat{U}^f (h_{t+n}, x^n) \) with \( U^i (h_{t+n}, x^n) \) and \( U^f (h_{t+n+1}, x^n) \) in the original program. In particular, we have that \( \hat{U}^f (h_{t+n}, x^n) \leq U^f (h_{t+n+1}, x^n) \) with strict inequality whenever after finitely many periods after node \((h_{t+n+1}, x^n)\) policy \( SA \) is implemented with positive probability. The reason is simple, \( \hat{U}^f (h_{t+n}, x^n) \) is generated by making exactly the same transfers with exactly the same probabilities to the agent, with one exception. Whenever the original program implemented \( SA \) (forever) then \( W' \) implements first \( JM \) and then continues with \( SA \) forever. Since the payment in \( JM \) is exactly the same as under \( SA \) in the original contract, we get two results. First, if at node \((h_{t+n}, x^n)\) the original program implemented \( SA \), i.e. we have \( U^{SA} (h_{t+n}, x^n) = \frac{u(c(h_{t+n}, x^n))}{1-\beta} \). Under \( W' \) we have \( JM \) and

\[
\hat{U}^{JM} (h_{t+n-1}, x^n) = u(c(h_{t+n}, x^n)) - e + \beta \frac{u(c(h_{t+n}, x^n))}{1-\beta} = U^{SA} (h_{t+n}, x^n) - e.
\]

We are now ready to verify that the new program satisfies the incentive constraint. Since \( \hat{U}^f (h_{t+n}, x^n) \leq
If $(h_{t+n+1}, x^n)$, and recalling that the payments upon employment are ‘the same’ as in the old program, the incentive compatibility constraint in (16), for example, can be written as

$$\pi \left[ \frac{\hat{z}^s (h_{t+1}, x)}{1 - \beta} - \hat{U}^f (h_{t+1}, x) \right] \geq \pi \left[ \frac{z^s (h_{t+2}, x)}{1 - \beta} - U^f (h_{t+2}, x^n) \right] \geq \frac{e}{\beta}.$$  

The first inequality is coming from $\hat{z}^s (h_{t+1}, x) = z^s (h_{t+2}, x)$, and $\hat{U}^f (h_{t+1}, x) \leq U^f (h_{t+2}, x^n)$ given the above discussion. The last inequality comes from the fact that at node $(h_{t+2}, x^n)$ the original program implemented $UI$, hence it must have satisfied the incentive compatibility constraint.

Whenever the incentive compatibility constraint is binding in the original contract we have that $z^s (h_{t+2}, x^n) > z (h_{t+1}, x^n) > z^f (h_{t+2}, x^n)$. Reducing $z^s$ and increasing either one of $z$ or $z^f$ or both in an actuarially fair manner improves agent’s welfare. As a consequence, the planner will be able to improve his return by adjusting payments (e.g. reducing $\hat{z}^s$ and increase $z$) while keeping agent’s ex-ante expected utility at the same level at any node of the unemployment state.

Notice that we have shown that there is an incentive feasible plan that improves the planner’s welfare. This delivers initial utilities $\hat{U}^{UI} (h_t, x_t, x_{t+1})$ in period $t$ by implementing policy $UI$. If we allow the planner to maximize both after the first period, and also reallocate utilities and policies in the first period, the planner’s welfare can actually be improved further. In other words, along the same lines of Proposition 1, it is easy to show that the planner’s value at node $(U_t, h_t)$ in the original plan is dominated by the quantity

$$[1 - \beta (1 - \pi)] V (\hat{U}^{B2}_t, h_t) + \beta (1 - \pi) V (\hat{U}^{B1}_t, h_t)$$

where $\hat{U}^{B2}_t = u (c_t) + \beta \pi \frac{u (c_t)}{1 - \beta} - Me$, while $\hat{U}^{B1}_t$ has been defined above. In turn, since we have shown that

$$[1 - \beta (1 - \pi)] \hat{U}^{B2}_t + \beta (1 - \pi) \hat{U}^{B1}_t = U_t,$$

the concavity of $V$ implies that this quantity is dominated by $V (U_t, h_t)$. This concludes the proof since we have generated a contradiction to the fact that the original program was optimal, only by using the fact that $W$ contemplated to switch between $JM$ and $UI$. Q.E.D.

References


Figure 1: The probabilities of implementing each of the three policies in the optimal program where the planner uses lotteries to randomize across policies. A value of 1 (red regions) means that lotteries are not used on that point in the state space. The graph suggests that, as expected, randomizations are mostly used around the borders delimiting different policy regions. It appears that lotteries between UI and JM are used very little, whereas lotteries between JM and SA are used more often, but primarily in regions that are not economically relevant (very high promised utilities).